

A PARAMETRIC FAMILY OF DIHEDRAL EXTENSIONS OF DEGREE 12 OVER $\mathbb{Q}$ AND AN INDEPENDENT SYSTEM OF UNITS

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RÉSUMÉ. Nous utilisons une relation de récurrence non linéaire pour construire une famille paramétrée de polynômes $P_{S,T}$ de degré 6, et nous montrons que le groupe de Galois du corps de décomposition de $P_{S,T}$ sur $\mathbb{Q}(S, T)$ est le groupe diédral D_6 d'ordre 12. Nous montrons qu'après avoir spécialisé en $S = -1$, les extensions diédrales obtenues sont totalement complexes et le groupe des unités est de rang 5. De plus, nous exhibons quatre unités indépendantes qui s'avèrent être exceptionnelles.

ABSTRACT. In this paper, using a non-linear recurrence relation, we construct a parametric family of polynomials $P_{S,T}$ of degree 6, and show that the Galois group of the splitting field of $P_{S,T}$ over $\mathbb{Q}(S, T)$ is the dihedral group D_6 of order 12. We show that after specializing at $S = -1$, the parametric dihedral extensions obtained are totally complex and the rank of the unit group is equal to 5. We also give explicitly four independent units which turn out to be exceptional units.

1. Introduction

Let $f(x) \in \mathbb{Z}[x]$ be a monic irreducible polynomial of degree n , and let α be a zero of f . Let $K = \mathbb{Q}(\alpha)$, let $\mathcal{O}_K$ be the ring of integers, and let $\mathcal{O}_K^*$ be the group of units in K . If f defines a parametric family of polynomials of degree n , it is difficult in general to find a fundamental system of units in a parametric form. There is no general method to find a fundamental system of parametric units for any given parametric family of number fields. To our best knowledge, the parametric number fields of highest degree in the literature for which a fundamental system of parametric units has been exhibited are parametric families of number fields of degree 10 (see [1], [2]). As far as the unit rank is concerned, the record is 7 and belongs to Shen [3].

In this paper, using a non-linear recurrence relation, we construct a parametric family of polynomials $P_{S,T}$ of degree 6, and show that the Galois group of the splitting field of $P_{S,T}$ over $\mathbb{Q}(S, T)$ is the dihedral group D_6 of order 12. We show that after specializing at $S = -1$ or $S = -3$, the parametric dihedral extensions obtained are totally complex and the rank of the unit group is equal to 5. We also exhibit four independent units which turn out to be exceptional units.

2. The polynomial $P_{S,T}$

Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence that satisfies the relation

$$(1) \quad x_{i-1}x_{i+1} = x_i.$$

To avoid trivial sequences, we add the restriction that $x_i \neq 0$ for any i .

Theorem 2.1. *The non linear recurrence that satisfies (1) is cyclic and of order 6.*

Proof. From (1), we have

$$(2) \quad x_i x_{i+2} = x_{i+1}$$

and

$$(3) \quad x_{i+1} x_{i+3} = x_{i+2}.$$

Substituting (2) into (3) gives $x_i x_{i+2} x_{i+3} = x_{i+2}$, which implies

$$(4) \quad x_i x_{i+3} = 1.$$

Setting $i = i + 3$ gives $x_{i+3} x_{i+6} = 1$; consequently, $x_i x_{i+3} = x_{i+3} x_{i+6}$, and

$$x_i = x_{i+6}. \quad \square$$

Given a sequence that satisfies (1), we define S and T as follows:

$$\begin{cases} S = x_1 + x_2 + x_3 + x_4 + x_5 + x_6, \\ T = x_1 x_2 + x_2 x_3 + x_3 x_4 + x_4 x_5 + x_5 x_6 + x_6 x_1. \end{cases}$$

Let $\sigma_1, \sigma_2, \dots, \sigma_6$ be the elementary functions of Newton. A calculations shows that

$$\begin{cases} \sigma_1 = S, \\ \sigma_2 = T + S + 3, \\ \sigma_3 = S^2 - 2T - 4, \\ \sigma_4 = T + S + 3, \\ \sigma_5 = S, \\ \sigma_6 = 1. \end{cases}$$

Then the polynomial $\prod_{i=1}^6 (x - x_i)$ is

$$(5) \quad P_{S,T}(x) = x^6 - Sx^5 + (T + S + 3)x^4 - (S^2 - 2T - 4)x^3 + (T + S + 3)x^2 - Sx + 1.$$

Using MAPLE, we found that the discriminant of $P_{S,T}$ is

$$D(S, T) = -(S + 2)^2(T^2 + 12ST + 24T + 36 - 4S^3 + 36S)^2(4T - S^2 + 12)^3.$$

3. The Galois group

Let G be the Galois group of $L = \mathbb{Q}(x_1, x_2, x_3, x_4, x_5, x_6)$ over $\mathbb{Q}(S, T)$

Theorem 3.1. *The group G is isomorphic to D_6 , the dihedral group of order 12.*

Proof. We will consider the roots of $P_{S,T}$ to be the vertices of a hexagon, as shown in Figure 1.

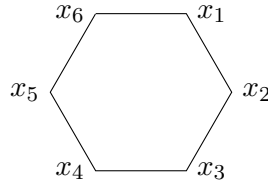


FIGURE 1. The roots of $f(x)$ as vertices of a hexagon

Assume $\sigma \in G$. Let $i \in \{1, \dots, 6\}$. Then $\sigma(x_i) = x_j$ for some $j \in \{1, \dots, 6\}$. From (4), $x_i x_{i+3} = 1$ for all i , so $\sigma(x_i) \sigma(x_{i+3}) = 1$, and if $\sigma(x_i) = x_j$, then $\sigma(x_{i+3}) = x_{j+3}$. Suppose that $\sigma(x_1) = x_i$, for some $i \in \{1, \dots, 6\}$. Then $\sigma(x_4) = x_{i+3}$. We will first show that either $\sigma(x_2) = x_{i+1}$ or $\sigma(x_2) = x_{i-1}$.

Assume first that $\sigma(x_2) = x_{i+2}$. Then $\sigma(x_5) = x_{i+5}$ and either $\sigma(x_3) = x_{i+1}$ and $\sigma(x_6) = x_{i+4}$, or $\sigma(x_3) = x_{i+4}$ and $\sigma(x_6) = x_{i+1}$, as in Figure 2.

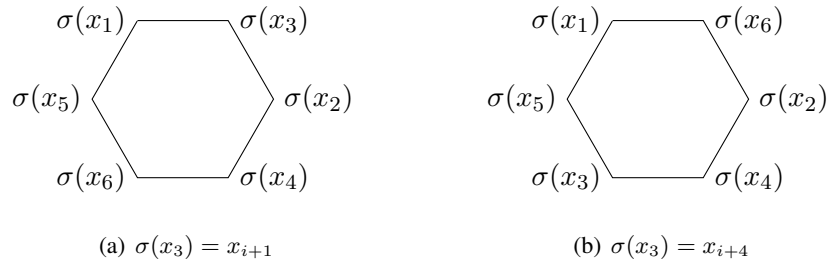


FIGURE 2. Possibilities for $\sigma(x_3)$ and $\sigma(x_6)$

If $\sigma(x_3) = x_{i+1}$ and $\sigma(x_6) = x_{i+4}$, then from (1) we have $x_i x_{i+2} = x_{i+1}$, which implies

$$(6) \quad \sigma(x_1) \sigma(x_2) = \sigma(x_3),$$

and $x_1 x_3 = x_2$, which implies

$$(7) \quad \sigma(x_1) \sigma(x_3) = \sigma(x_2).$$

Substituting (7) into (6) gives $\sigma(x_1) \sigma(x_1) \sigma(x_3) = \sigma(x_3)$, implying $\sigma(x_1)^2 = 1$, or $x_i^2 = 1$. This is a contradiction since x_i is the root of an irreducible polynomial of

degree 6. Thus, we must have $\sigma(x_3) = x_{i+4}$ and $\sigma(x_6) = x_{i+1}$. From (1), we have $x_i x_{i+2} = x_{i+1}$, $x_{i+1} x_{i+3} = x_{i+2}$ and $x_{i+2} x_{i+4} = x_{i+3}$, which implies

$$(8) \quad \sigma(x_1)\sigma(x_2) = \sigma(x_6),$$

$$(9) \quad \sigma(x_6)\sigma(x_4) = \sigma(x_2),$$

and

$$(10) \quad \sigma(x_2)\sigma(x_3) = \sigma(x_4).$$

Substituting (8) and (10) into (9) gives $\sigma(x_1)\sigma(x_2)\sigma(x_2)\sigma(x_3) = \sigma(x_2)$, implying

$$\sigma(x_1)\sigma(x_2)\sigma(x_3) = 1.$$

Since $x_1 x_3 = x_2$ implies $\sigma(x_1)\sigma(x_3) = \sigma(x_2)$, we are left with $\sigma(x_2)^2 = 1$, so $x_{i+2}^2 = 1$. Again, this is a contradiction, since x_{i+2} is the root of an irreducible polynomial of degree 6; consequently, $\sigma(x_2) = x_{i+2}$ is impossible. The case where $\sigma(x_2) = x_{i+4}$ is similar due to symmetry, and also leads to contradictions; thus, either $\sigma(x_2) = x_{i+1}$ or $\sigma(x_2) = x_{i-1}$.

Next we will show that σ permutes the hexagon. Assume that $\sigma(x_2) = x_{i+1}$. Then $\sigma(x_5) = x_{i+4}$ and, either $\sigma(x_3) = x_{i+2}$ and $\sigma(x_6) = x_{i+5}$, or $\sigma(x_3) = x_{i+5}$ and $\sigma(x_6) = x_{i+2}$, as in Figure 3.

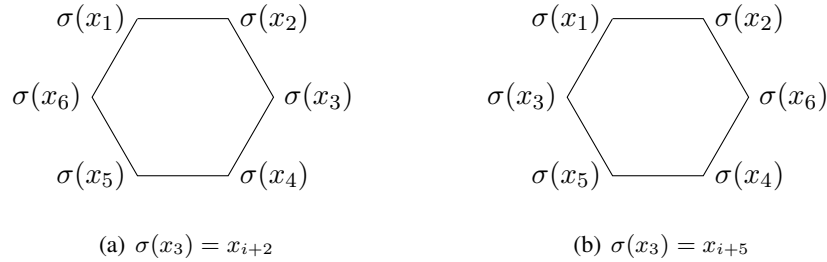


FIGURE 3. Possibilities for $\sigma(x_3)$ and $\sigma(x_6)$

Assume that $\sigma(x_3) = x_{i+5}$. Then (1) gives $x_i x_{i+2} = x_{i+1}$, which means

$$\sigma(x_1)\sigma(x_6) = \sigma(x_2).$$

Since $x_1 x_3 = x_2$ implies $\sigma(x_1)\sigma(x_3) = \sigma(x_2)$, we have $\sigma(x_1)\sigma(x_6) = \sigma(x_1)\sigma(x_3)$, so

$$\sigma(x_6) = \sigma(x_3) = x_{i+2}.$$

Thus, when $\sigma(x_2) = x_{i+1}$, we must have $\sigma(x_3) = x_{i+2}$. The case where $\sigma(x_2) = x_{i-1}$ can be handled similarly due to symmetry, and results in $\sigma(x_3) = x_{i-2}$. We conclude that σ permutes the original hexagon, and $\sigma \in D_6$, proving that $G \subseteq D_6$.

Finally, set $S = -1$ and $T = 1$, and let the splitting field of the corresponding polynomial be K . Calculating the Galois group gives $\text{Gal}(K/\mathbb{Q}) = D_6$. Hence, the Galois group G of L over $\mathbb{Q}(S, T)$ is D_6 , the dihedral group of order 12. Then, if we specialize S at -1 , there exist infinitely many integers T such that the Galois group of the splitting field of the corresponding polynomial over $\mathbb{Q}$ is D_6 . $\square$

Given $G = D_6$, we define the following automorphisms: let e be the identity automorphism, let σ be the automorphism given by $\sigma(x_1) = x_2$ and $\sigma(x_2) = x_3$, and let τ be the automorphism given by $\tau(x_1) = x_3$ and $\tau(x_2) = x_2$. Then

$$G = \{e, \sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3, \tau\sigma^4, \tau\sigma^5\}.$$

4. Units in L

Suppose that S and T are now integers. Then, our number field is $L = \mathbb{Q}(x_1, x_2)$ and the Galois group is $G = \text{Gal}(L/\mathbb{Q})$.

Theorem 4.1. *For $S, T \in \mathbb{Z}$, the elements x_i are units of L , for $i = 1, \dots, 6$.*

Proof. Since S and T are integers, the polynomial in (5) is a monic polynomial with integer coefficients, so its roots are algebraic integers. Also, for each i , the norm of x_i is $N(x_i) = (x_1 x_2 x_3 x_4 x_5 x_6)^2 = 1$. $\square$

If we specialize S at -1 and $T > 0$, then the discriminant D is less than zero. Hence, some x_i is complex, and all of the embeddings are complex. Since $|G| = 12$, there are 6 pairs of complex embeddings, and by the Dirichlet Unit Theorem, a fundamental system of units will have rank 5.

Theorem 4.2. *For $S = -1$ and $T \in \mathbb{Z}$, $x_i + 1$ is a unit of L , for $i = 1, \dots, 6$.*

Proof. Given any $i \in \{1, \dots, 6\}$, since x_i is an algebraic integer, $x_i + 1$ is also an algebraic integer. Calculating the norm $N(x_i + 1)$ and simplifying using (1) gives $N(x_i + 1) = (4 + 4S + S^2)^2$. Then, $N(x_i + 1) = 1$ when $S = -1$. $\square$

Proposition 4.3. *The following equality holds true:*

$$(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = x_{i+1}.$$

Proof. We will show that

$$x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = 1,$$

and it will follow from (2) that $(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = x_{i+1}$. Expanding $x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1)$ gives

$$\begin{aligned} x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) &= x_{i+4}x_i x_{i+1} x_{i+2} + x_{i+4}x_i x_{i+1} + x_{i+4}x_i x_{i+2} \\ &\quad + x_{i+4}x_i + x_{i+4}x_{i+1} x_{i+2} + x_{i+4}x_{i+1} \\ &\quad + x_{i+4}x_{i+2} + x_{i+4}. \end{aligned}$$

Simplifying with the help of (4), we obtain

$$\begin{aligned} x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) &= x_i x_{i+2} + x_i + x_{i+4}x_i x_{i+2} + x_{i+4}x_i + x_{i+2} + 1 \\ &\quad + x_{i+4}x_{i+2} + x_{i+4}. \end{aligned}$$

Applying (2) gives

$$x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = x_{i+1} + x_i + x_{i+5}x_{i+2} + x_{i+5} + x_{i+2} + 1 + x_{i+3} + x_{i+4}.$$

Applying (4) once more gives

$$x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = x_{i+1} + x_i + 1 + x_{i+5} + x_{i+2} + 1 + x_{i+3} + x_{i+4},$$

which simplifies to $x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4} + x_{i+5} + 2$, namely $S + 2$.

When $S = -1$, this gives

$$x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = 1$$

and

$$(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = x_{i+1}. \quad \square$$

From the equality $x_{i+4}(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = 1$, we deduce that

$$\frac{1}{x_i + 1} = x_{i+4}(x_{i+1} + 1)(x_{i+2} + 1),$$

providing another proof that $x_i + 1$ is a unit of L .

We would like to find five independent units. We first note that any set of five units of the form x_i and $x_i + 1$ cannot be independent. This can be seen by considering the relations

$$(x_i + 1)(x_{i+1} + 1)(x_{i+2} + 1) = x_{i+1} \quad \text{and} \quad \frac{x_i + 1}{x_{i+3} + 1} = x_i$$

along with (1). We will show, however, that the units we have found provide us with four independent units.

Theorem 4.4. *For $S = -1$, the units $x_1, x_2, x_1 + 1$ and $x_2 + 1$ are independent.*

Proof. Assume that $x_1^a x_2^b (x_1 + 1)^c (x_2 + 1)^d = 1$.

Part 1: *Proving $d = 0$.* Let $A = x_1^a x_2^b (x_1 + 1)^c (x_2 + 1)^d$. Then

$$\begin{cases} \tau\sigma^2(A) &= x_1^a x_6^b (x_1 + 1)^c (x_6 + 1)^d = 1, \\ \tau\sigma^5(A) &= x_4^a x_3^b (x_4 + 1)^c (x_3 + 1)^d = 1. \end{cases}$$

Clearly $\tau\sigma^2(A)\tau\sigma^5(A) = 1$. Using (4), we obtain

$$\begin{cases} x_1^a x_6^b (x_1 + 1)^c (x_6 + 1)^d x_4^a x_3^b (x_4 + 1)^c (x_3 + 1)^d &= 1, \\ (x_1 x_4)^a (x_6 x_3)^b (x_1 + 1)^c (x_6 + 1)^d (x_4 + 1)^c (x_3 + 1)^d &= 1, \end{cases}$$

and

$$(11) \quad (x_1 + 1)^c (x_6 + 1)^d (x_4 + 1)^c (x_3 + 1)^d = 1.$$

Applying $\tau\sigma^2$ to (11) gives

$$(12) \quad (x_1 + 1)^c (x_2 + 1)^d (x_4 + 1)^c (x_5 + 1)^d = 1.$$

The equations (11) and (12) give

$$(x_1 + 1)^c (x_6 + 1)^d (x_4 + 1)^c (x_3 + 1)^d = (x_1 + 1)^c (x_2 + 1)^d (x_4 + 1)^c (x_5 + 1)^d,$$

so that

$$(x_6 + 1)^d (x_3 + 1)^d = (x_2 + 1)^d (x_5 + 1)^d$$

and

$$(13) \quad \left(\frac{(x_6 + 1)(x_3 + 1)}{(x_2 + 1)(x_5 + 1)} \right)^d = 1.$$

If $d \neq 0$, then

$$\frac{(x_6 + 1)(x_3 + 1)}{(x_2 + 1)(x_5 + 1)}$$

is a d -th root of 1. Since

$$\frac{(x_6 + 1)(x_3 + 1)}{(x_2 + 1)(x_5 + 1)} \in L,$$

the cyclotomic field $\mathbb{Q} \left(\frac{(x_6 + 1)(x_3 + 1)}{(x_2 + 1)(x_5 + 1)} \right)$ is a subfield of L , with degree over $\mathbb{Q}$ a divisor of 12 and of $\phi(d)$. The abelian subgroups of $G = D_6$ have orders 1, 2, 3, 4 and 6; consequently the possible degrees of the corresponding subfields of L over $\mathbb{Q}$ are 12, 6, 4, 3 and 2. Thus, there are only a finite number of possible values of d .

Let $B = (x_2 + 1)(x_5 + 1)$. From (13), $((x_6 + 1)(x_3 + 1))^d = ((x_2 + 1)(x_5 + 1))^d$, and applying σ gives

$$((x_1 + 1)(x_4 + 1))^d = ((x_3 + 1)(x_6 + 1))^d.$$

Then

$$B^d = ((x_1 + 1)(x_4 + 1))^d = ((x_2 + 1)(x_5 + 1))^d = ((x_3 + 1)(x_6 + 1))^d.$$

It is clear that B^d is fixed by the Galois group, so $B^d \in \mathbb{Q}$. Since $N(x_i + 1) = 1$, we have

$$\begin{aligned} B^{6d} &= ((x_1 + 1)(x_2 + 1)(x_3 + 1)(x_4 + 1)(x_5 + 1)(x_6 + 1))^{2d} \\ &= (N(x_i + 1))^d \\ &= 1. \end{aligned}$$

Now $B^d \in \mathbb{Q}$ and $B^{6d} = 1$, so $B^d = \pm 1$. From (4), $x_5 = \frac{1}{x_2}$, so

$$B = (x_2 + x_5 + 2) = \frac{x_2^2 + 1 + 2x_2}{x_2}.$$

If $B^d = 1$, then $\left(\frac{x_2^2 + 1 + 2x_2}{x_2} \right)^d = 1$, or equivalently, $(x_2^2 + 1 + 2x_2)^d - x_2^d = 0$. Then x_2 is a root of $g(x) = (x^2 + 1 + 2x)^d - x^d$, and is also a root of an irreducible polynomial of degree 6. This implies that $g(x)$ must be irreducible of degree 6, or have an irreducible factor of degree 6. Factoring $g(x)$ for all possible values of d shows that $g(x)$ is never irreducible of degree 6, and never has an irreducible factor of degree 6. Thus, $B^d = -1$, which implies

$$\left(\frac{x_2^2 + 1 + 2x_2}{x_2} \right)^d = -1 \quad \text{or} \quad (x_2^2 + 1 + 2x_2)^d + x_2^d = 0.$$

Then x_2 is a root of $h(x) = (x^2 + 1 + 2x)^d + x^d$, which implies that $h(x)$ must be irreducible of degree 6, or have an irreducible factor of degree 6. Factoring $h(x)$ for all

possible values of d shows that this never holds. Consequently, $\frac{(x_6 + 1)(x_3 + 1)}{(x_2 + 1)(x_5 + 1)}$ is not a root of 1, and $d = 0$.

Part 2: Proving $c = 0$. Since $d = 0$, we have

$$\begin{cases} A &= x_1^a x_2^b (x_1 + 1)^c = 1, \\ \sigma^3(A) &= x_4^a x_5^b (x_4 + 1)^c = 1, \\ \tau\sigma(A) &= x_2^a x_1^b (x_2 + 1)^c = 1, \\ \tau\sigma^4(A) &= x_5^a x_4^b (x_5 + 1)^c = 1. \end{cases}$$

Then $A\sigma^3(A) = \tau\sigma(A)\tau\sigma^4(A)$ gives

$$x_1^a x_2^b (x_1 + 1)^c x_4^a x_5^b (x_4 + 1)^c = x_2^a x_1^b (x_2 + 1)^c x_5^a x_4^b (x_5 + 1)^c,$$

which can be simplified to

$$(x_1 x_4)^a (x_2 x_5)^b (x_1 + 1)^c (x_4 + 1)^c = (x_2 x_5)^a (x_1 x_4)^b (x_2 + 1)^c (x_5 + 1)^c.$$

Using (4), this becomes $(x_1 + 1)^c (x_4 + 1)^c = (x_2 + 1)^c (x_5 + 1)^c$, or

$$\left(\frac{(x_1 + 1)(x_4 + 1)}{(x_2 + 1)(x_5 + 1)} \right)^c = 1.$$

Applying τ gives

$$\left(\frac{(x_3 + 1)(x_6 + 1)}{(x_2 + 1)(x_5 + 1)} \right)^c = 1.$$

However, as in Part 1, $\frac{(x_3 + 1)(x_6 + 1)}{(x_2 + 1)(x_5 + 1)}$ cannot be a c -th root of 1, so we must have $c = 0$.

Part 3: Proving $a = 0$. Since $c = d = 0$, we have

$$\begin{cases} A &= x_1^a x_2^b = 1, \\ \tau(A) &= x_3^a x_2^b = 1. \end{cases}$$

It is clear that $x_1^a x_2^b = x_3^a x_2^b$, so $x_1^a = x_3^a$. Applying σ gives $x_2^a = x_4^a$, and multiplying these equations together gives $x_1^a x_2^a = x_3^a x_4^a$, which simplifies to $x_1^a x_1^a x_3^a = x_3^a \frac{1}{x_1^a}$ by (1) and (4). Thus, we have $x_1^{3a} = 1$. If $x_1 = 1$, then applying $\sigma, \sigma^2, \sigma^3, \sigma^4$, and σ^5 , we obtain that $x_i = 1$, for $i = 1, \dots, 6$, which is impossible because $S = -1$. We can easily prove that $x_i \neq -1$. So if $a \neq 0$, then $x_1 \neq \pm 1$ but x_1 is a root of unity. Then, by applying the automorphisms, we obtain that each x_i , for $i = 1, \dots, 6$, is also a root of unity, and L is the cyclotomic field $\mathbb{Q}(x_1)$. If x_1 is an n -th root of unity, then the Galois group is abelian; however, $G = D_6$, is not abelian. We conclude that x_1 cannot be a root of unity, and $a = 0$.

Part 4: Proving $b = 0$. Since $a = c = d = 0$, we now have $A = x_2^b = 1$, which implies that x_2 is a root of unity if $b \neq 0$. As shown in Part 3, this is impossible. Consequently, we have $b = 0$.

We have now shown that $a = b = c = d = 0$, and then, the units $x_1, x_2, (x_1 + 1)$ and $(x_2 + 1)$ are independent. $\square$

It would be interesting to find a fifth unit ϵ such that the units $x_1, x_2, (x_1+1), (x_2+1)$ and ϵ are independent.

Acknowledgements. The authors express their gratitude to the anonymous referee for constructive suggestions which improved the quality of the paper. The first author was supported in part by NSERC.

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