

THE COMPOSITION PRODUCT ON POLYNOMIALS AND HERMITE BASIS POLYNOMIALS

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RÉSUMÉ. Une structure de produit, appelée *produit de composition*, est définie sur les polynômes. Le produit de composition est une généralisation d’une structure de produit classique due à N. G. de Bruijn. Nous en donnons des applications pour la localisation des zéros pour le produit de composition, la divisibilité algébrique et géométrique pour ce produit, et nous étendons ces propriétés aux fonctions entières.

ABSTRACT. A product structure, termed the *composition product*, is defined on polynomials. The composition product is a generalization of a classic product structure defined on the Laguerre-Pólya class, by N. G. de Bruijn. Applications are given to location of zeros of the composition product, to formulations of algebraic and geometric divisibility for this product, with extensions to entire functions.

1. Introduction and background

N. G. de Bruijn introduced an implicit form of the composition product applied to functions in the Laguerre-Pólya class [5]. This class consists of all entire functions that are uniform limits on compacta of polynomials having only real zeros. The composition product developed here generalizes de Bruijn’s product for polynomials [3] by removing the restriction that their zeros be real, and by introducing a real valued parameter. De Bruijn’s product is then a special case of the composition product. To this end, the *Weierstrass operators* $e^{\nu D^2}$ and $e^{-\nu D^2}$, $\nu \geq 0$, are both employed in defining the composition product. These operators were introduced by I. I. Hirschman and D. V. Widder who studied these operators with the emphasis on obtaining an inversion formula for $e^{\nu D^2}$, appropriately denoted $e^{-\nu D^2}$, for various classes of functions [9]. In other sources, the integrals defining the operators $e^{\nu D^2}$ and $e^{-\nu D^2}$ are the (parametrized) Weierstrass transform and inverse Weierstrass transform, respectively [19]. The author has investigated the composition product for the case $\nu \geq 0$ in [7]. This work showed closure of the composition product for entire functions in Korevaar classes [6], with applications to $e^{\nu D^2}$, for $\nu \geq 0$. In contrast, the results below remove the restriction $\nu \geq 0$ by restricting the investigation of the composition product to polynomials, and formulating the notion of division for this product.

If λ is complex, then the *translation operator* is defined by

$$e^{\lambda D} := \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} D^k,$$

where D denotes *differentiation*. For polynomials $P(z) = \sum_{k=0}^n a_k z^k$, translation gives the series

$$e^{\lambda D} P(z) = \sum_{k=0}^n \frac{\lambda^k}{k!} (D^k P)(z) = P(z + \lambda).$$

For $\nu > 0$, formally replacing the variable z with operator D in the integrals

$$e^{\nu z^2} = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} e^{-\tau z - \tau^2/4\nu} d\tau \quad \text{and} \quad e^{-\nu z^2} = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} e^{i\tau z - \tau^2/4\nu} d\tau,$$

defines operations on polynomials, given by

$$\begin{cases} e^{\nu D^2} P(z) = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} P(z - \tau) e^{-\tau^2/4\nu} d\tau, \\ e^{-\nu D^2} P(z) = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} P(z + i\tau) e^{-\tau^2/4\nu} d\tau. \end{cases}$$

The Cauchy integral formula [4] applied to these integrals yields *Weierstrass operators*

$$\begin{cases} e^{\nu D^2} P(z) = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} P(t) e^{-(z-t)^2/4\nu} dt, \\ e^{-\nu D^2} P(z) = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} P(it) e^{-(iz+t)^2/4\nu} dt, \end{cases}$$

which clearly exist for polynomials. In view of these integrals, it is seen that

$$\left(e^{-\nu D^2} P(z) \right)^{\sim} = e^{\nu D^2} \tilde{P}(z),$$

where $\tilde{P}(z) := P(iz)$.

Applied to ordinary basis polynomials, these operators yield the parametrized Hermite polynomials

$$e^{\pm\nu D^2} z^n = e^{\mp z^2/4\nu} (\pm 2\nu D)^n e^{\pm z^2/4\nu} = \sqrt{\mp\nu}^n H_n \left(\frac{z}{\sqrt{\mp 4\nu}} \right),$$

where $H_n(z) := e^{z^2}(-D)^n e^{-z^2}$, $n = 0, 1, \dots$, is the n -th *Hermite polynomial* [10]. The first few are given by

$$\left\{ \begin{array}{l} e^{-\nu D^2} 1 = \sqrt{\nu}^0 H_0 \left(\frac{z}{\sqrt{4\nu}} \right) = 1, \\ e^{-\nu D^2} z = \sqrt{\nu}^1 H_1 \left(\frac{z}{\sqrt{4\nu}} \right) = z, \\ e^{-\nu D^2} z^2 = \sqrt{\nu}^2 H_2 \left(\frac{z}{\sqrt{4\nu}} \right) = z^2 - 2\nu, \\ e^{-\nu D^2} z^3 = \sqrt{\nu}^3 H_3 \left(\frac{z}{\sqrt{4\nu}} \right) = z^3 - 6\nu z, \\ e^{-\nu D^2} z^4 = \sqrt{\nu}^4 H_4 \left(\frac{z}{\sqrt{4\nu}} \right) = z^4 - 12\nu z^2 + 12\nu^2, \\ e^{-\nu D^2} z^5 = \sqrt{\nu}^5 H_5 \left(\frac{z}{\sqrt{4\nu}} \right) = z^5 - 20\nu z^3 + 60\nu^2 z, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} e^{\nu D^2} 1 = \sqrt{-\nu}^0 H_0 \left(\frac{z}{\sqrt{-4\nu}} \right) = 1, \\ e^{\nu D^2} z = \sqrt{-\nu}^1 H_1 \left(\frac{z}{\sqrt{-4\nu}} \right) = z, \\ e^{\nu D^2} z^2 = \sqrt{-\nu}^2 H_2 \left(\frac{z}{\sqrt{-4\nu}} \right) = z^2 + 2\nu, \\ e^{\nu D^2} z^3 = \sqrt{-\nu}^3 H_3 \left(\frac{z}{\sqrt{-4\nu}} \right) = z^3 + 6\nu z, \\ e^{\nu D^2} z^4 = \sqrt{-\nu}^4 H_4 \left(\frac{z}{\sqrt{-4\nu}} \right) = z^4 + 12\nu z^2 + 12\nu^2, \\ e^{\nu D^2} z^5 = \sqrt{-\nu}^5 H_5 \left(\frac{z}{\sqrt{-4\nu}} \right) = z^5 + 20\nu z^3 + 60\nu^2 z. \end{array} \right.$$

Since the relationship between transforms $e^{\pm\nu D^2} z^n$ is a formal exchange of parameter $-\nu \leftrightarrow \nu$, for $n = 0, 1, \dots$, the presentation is simplified if ν is taken to be real. If ν is real, then the transforms for polynomials $P(z) = \sum_{k=0}^n p_k z^k$ are given by

$$e^{-\nu D^2} P(z) = \sum_{k=0}^n p_k e^{-\nu D^2} z^k = \sum_{k=0}^n p_k H_k^\nu(z),$$

by denoting parametrized Hermite polynomials

$$H_n^\nu(z) := e^{-\nu D^2} z^n = \sqrt{\nu}^n H_n \left(\frac{z}{\sqrt{4\nu}} \right),$$

for $n = 0, 1, \dots$. Denoting the operator

$$D_\nu := e^{-z^2/4\nu} (2\nu D) e^{z^2/4\nu} = z + 2\nu D,$$

it follows that $(D_\nu)^j 1 = H_j^{-\nu}(z)$, and more generally

$$(D_\nu)^j H_k^{-\nu}(z) = H_{j+k}^{-\nu}(z),$$

for $j, k = 0, 1, \dots$. The Weierstrass operator has the operational property

$$(D_\nu)^k e^{\nu D^2} = e^{\nu D^2} z^k,$$

for $k = 0, 1, \dots$, whenever ν is real. Thus, for $P(z)$ given above, the polynomial operator

$$P(D)_\nu := P(D_\nu) = \sum_{k=0}^n p_k (D_\nu)^k,$$

gives the operational property

$$P(D)_\nu e^{\nu D^2} = e^{\nu D^2} P(z).$$

The Weierstrass operator has a series representation

$$e^{\nu D^2} = \sum_{k=0}^{\infty} \frac{\nu^k}{k!} D^{2k},$$

whenever ν real. Indeed, it suffices to validate this result for z^n , $n = 0, 1, \dots$. The result clearly holds for $n = 0$. Assuming the result holds for z^{n-1} , it follows that

$$\begin{aligned} e^{\nu D^2} z^n &= D_\nu e^{\nu D^2} z^{n-1} \\ &= (z + 2\nu D) \sum_{k=0}^{\infty} \frac{\nu^k}{k!} D^{2k} z^{n-1} \\ &= z^n + \frac{\nu}{1!} (z D^2 + 2D) z^{n-1} + \frac{\nu^2}{2!} (z D^4 + 4D^3) z^{n-1} + \dots \\ &= z^n + \frac{\nu}{1!} D^2 z^n + \frac{\nu^2}{2!} D^4 z^n + \frac{\nu^3}{3!} D^6 z^n + \dots \\ &= \sum_{k=0}^{\infty} \frac{\nu^k}{k!} D^{2k} z^n. \end{aligned}$$

The result now follows by induction. Thus, for $P(z)$ given above, this representation gives

$$e^{\nu D^2} P(z) = \sum_{k=0}^n \frac{\nu^k}{k!} P^{(2k)}(z),$$

whenever ν real. In view of this representation, we have $e^{\nu D^2} P(z) \rightarrow P(z)$ as $\nu \rightarrow 0$, uniformly on compacta, where the identity operator corresponds to $\nu = 0$. Operators $e^{\nu_1 D^2}$, $e^{\nu_2 D^2}$, with ν_1, ν_2 real, compose by additivity

$$e^{\nu_1 D^2} e^{\nu_2 D^2} = e^{(\nu_1 + \nu_2) D^2}.$$

Indeed, applied to polynomials, these operations are finite sums, so that

$$\begin{aligned}
 e^{(\nu_1+\nu_2)D^2} &= \sum_{k=0}^{\infty} \frac{(\nu_1+\nu_2)^k}{k!} D^{2k} = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\sum_{l=0}^k \binom{k}{l} \nu_1^l \nu_2^{k-l} \right] D^{2k} \\
 &= \sum_{l=0}^{\infty} \nu_1^l \left[\sum_{k=l}^{\infty} \frac{1}{k!} \binom{k}{l} \nu_2^{k-l} D^{2k} \right] = \sum_{l=0}^{\infty} \frac{\nu_1^l}{l!} \left[\sum_{j=0}^{\infty} \frac{\nu_2^j}{j!} D^{2j} \right] D^{2l} \\
 &= \sum_{l=0}^{\infty} \frac{\nu_1^l}{l!} D^{2l} \sum_{j=0}^{\infty} \frac{\nu_2^j}{j!} D^{2j} = e^{\nu_1 D^2} e^{\nu_2 D^2}.
 \end{aligned}$$

The *inversion formulae*, given by

$$e^{-\nu D^2} (e^{\nu D^2} P) = P \quad \text{and} \quad e^{\nu D^2} (e^{-\nu D^2} P) = P,$$

now follow directly. For polynomials P , it is a routine exercise to verify that the Weierstrass operator commutes with the translation and reflection operators, so that

$$\begin{cases} e^{\lambda D} (e^{\nu D^2} P) = e^{\nu D^2} (e^{\lambda D} P), \\ (e^{\nu D^2} P)^- = e^{\nu D^2} (P^-), \end{cases}$$

with λ complex and ν real, where $P^-(z) := P(-z)$.

For ν real, the *composition product* of two polynomials P_1 and P_2 is denoted by $[P_1, P_2]_{\nu}$, and defined by

$$[P_1, P_2]_{\nu} = e^{\nu D^2} \left[(e^{-\nu D^2} P_1) (e^{-\nu D^2} P_2) \right].$$

The extension of these products to polynomials P_k , for $k = 1, 2, \dots, n$, is given recursively by

$$[P_1, P_2, \dots, P_n]_{\nu} = \left[[P_1, P_2, \dots, P_{n-1}]_{\nu}, P_n \right]_{\nu},$$

for $n = 3, 4, \dots$. Since the Weierstrass operator commutes with the translation and reflection operators, the composition product inherits the commutative property, for polynomials P_k , $k = 1, 2, \dots, n$, so that

$$\begin{cases} e^{\lambda D} [P_1, P_2, \dots, P_n]_{\nu} = [e^{\lambda D} P_1, e^{\lambda D} P_2, \dots, e^{\lambda D} P_n]_{\nu}, \\ [P_1, P_2, \dots, P_n]_{\nu}^- = [P_1^-, P_2^-, \dots, P_n^-]_{\nu}, \end{cases}$$

with λ complex and ν real. For polynomials P_1 and P_2 , the expression $P_1(D)_{\nu} P_2$ is given in terms of the composition product. Indeed, in view of the operational property for $P(D)_{\nu}$ given above, it follows that

$$P_1(D)_{\nu} P_2 = P_1(D)_{\nu} e^{\nu D^2} \left[e^{-\nu D^2} P_2 \right] = e^{\nu D^2} \left[P_1 e^{-\nu D^2} P_2 \right] = [e^{\nu D^2} P_1, P_2]_{\nu},$$

whenever ν is real. The next two lemmas are needed in the next section, and give the composition product in terms of Hermite expansions.

Lemma 1.1. *If ν is real,*

$$P_1(z) = \sum_{k=0}^n p_{1k} H_k^{-\nu}(z) \quad \text{and} \quad P_2(z) = \sum_{k=0}^n p_{2k} H_k^{-\nu}(z),$$

then the composition product $[P_1, P_2]_\nu$ is given by the Hermite expansion

$$[P_1, P_2]_\nu(z) = \sum_{k=0}^{2n} r_k H_k^{-\nu}(z),$$

where $r_k = \sum p_{1k_1} p_{2k_2}$, with $k = k_1 + k_2$, is the Cauchy product of the coefficients of P_1 and P_2 .

Proof. Since $e^{-\nu D^2} P_1(z) = \sum_{k=0}^n p_{1k} z^k$ and $e^{-\nu D^2} P_2(z) = \sum_{k=0}^n p_{2k} z^k$, it follows that

$$[P_1(z), P_2(z)]_\nu = e^{\nu D^2} \left[e^{-\nu D^2} P_1(z), e^{-\nu D^2} P_2(z) \right] = \sum_{k=0}^{2n} r_k H_k^{-\nu}(z),$$

where $r_k, k = 0, 1, \dots, 2n$, is the Cauchy product of p_{1k} and $p_{2k}, k = 0, 1, \dots, n$. $\square$

The last lemma extends by an induction argument to generalized Cauchy products of the coefficients of polynomials.

Lemma 1.2. *If ν is real, and if P_l has a Hermite expansion*

$$P_l(z) = \sum_{k=0}^n p_{lk} H_k^{-\nu}(z),$$

for $l = 1, 2, \dots, m$, then the composition product $[P_1, P_2, \dots, P_m]_\nu$ is given by the Hermite expansion

$$[P_1(z), P_2(z), \dots, P_m(z)]_\nu = \sum_{k=0}^{mn} r_k H_k^{-\nu}(z),$$

where the generalized Cauchy product of the coefficients is given by

$$r_k = \sum_{k_1 + \dots + k_m = k} p_{1k_1} p_{2k_2} \cdots p_{mk_m},$$

for $k = 0, 1, \dots, mn$.

The composition product gives an alternate formulation of divisibility. The *division algorithm* for polynomials states that if $P(z)$ and $S(z)$ are polynomials with $S(z) \neq 0$, then there exist unique polynomials $Q(z)$ and $R(z)$, called respectively the *quotient* and *remainder*, such that

$$P(z) = S(z)Q(z) + R(z),$$

where $R(z) = 0$ or $\deg(R(z)) < \deg(S(z))$ [15]. The next theorem is an equivalent formulation of this algorithm in terms of the composition product. It is convenient below to employ the notation $P_{-\nu} = e^{\nu D^2} P$ and $P_\nu = e^{-\nu D^2} P$, for ν real.

Theorem 1.3. Let ν be real. If $P(z)$ and $S(z)$ are polynomials with $S(z) \neq 0$, and $Q(z)$ and $R(z)$ are the corresponding quotient and remainder polynomials given above, then

$$\begin{cases} P(z) = S(z)Q(z) + R(z) \iff P_{-\nu}(z) = [S_{-\nu}(z), Q_{-\nu}(z)]_{\nu} + R_{-\nu}(z), \\ P_{\nu}(z) = S_{\nu}(z)Q_{\nu}(z) + R_{\nu}(z) \iff P(z) = [S(z), Q(z)]_{\nu} + R(z). \end{cases}$$

Proof. The proof of the first equivalence follows directly by applying the Weierstrass operator $e^{\nu D^2}$ to both sides of the division algorithm. Replacing P , S , Q , and R , with P_{ν} , S_{ν} , Q_{ν} , and R_{ν} in the first equivalence yields the second equivalence. $\square$

The first equivalence is obvious in view of the formula $(D_{\nu})^j H_k^{-\nu}(z) = H_{j+k}^{-\nu}(z)$ for $j, k = 0, 1, \dots$, whenever ν is real. Indeed, putting

$$S(z) = \sum_{j=0}^n s_j z^j \quad \text{and} \quad Q(z) = \sum_{k=0}^n q_k z^k$$

gives the Cauchy product form

$$\begin{aligned} [S_{-\nu}(z), Q_{-\nu}(z)]_{\nu} &= S(D)_{\nu} Q_{-\nu}(z) \\ &= \sum_{j=0}^n s_j D_{\nu}^j \sum_{k=0}^n q_k H_k^{-\nu}(z) \\ &= \sum_{j+k=0}^{2n} s_j t_k H_{j+k}^{-\nu}(z), \end{aligned}$$

which is also the product form of $S(z)Q(z)$. An example of the first equivalence for linear factors $S(z) = z - \tau = S_{-\nu}(z)$ and $R(z) = P(\tau) = R_{-\nu}(z)$, where τ is complex, is given in Corollary 1.4 below. An example of the second equivalence for linear factors $S(z) = z - \tau = S_{\nu}(z)$ and $R(z) = P_{\nu}(\tau) = R_{\nu}(z)$, is detailed below in Theorem 3.4.

Corollary 1.4. Let ν be real, τ be complex, and let $e^{\tau D} P(z) = \sum_{k=0}^n p_k z^k$. If $S(z) = z - \tau$ and $R(z) = P(\tau)$, then $S_{-\nu}(z) = z - \tau$, $R_{-\nu}(z) = P(\tau)$ and

$$Q_{-\nu}(z) = \sum_{k=1}^n p_k H_{k-1}^{-\nu}(z - \tau).$$

Proof. The formula for the quotient follows from

$$\begin{aligned} Q_{-\nu}(z) &= e^{\nu D^2} \left[\frac{P(z) - P(\tau)}{z - \tau} \right] = e^{\nu D^2} \left[e^{-\tau D} \frac{P(z + \tau) - P(\tau)}{z} \right] \\ &= e^{-\tau D} e^{\nu D^2} \left[\frac{P(z + \tau) - P(\tau)}{z} \right] = e^{-\tau D} e^{\nu D^2} \left[\sum_{k=1}^n p_k z^{k-1} \right] \\ &= \sum_{k=1}^n p_k H_{k-1}^{-\nu}(z - \tau), \end{aligned}$$

and the other formulae follow directly from $S_{-\nu} = e^{\nu D^2} S$ and $R_{-\nu} = e^{\nu D^2} R$. $\square$

The operational property preceeding Lemma 1.1 yields an alternate calculation of the quotient formula in Corollary 1.4. Indeed, this property gives

$$P_1(D)_\nu P_2 = P_1(D)_\nu e^{\nu D^2} [e^{-\nu D^2} P_2] = e^{\nu D^2} [P_1 e^{-\nu D^2} P_2] = [e^{\nu D^2} P_1, P_2]_\nu$$

and

$$[S_{-\nu}(z), Q_{-\nu}(z)]_\nu = (D - \tau)_\nu Q_{-\nu}(z) = (2\nu D + (z - \tau))Q_{-\nu}(z).$$

The division algorithm is equivalent to the first order equation

$$(D + (z - \tau)/2\nu)Q_{-\nu}(z) = (P_{-\nu}(z) - P(\tau))/2\nu,$$

which yields an expression for the quotient, given by

$$Q_{-\nu}(z) = \frac{e^{-(z-\tau)^2/4\nu}}{2\nu} \int_\tau^z e^{(s-\tau)^2/4\nu} (P_{-\nu}(s) - P(\tau)) ds + Q_{-\nu}(\tau) e^{-(z-\tau)^2/4\nu}.$$

A routine calculation verifies the integral formula

$$\int_0^\omega e^{t^2/4\nu} H_k^{-\nu}(t) dt = \begin{cases} 2\nu e^{\omega^2/4\nu} H_{k-1}^{-\nu}(\omega) & \text{for } k = 2j, \\ 2\nu e^{\omega^2/4\nu} H_{k-1}^{-\nu}(\omega) - \kappa_k & \text{for } k = 2j - 1, \end{cases}$$

where $\kappa_k = 2\nu H_{k-1}^{-\nu}(0)$, for $j = 1, 2, \dots$. Substituting these formulae into the above expression yields the quotient formula, and the value

$$Q_{-\nu}(\tau) = \sum_{k=1,3,\dots} p_k H_{k-1}^{-\nu}(0).$$

The next corollary addresses divisibility for the second equivalence in Theorem 1.3 for quadratic factors. For this case, let $\nu \geq 0$ and $\beta > 0$, and put $\gamma = \sqrt{\beta^2 + 2\nu}$ and $S(z) = (z - \alpha)^2 + \gamma^2$, with α real, so that $S_\nu(z) = (z - \alpha)^2 + \beta^2$. If $R(z) = R_\nu(z) = 0$, then

$$P_\nu(z) = S_\nu(z)Q_\nu(z) \iff P(z) = [S(z), Q(z)]_\nu \iff S_\nu(D)_\nu Q(z) = P(z),$$

by Theorem 1.3. Thus, divisibility of $P_\nu(z)$ by $S_\nu(z)$ is equivalent to solvability of the differential equation

$$S_\nu(D)_\nu Q(z) = P(z),$$

where $S_\nu(D)_\nu = ((D - \alpha)^2 + \beta^2)_\nu$. Factoring the translation operator out of $S_\nu(D)_\nu$ simplifies articulation of a solution to the differential equation, given below. Indeed, it follows that

$$\begin{aligned} [(z - \alpha)^2 + \gamma^2, Q(z)]_\nu &= [e^{-\alpha D}(z^2 + \gamma^2), Q(z)]_\nu \\ &= e^{-\alpha D}[z^2 + \gamma^2, e^{\alpha D}Q(z)]_\nu \\ &= e^{-\alpha D}(D^2 + \beta^2)_\nu e^{\alpha D}Q(z), \end{aligned}$$

giving that

$$((D - \alpha)^2 + \beta^2)_\nu Q(z) = P(z) \iff (D^2 + \beta^2)_\nu e^{\alpha D}Q(z) = e^{\alpha D}P(z),$$

where

$$(D^2 + \beta^2)_\nu := (D_\nu^2 + \beta^2) = 4\nu^2 D^2 + 4\nu z D + (z^2 + \gamma^2).$$

Without loss of generality, assume that

$$\hat{P}(z) = e^{\alpha D} P(z) = \sum_{k=0}^{2n} \hat{p}_k z^k \quad \text{and} \quad \hat{Q}(z) = e^{\alpha D} Q(z) = \sum_{k=0}^{2n-2} \hat{q}_k z^k,$$

by putting $p_{2n} = 0$, if necessary. Substitution into the equation above yields the recursive relations

$$\left\{ \begin{array}{l} 8\nu^2 \hat{q}_2 + \beta_\nu^2 \hat{q}_0 = \hat{p}_0, \\ 24\nu^2 \hat{q}_3 + (4\nu + \beta_\nu^2) \hat{q}_1 = \hat{p}_1, \\ 4\nu^2(k+2)(k+1) \hat{q}_{k+2} + (4\nu k + \beta_\nu^2) \hat{q}_k + \hat{q}_{k-2} = \hat{p}_k, \quad \text{for } k = 2, 3, \dots, 2n-4, \\ (4\nu(2n-3) + \beta_\nu^2) \hat{q}_{2n-3} + \hat{q}_{2n-5} = \hat{p}_{2n-3}, \\ (4\nu(2n-2) + \beta_\nu^2) \hat{q}_{2n-2} + \hat{q}_{2n-4} = \hat{p}_{2n-2}, \\ \hat{q}_{2n-3} = \hat{p}_{2n-1}, \\ \hat{q}_{2n-2} = \hat{p}_{2n}. \end{array} \right.$$

Consider the $(n+1) \times n$ matrix

$$\mathcal{S}_n^\nu = \begin{bmatrix} 2\nu + \beta^2 & 8\nu^2 & 0 & \cdots & 0 \\ 1 & 10\nu + \beta^2 & 48\nu^2 & \cdots & 0 \\ 0 & 1 & 18\nu + \beta^2 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4\nu^2(2n-2)(2n-3) \\ 0 & 0 & 0 & \cdots & 2\nu(4n-3) + \beta^2 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and the $n \times (n-1)$ matrix

$$\mathcal{T}_n^\nu = \begin{bmatrix} 6\nu + \beta^2 & 24\nu^2 & 0 & \cdots & 0 \\ 1 & 14\nu + \beta^2 & 80\nu^2 & \cdots & 0 \\ 0 & 1 & 22\nu + \beta^2 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4\nu^2(2n-3)(2n-4) \\ 0 & 0 & 0 & \cdots & 2\nu(4n-5) + \beta^2 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and the vectors

$$\begin{cases} \hat{P}_e = [\hat{p}_0 \ \hat{p}_2 \ \hat{p}_4 \ \dots \ \hat{p}_{2n-2} \ \hat{p}_{2n}]^T, & \hat{P}_o = [\hat{p}_1 \ \hat{p}_3 \ \hat{p}_5 \ \dots \ \hat{p}_{2n-3} \ \hat{p}_{2n-1}]^T, \\ \hat{Q}_e = [\hat{q}_0 \ \hat{q}_2 \ \hat{q}_4 \ \dots \ \hat{q}_{2n-4} \ \hat{q}_{2n-2}]^T, & \hat{Q}_o = [\hat{q}_1 \ \hat{q}_3 \ \hat{q}_5 \ \dots \ \hat{q}_{2n-5} \ \hat{q}_{2n-3}]^T. \end{cases}$$

The recursion relations translate as an uncoupled system of two matrix equations, given by

$$\mathcal{S}_n^\nu \hat{Q}_e = \hat{P}_e \quad \text{and} \quad \mathcal{T}_n^\nu \hat{Q}_o = \hat{P}_o.$$

A possible solution is obtained by back-substitution of bottom values $\hat{q}_{2n-3} = \hat{p}_{2n-1}$ and $\hat{q}_{2n-2} = \hat{p}_{2n}$, and verifying top equations

$$8\nu^2 \hat{q}_2 + (2\nu + \beta^2) \hat{q}_0 = \hat{p}_0 \quad \text{and} \quad 24\nu^2 \hat{q}_3 + (6\nu + \beta^2) \hat{q}_1 = \hat{p}_1.$$

If the solution to the system exists, then $P_\nu(z)$ is divisible by $(z - \alpha)^2 + \beta^2$, with $Q(z) = e^{-\alpha D} \sum_{k=0}^{2n-2} \hat{q}_k z^k$ and with quotient

$$Q_\nu(z) = e^{-\nu D^2} Q(z) = e^{-\alpha D} e^{-\nu D^2} \sum_{k=0}^{2n-2} \hat{q}_k z^k = \sum_{k=0}^{2n-2} \hat{q}_k H_k^\nu(z - \alpha).$$

Moreover, a solution to the matrix system exists if and only if vectors $\hat{P}_e$ and $\hat{P}_o$ are in the column subspaces of the matrices $\mathcal{S}_n^\nu$ and $\mathcal{T}_n^\nu$, respectively.

Performing elementary column operations, the matrices $\mathcal{S}_n^\nu$ and $\mathcal{T}_n^\nu$ reduce to the matrices

$$\mathcal{U}_n^\nu = \begin{bmatrix} -H_2^\nu(i\beta) & -H_4^\nu(i\beta) & -H_6^\nu(i\beta) & \dots & -H_{2n-2}^\nu(i\beta) & -H_{2n}^\nu(i\beta) \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

and

$$\mathcal{V}_n^\nu = \begin{bmatrix} i\tilde{H}_3^\nu(i\beta) & i\tilde{H}_5^\nu(i\beta) & i\tilde{H}_7^\nu(i\beta) & \dots & i\tilde{H}_{2n-3}^\nu(i\beta) & i\tilde{H}_{2n-1}^\nu(i\beta) \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

respectively, where $\tilde{H}_k^\nu(i\beta) = \beta^{-1}H_k^\nu(i\beta)$, for $k = 3, 5, \dots, 2n-1$. If $\tilde{Q}_e$ and $\tilde{Q}_o$ are quotient vectors, corresponding to new basis polynomials, for the matrices $\mathcal{U}_n^\nu$ and $\mathcal{V}_n^\nu$, respectively, then

$$\begin{aligned}\hat{P}_\nu(z) = \hat{S}_\nu(z)\hat{Q}_\nu(z) &\iff \hat{P}(z) = [\hat{S}(z), \hat{Q}(z)]_\nu \\ &\iff \mathcal{U}_n^\nu \tilde{Q}_e = \hat{P}_e \text{ and } \mathcal{V}_n^\nu \tilde{Q}_o = \hat{P}_o,\end{aligned}$$

where $\hat{S}(z) = e^{\alpha D}S(z)$, so that $\hat{S}_\nu(z) = z^2 + \beta^2$. Some details of the discussion above are employed in the proof of the next corollary.

Corollary 1.5. *Let $\nu \geq 0$, $\beta > 0$, and let α be real. Let*

$$P(z) = \sum_{k=0}^{2n} p_k z^k \quad \text{and} \quad P_\nu(z) = e^{-\nu D^2} P(z) = \sum_{k=0}^{2n} p_k H_k^\nu(z).$$

Put

$$\hat{P}(z) = e^{\alpha D} P(z) = \sum_{k=0}^{2n} \hat{p}_k z^k, \quad \hat{P}_e(z) = \sum_{k=0}^{2n} \hat{p}_{2k} z^{2k}$$

and

$$\hat{P}_o(z) = \sum_{k=0}^{2n} \hat{p}_{2k+1} z^{2k+1}.$$

A necessary and sufficient condition for having the equalities

$$P_\nu(z) = ((z - \alpha)^2 + \beta^2) Q_\nu(z)$$

is given by $\hat{P}_{e\nu}(i\beta) = 0$ and $\hat{P}_{o\nu}(i\beta) = 0$, where

$$\hat{P}_{e\nu}(z) = e^{-\nu D^2} \hat{P}_e(z) = \sum_{k=0}^n \hat{p}_{2k} H_{2k}^\nu(z)$$

and

$$\hat{P}_{o\nu}(z) = e^{-\nu D^2} \hat{P}_o(z) = \sum_{k=0}^{n-1} \hat{p}_{2k+1} H_{2k+1}^\nu(z).$$

Moreover, denoting basis polynomials by

$$B_k(z) = \begin{cases} \frac{H_k^\nu(z - \alpha) - H_k^\nu(i\beta)}{(z - \alpha)^2 + \beta^2} & \text{for } k = 2, 4, \dots, 2n, \\ \frac{H_k^\nu(z - \alpha) + i(z - \alpha)\tilde{H}_k^\nu(i\beta)}{(z - \alpha)^2 + \beta^2} & \text{for } k = 3, 5, \dots, 2n-1, \end{cases}$$

the quotient is given by $Q_\nu(z) = \sum_{k=2}^{2n} p_k B_k(z)$.

Proof. The divisibility of $P_\nu(z)$ by $(z - \alpha)^2 + \beta^2$ is equivalent to the solvability of the system of matrix equations $\mathcal{U}_n^\nu \tilde{Q}_e = \hat{P}_e$ and $\mathcal{V}_n^\nu \tilde{Q}_o = \hat{P}_o$. Back-substitution of these matrix equations from bottom to top, yields quotient vectors $\tilde{Q}_e = [p_2 \ p_4 \ \dots \ p_{2n}]^T$

and $\tilde{Q}_0 = [p_3 \ p_5 \ \dots \ p_{2n-1}]^T$. Substitution of these values into the top equations gives equations

$$-\sum_{k=1}^n p_{2k} H_{2k}^\nu(i\beta) = p_0 \quad \text{and} \quad \sum_{k=1}^{n-1} p_{2k+1} i\tilde{H}_{2k+1}^\nu(i\beta) = p_1,$$

or equivalently $\hat{P}_{e\nu}(i\beta) = 0$ and $\hat{P}_{o\nu}(i\beta) = 0$, respectively. This proves necessity.

Conversely, if $\hat{P}_{e\nu}(i\beta) = 0$ and $\hat{P}_{o\nu}(i\beta) = 0$, then

$$P_\nu(\alpha + i\beta) = \hat{P}_\nu(i\beta) = \hat{P}_{e\nu}(i\beta) + \hat{P}_{o\nu}(i\beta) = 0.$$

The replacement $\beta \rightarrow -\beta$ yields $\hat{P}_{e\nu}(-i\beta) = 0$ and $\hat{P}_{o\nu}(-i\beta) = 0$. Similarly, it follows that $P_\nu(\alpha - i\beta) = 0$. Thus, $P_\nu(z)$ is divisible by $(z - \alpha)^2 + \beta^2$, which proves sufficiency. To establish the quotient, let

$$A_e(z) = [1 \ z^2 \ \dots \ z^{2n}] \quad \text{and} \quad A_o(z) = [z \ z^3 \ \dots \ z^{2n-1}].$$

Then

$$\begin{aligned} \hat{P}(z) &= A_e(z)\hat{P}_e + A_o(z)\hat{P}_o \\ &= A_e(z)\mathcal{U}_n^\nu \tilde{Q}_e + A_o(z)\mathcal{V}_n^\nu \tilde{Q}_o \\ &= \sum_{k=1}^n \hat{p}_{2k} (z^{2k} - H_{2k}^\nu(i\beta)) + \sum_{k=1}^{n-1} \hat{p}_{2k+1} (z^{2k+1} + iz\tilde{H}_{2k+1}^\nu(i\beta)), \end{aligned}$$

and thus

$$\hat{P}_\nu(z) = \sum_{k=1}^n \hat{p}_{2k} [H_{2k}^\nu(z) - H_{2k}^\nu(i\beta)] + \sum_{k=1}^{n-1} \hat{p}_{2k+1} [H_{2k+1}^\nu(z) + iz\tilde{H}_{2k+1}^\nu(i\beta)].$$

Since $\hat{P}_\nu(z) = e^{-\nu D^2} e^{\alpha D} P(z) = e^{\alpha D} P_\nu(z)$, the polynomial $P_\nu(z) = e^{-\alpha D} \hat{P}_\nu(z)$ is given by the expression

$$\sum_{k=1}^n \hat{p}_{2k} (H_{2k}^\nu(z - \alpha) - H_{2k}^\nu(i\beta)) + \sum_{k=1}^{n-1} \hat{p}_{2k+1} (H_{2k+1}^\nu(z - \alpha) + i(z - \alpha)\tilde{H}_{2k+1}^\nu(i\beta)),$$

from which the quotient formula follows. $\square$

2. Zeros of the composition product

The next lemma, expressing the composition product as a sum, is due to N. G. de Bruijn [3].

Lemma 2.1. *Let ν be real. If P_1 and P_2 are polynomials, then*

$$[P_1, P_2]_\nu = \sum_{k=0}^{\infty} \frac{(2\nu)^k}{k!} (D^k P_1)(D^k P_2),$$

where the sum eventually truncates.

Proof. In the sum, subsequent terms employ derivatives of the polynomials of increasing order, so the sum eventually truncates. Putting $P_1(z) = e^{\nu D^2} T(z)$, where $T(z) = \sum_{j=0}^n c_j z^j$, gives

$$[P_1, P_2]_{\nu}(z) = [e^{\nu D^2} T, P_2]_{\nu}(z) = T(D)_{\nu} P_2(z) = \sum_{j=0}^n c_j (D_{\nu})^j P_2(z),$$

where

$$\begin{aligned} (D_{\nu})^j P_2(z) &= (2\nu)^j e^{-z^2/4\nu} D^j (e^{z^2/4\nu} P_2(z)) \\ &= (2\nu)^j e^{-z^2/4\nu} \sum_{k=0}^j \binom{j}{k} D^{j-k} (e^{z^2/4\nu}) D^k P_2(z), \end{aligned}$$

for $j = 0, 1, \dots, n$, and where

$$D^{j-k} (e^{z^2/4\nu}) = e^{z^2/4\nu} \left(\frac{1}{\sqrt{-4\nu}} \right)^{j-k} H_{j-k}^{-\nu} \left(\frac{z}{\sqrt{-4\nu}} \right),$$

for $k = 0, 1, \dots, j$. It follows that

$$\begin{aligned} [P_1, P_2]_{\nu}(z) &= \sum_{j=0}^n c_j (2\nu)^j \left[\sum_{k=0}^j \binom{j}{k} \left(\frac{1}{\sqrt{-4\nu}} \right)^{j-k} H_{j-k}^{-\nu} \left(\frac{z}{\sqrt{-4\nu}} \right) D^k P_2(z) \right] \\ &= \sum_{k=0}^n \frac{(2\nu)^k}{k!} \left[\sum_{j=k}^n k! \binom{j}{k} c_j H_{j-k}^{-\nu}(z) \right] D^k P_2(z), \end{aligned}$$

where the last bracketed expression is

$$D^k P_1(z) = e^{\nu D^2} D^k T(z) = \sum_{j=k}^n k! \binom{j}{k} c_j e^{\nu D^2} z^{j-k} = \sum_{j=k}^n k! \binom{j}{k} c_j H_{j-k}^{-\nu}(z),$$

which yields the result. $\square$

Lemma 2.1 yields the limit $[P_1, P_2]_{\nu} \rightarrow P_1 P_2$, uniformly on compacta, as $\nu \rightarrow 0$. For several polynomials, an induction argument gives that

$$[P_1, P_2, \dots, P_n]_{\nu} \rightarrow P_1 P_2 \cdots P_n,$$

uniformly on compacta, as $\nu \rightarrow 0$, for $n = 2, 3, \dots$. The composition product is thus a parametric extension of the ordinary product.

Denote the closed half-planes by

$$\begin{cases} \bigwedge[\sigma', \infty) = \{z \mid \Re(z) \geq \sigma'\} & \text{and} & \bigwedge(-\infty, \sigma] = \{z \mid \Re(z) \leq \sigma\}, \\ \bigvee[\rho', \infty) = \{z \mid \Im(z) \geq \rho'\} & \text{and} & \bigvee(-\infty, \rho] = \{z \mid \Im(z) \leq \rho\}. \end{cases}$$

and closed strips by

$$\bigwedge[\sigma', \sigma] = \{z \mid \sigma' \leq \Re(z) \leq \sigma\} \quad \text{and} \quad \bigvee[\rho', \rho] = \{z \mid \rho' \leq \Im(z) \leq \rho\}.$$

If the zeros of a polynomial P are located in $\bigwedge(-\infty, \sigma]$, then the Gauss-Lucas theorem [12] gives that the zeros of DP are located in $\bigwedge(-\infty, \sigma]$. A variation of this result is given in the next lemma.

Lemma 2.2. *If the zeros of polynomials P and Q are located in $\bigwedge(-\infty, 0]$ and $\bigwedge(-\infty, \sigma]$, respectively, then the zeros of $P(D)Q$ are located in $\bigwedge(-\infty, \sigma]$.*

Proof. First, let $P(z) = z + \tau$, with $\Re(\tau) \geq 0$. Since $P(D)Q$ is given as the limit

$$\begin{aligned} P(D)Q(z) &= (D + \tau)Q(z) = e^{-\tau z} D(e^{\tau z} Q(z)) \\ &= e^{-\tau z} D\left(\lim_{m \rightarrow \infty} \left(1 + \frac{\tau z}{m}\right)^m Q(z)\right) \\ &= e^{-\tau z} \lim_{m \rightarrow \infty} D\left(\left(1 + \frac{\tau z}{m}\right)^m Q(z)\right), \end{aligned}$$

where convergence is uniform on compacta, the result now follows by the Gauss-Lucas theorem and Hurwitz's theorem [4]. The general case for $P(z) = \gamma \prod_{k=1}^n (z + \tau_k)$, with γ complex and $\Re(\tau_k) \geq 0$, $k = 1, 2, \dots, n$, follows by induction on n . $\square$

The original proof of the next lemma from [3] is based on a generalization of a classic composition result for polynomials [2]. A direct proof is given to avoid this digression.

Lemma 2.3. *If the zeros of polynomials P and Q are located in $\bigwedge(-\infty, 0]$, then the zeros of $[P, Q]_\nu$ are located in $\bigwedge(-\infty, 0]$, for all $\nu \geq 0$.*

Proof. The case $\nu = 0$ clearly holds, since this case is the ordinary product. Let $\nu > 0$, and let w be complex, with $\sigma = \Re(w) > 0$. Since P and Q are polynomials, the expansions

$$P'(z) = P(2\nu z + w) = \sum_{k=0}^{\infty} \frac{(2\nu z)^k}{k!} (D^k P)(w)$$

and

$$Q'(z) = Q(z + w) = \sum_{k=0}^{\infty} \frac{z^k}{k!} (D^k Q)(w)$$

eventually truncate. It follows that $P'(z)$ and $Q'(z)$ are polynomials having all zeros in $\bigwedge(-\infty, 0]$ and $\bigwedge(-\infty, -\sigma]$, respectively. By Lemma 2.2, the zeros of $P'(D)Q'$ are located in $\bigwedge(-\infty, -\sigma]$, so that $P'(D)Q'(0) \neq 0$. A calculation allows to verify the relation

$$P'(D)Q'(0) = \sum_{k=0}^{\infty} \frac{(2\nu)^k}{k!} (D^k P)(w)(D^k Q)(w),$$

which gives $P'(D)Q'(0) = [P, Q]_\nu(w)$, by Lemma 2.1. Thus, the zeros of $[P, Q]_\nu$ are all located in $\bigwedge(-\infty, 0]$. $\square$

Theorem 2.4. *Let the zeros of polynomials P_1 and P_2 be located in $\bigwedge[\sigma', \sigma]$ and $\bigvee[\rho', \rho]$. Then, the zeros of $[P_1, P_2]_\nu$ are located in $\bigwedge[\sigma', \sigma]$ for all $\nu \geq 0$, and are located in $\bigvee[\rho', \rho]$ for all $\nu \leq 0$.*

Proof. Let $\nu \geq 0$. The zeros of the polynomials

$$P'_1(z) = e^{\sigma D} P_1(z) \quad \text{and} \quad P'_2(z) = e^{\sigma D} P_2(z)$$

are in the half-plane $\bigwedge(-\infty, 0]$. By Lemma 2.3, the zeros of the composition $[P'_1, P'_2]_\nu$ are in the half-plane $\bigwedge(-\infty, 0]$. By the translation property of the composition product, the zeros of the composition $[P_1, P_2]_\nu = e^{-\sigma D}[P'_1, P'_2]_\nu$ are in the half-plane $\bigwedge(-\infty, \sigma]$. Since the zeros of the polynomials

$$P_1^-(z) = P_1(-z) \quad \text{and} \quad P_2^-(z) = P_2(-z)$$

are in $\bigwedge(-\infty, -\sigma']$, it follows that the zeros of the composition $[P_1^-, P_2^-]_\nu$ are in the half-plane $\bigwedge(-\infty, -\sigma']$. By the reflection property of the composition product, the zeros of the composition $[P_1, P_2]_\nu = [P_1^-, P_2^-]_\nu^-$ are in the half-plane $\bigwedge[\sigma', \infty)$. Thus, the zeros of the composition $[P_1, P_2]_\nu$ are in the strip $\bigwedge[\sigma', \sigma]$. Let $\nu \leq 0$. Since the zeros of the polynomials $\tilde{P}_1(z) = P_1(iz)$ and $\tilde{P}_2(z) = P_2(iz)$ are in $\bigwedge[\rho', \rho]$, it follows that the zeros of the composition $[\tilde{P}_1, \tilde{P}_2]_{-\nu}$ are in $\bigwedge[\rho', \rho]$. The relation $[P_1, P_2]_\nu(z) = [\tilde{P}_1, \tilde{P}_2]_{-\nu}(-iz)$ follows from Lemma 2.1, and gives that the zeros of $[P_1, P_2]_\nu$ are in $\bigvee[\rho', \rho]$. $\square$

In view of Theorem 2.4, the composition product is a parametric extension of the ordinary product that preserves the location of the zeros to closed half-planes and strips. A routine induction extends Theorem 2.4 for arbitrary composition products, given in the next corollary.

Corollary 2.5. *Let $n = 2, 3, \dots$, and let the zeros of the polynomials P_k , for $k = 1, 2, \dots, n$, be located in closed strips $\bigwedge[\sigma', \sigma]$ and $\bigvee[\rho', \rho]$. Then, the zeros of $[P_1, P_2, \dots, P_n]_\nu$ are located in $\bigwedge[\sigma', \sigma]$ for all $\nu \geq 0$, and are located in $\bigvee[\rho', \rho]$, for all $\nu \leq 0$.*

A result in terms of Hermite expansions, that corresponds to Theorem 2.4, is given in the next corollary.

Corollary 2.6. *Let ν be real, and let P_1 and P_2 have the Hermite expansions*

$$P_1(z) = \sum_{k=0}^n p_{1k} H_k^{-\nu}(z) \quad \text{and} \quad P_2(z) = \sum_{k=0}^n p_{2k} H_k^{-\nu}(z).$$

Then, the composition product $[P_1, P_2]_\nu$ is given by the Hermite expansion

$$[P_1, P_2]_\nu(z) = \sum_{k=0}^{2n} r_k H_k^{-\nu}(z),$$

where $r_k = \sum p_{1k_1} p_{2k_2}$, with $k = k_1 + k_2$, is the Cauchy product of the coefficients of P_1 and P_2 . Moreover, if the zeros of P_1 and P_2 are located in $\bigwedge[\sigma', \sigma]$ and $\bigvee[\rho', \rho]$, then the zeros $[P_1, P_2]_\nu$ are in $\bigwedge[\sigma', \sigma]$ for all $\nu \geq 0$, and are in $\bigvee[\rho', \rho]$ for all $\nu \leq 0$.

Proof. The result follows directly from Lemma 1.1 and Theorem 2.4. $\square$

In view of Corollary 2.6, the Cauchy coefficients play the same role for the composition product of Hermite-basis polynomials, as for the ordinary product of polynomials in the usual basis. A special case further illustrates this result in the next corollary.

Corollary 2.7. Suppose that the zeros of the polynomials

$$P(z) = \sum_{k=0}^n p_k H_k(z) \quad \text{and} \quad Q(z) = \sum_{k=0}^n q_k H_k(z)$$

are located in $\bigvee[\rho', \rho]$. Then the zeros of the polynomial

$$R(z) = \sum_{k=0}^{2n} r_k H_n(z),$$

where $r_k = \sum_{j=0}^k p_j q_{k-j}$, for $k = 0, 1, \dots, 2n$, are located in $\bigvee[\rho', \rho]$.

Proof. The zeros of the polynomials $P'(z) = P(z/2)$ and $Q'(z) = Q(z/2)$ are located in the strip $\bigvee[2\rho', 2\rho]$. By Corollary 2.6, the zeros of $R'(z) = R(z/2)$ are in the strip $\bigvee[2\rho', 2\rho]$. Thus, the zeros of R are located in $\bigvee[\rho', \rho]$. $\square$

The following extension of Corollary 2.6 follows directly from Lemma 1.2 and Corollary 2.5.

Corollary 2.8. Let ν be real, and let P_l have Hermite expansion

$$P_l(z) = \sum_{k=0}^n p_{lk} H_k^{-\nu}(z),$$

for $l = 1, 2, \dots, m$. Then, the composition product $[P_1, P_2, \dots, P_m]_\nu$ is given by the Hermite expansion

$$[P_1(z), P_2(z), \dots, P_n(z)]_\nu = \sum_{k=0}^{mn} r_k H_k^{-\nu}(z),$$

where the generalized Cauchy product of the coefficients is given by

$$r_k = \sum_{k_1 + \dots + k_m = k} p_{1k_1} p_{2k_2} \cdots p_{mk_m},$$

for $k = 0, 1, \dots, mn$. Moreover, if the zeros of P_l are located in $\bigwedge[\sigma', \sigma]$ and $\bigvee[\rho', \rho]$, for $l = 1, 2, \dots, m$, then the zeros of $[P_1, P_2, \dots, P_m]_\nu$ are in $\bigwedge[\sigma', \sigma]$ for all $\nu \geq 0$, and are in $\bigvee[\rho', \rho]$ for all $\nu \leq 0$.

3. Algebraic formulation of ν -divisibility

Let ν be real and τ be complex. If $P(z)$ is a polynomial, then $P(z)$ is called ν -divisible by $z - \tau$, which we denote by $z - \tau|_\nu P(z)$, provided that

$$P(z) = [z - \tau, Q(z)]_\nu,$$

for some polynomial $Q(z)$, called the ν -quotient. Clearly, ordinary divisibility corresponds to $\nu = 0$.

Lemma 3.1. Let ν be real and τ be complex. Let $P(z)$ be an n -th degree polynomial. If $P(z)$ is ν -divisible by $z - \tau$, then the ν -quotient $Q(z)$ has degree $n - 1$, and is unique.

Proof. Put $P_\nu(z) = e^{-\nu D^2} P(z)$ and $Q_\nu(z) = e^{-\nu D^2} Q(z)$. By Theorem 1.3, it follows that

$$P_\nu(z) = (z - \tau)Q_\nu(z),$$

and thus $Q(z)$ has degree $n - 1$. For uniqueness, let $Q_1(z)$ and $Q_2(z)$ be ν -quotients, so that $[z - \tau, Q_1(z)]_\nu = [z - \tau, Q_2(z)]_\nu$. Applying the operator $e^{-\nu D^2}$ to both sides of this relation gives

$$e^{-\nu D^2} Q_1(z) = e^{-\nu D^2} Q_2(z),$$

which implies $Q_1(z) = Q_2(z)$. $\square$

The next lemma gives the correspondence between ν -divisibility and ordinary divisibility.

Lemma 3.2. *Let ν be real and τ be complex. If $P(z)$ is an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ is the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$, then*

$$\begin{aligned} (z - \tau) | P_\nu(z) &\iff P_\nu(z) = (z - \tau)Q_\nu(z) \\ &\iff P(z) = [z - \tau, Q(z)]_\nu \\ &\iff (z - \tau) |_\nu P(z), \end{aligned}$$

where quotients $Q(z)$ and $Q_\nu(z) = e^{-\nu D^2} Q(z)$ are unique polynomials having degree $n - 1$. The case $\nu = 0$ corresponds to ordinary divisibility

$$(z - \tau) | P(z) \iff P(z) = (z - \tau)Q(z).$$

Proof. The middle equivalence follows from

$$\begin{aligned} P_\nu(z) = (z - \tau)Q_\nu(z) &\iff e^{\nu D^2} P_\nu(z) = e^{\nu D^2} [(z - \tau)Q_\nu(z)] \\ &\iff P(z) = [z - \tau, Q(z)]_\nu. \end{aligned}$$

The first equivalence follows from the definition of ordinary divisibility, and the last equivalence follows from the definition of ν -divisibility. $\square$

For ν real and τ complex, the matrix formulation of ν -divisibility uses the tridiagonal $(n + 1) \times n$ matrix

$$\mathcal{F}_n^\nu(\tau) = \begin{bmatrix} -\tau & 2\nu & 0 & 0 & \cdots & 0 & 0 \\ 1 & -\tau & 4\nu & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\tau & 6\nu & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\tau & 2(n-1)\nu \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\tau \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

for $n = 2, 3, \dots$. The column vectors of $\mathcal{F}_n^\nu(\tau)$ in $\mathbb{C}^{n+1}$, denoted by $F_0^\nu(\tau), F_1^\nu(\tau), \dots, F_{n-1}^\nu(\tau)$, are linearly independent, so that $\text{Rank}(\mathcal{F}_n^\nu(\tau)) = n$. The quotient subspace

$$\mathcal{Q}_n^\nu(\tau) = \text{Rng}[\mathcal{F}_n^\nu(\tau)] = \text{Span}\{F_0^\nu(\tau), F_1^\nu(\tau), \dots, F_{n-1}^\nu(\tau)\},$$

corresponding to $\mathcal{F}_n^\nu(\tau)$ is thus an n -dimensional subspace of $\mathbb{C}^{n+1}$. In the matrix formulation of ν -divisibility given below, the complex polynomial $P(z) = \sum_{k=0}^n p_k z^k$ is associated with the complex vector $P = [p_0 \ p_1 \ \dots \ p_n]^\top$ in $\mathbb{C}^{n+1}$. The next lemma expresses ν -divisibility in terms of the quotient subspace.

Lemma 3.3. *Let ν be real and τ be complex. If*

$$P(z) = \sum_{k=0}^n p_k z^k$$

is an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ is the associated polynomial vector, then

$$(z - \tau)|_\nu P(z) \iff P(z) = [z - \tau, Q(z)]_\nu \iff \mathcal{F}_n^\nu(\tau)Q = P \iff P \in \mathcal{Q}_n^\nu(\tau),$$

where the ν -quotient

$$Q(z) = \sum_{k=0}^{n-1} q_k z^k$$

is unique, with

$$P = \sum_{k=0}^{n-1} q_k F_k^\nu(\tau).$$

The case $\nu = 0$ corresponds to ordinary divisibility

$$(z - \tau)|P(z) \iff P(z) = (z - \tau)Q(z) \iff \mathcal{F}_n^0(\tau)Q = P \iff P \in \mathcal{Q}_n^0(\tau),$$

$$\text{with } P = \sum_{k=0}^{n-1} q_k F_k^0(\tau).$$

Proof. The first equivalence follows from the definition of ν -divisibility. The last equivalence follows from the definition of the quotient subspace. The operational property for expressing $(D - \tau)_\nu$ as composition product yields

$$[z - \tau, Q(z)]_\nu = (D - \tau)_\nu Q(z) = (2\nu D + (z - \tau))Q(z).$$

Putting

$$Q(z) = \sum_{k=0}^{n-1} q_k z^k \quad \text{and} \quad P(z) = \sum_{k=0}^n p_k z^k,$$

the equation $[z - \tau, Q(z)]_\nu = P(z)$ becomes

$$\begin{aligned} \sum_{k=0}^n p_k z^k &= (2\nu q_1 - \tau q_0)z^0 + \sum_{k=1}^{n-2} [2\nu q_{k+1} + q_{k-1} - \tau q_k]z^k \\ &\quad + (q_{n-2} - \tau q_{n-1})z^{n-1} + q_{n-1}z^n, \end{aligned}$$

giving $\mathcal{F}_n^\nu(\tau)Q = P$, and conversely. This establishes the middle equivalence. The case for ordinary divisibility follows by putting $\nu = 0$. $\square$

The next theorem is the ν -analogue for the Remainder Theorem [15].

Theorem 3.4. *Let ν be real and τ be complex. If $P(z)$ is an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ is the associated polynomial vector and $P_\nu(z) = e^{-\nu D^2} P(z)$, then*

$$\begin{aligned} P_\nu(z) = (z - \tau)Q_\nu(z) + P_\nu(\tau) &\iff P(z) = [z - \tau, Q(z)]_\nu + P_\nu(\tau) \\ &\iff P = \mathcal{F}_n^\nu(\tau)Q + R, \end{aligned}$$

where $R = [P_\nu(\tau) \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{C}^{n+1}$, and the quotients $Q(z)$ and $Q_\nu(z)$ are unique polynomials having degree $n - 1$, with $Q_\nu(z) = e^{-\nu D^2} Q(z)$, and quotient vector $Q \in \mathbb{C}^n$.

Proof. Consider the polynomial $T(z) = P(z) - P_\nu(\tau)$ and the associated vector $T = P - R$. Then, $T_\nu(\tau) = 0$, where $T_\nu(z) = e^{-\nu D^2} T(z) = P_\nu(z) - P_\nu(\tau)$, so that $(z - \tau) \mid T_\nu(z)$. Combining Lemma 3.2 and Lemma 3.3 gives equivalences

$$T_\nu(z) = (z - \tau)Q_\nu(z) \iff T(z) = [z - \tau, Q(z)]_\nu \iff \mathcal{F}_n^\nu(\tau)Q = T,$$

where $Q(z)$ and $Q_\nu(z)$ are unique polynomials having degree $n - 1$, with

$$Q_\nu(z) = e^{-\nu D^2} Q(z),$$

so that $Q \in \mathbb{C}^n$. Substituting for $T(z)$, $T_\nu(z)$, and T in these equivalences yield the equivalences in the theorem. $\square$

The next corollary follows directly from Theorem 3.4, and is the ν -analogue of the Factor Theorem [15].

Corollary 3.5. *Let ν be real and τ be complex. Let $P(z)$ be an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. If $P_\nu(\tau) = 0$, then*

$$P = \mathcal{F}_n^\nu(\tau)Q \iff P(z) = [z - \tau, Q(z)]_\nu \iff P_\nu(z) = (z - \tau)Q_\nu(z),$$

where the quotients $Q(z)$ and $Q_\nu(z)$ are unique polynomials having degree $n - 1$, with $Q_\nu(z) = e^{-\nu D^2} Q(z)$, and quotient vector $Q \in \mathbb{C}^n$.

The next corollary is the ν -analogue of the method of synthetic division.

Corollary 3.6. *Let ν be real and τ be complex. Let $P(z)$ and $P_\nu(z)$ be the polynomials*

$$P(z) = \sum_{k=0}^n p_k z^k \quad \text{and} \quad P_\nu(z) = e^{-\nu D^2} P(z) = \sum_{k=0}^n p_k H_k^\nu(z).$$

Then, the quotient-remainder forms given in Theorem 3.4, with $Q(z) = \sum_{k=0}^{n-1} q_k z^k$ and

$$Q_\nu(z) = e^{-\nu D^2} Q(z) = \sum_{k=0}^{n-1} q_k H_k^\nu(z),$$

for values $q_0, q_1, \dots, q_{n-1}$, are given recursively by

$$q_{n-k} = p_{n-k+1} + \tau q_{n-k+1} - 2(n-k+2)\nu q_{n-k+2},$$

for $k = 1, 2, \dots, n+1$, with $q_n = q_{n+1} = 0$, and where $q_{-1} = P_\nu(\tau)$, for $n = 1, 2, \dots$.

Proof. Let $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector. By Theorem 3.4, the equation $P_\nu(z) = (z - \tau)Q_\nu(z) + P_\nu(\tau)$ is equivalent to the system of $n+1$ equations $P = \mathcal{F}_n^\nu(\tau)Q + R$, given by the system

$$\begin{bmatrix} -\tau & 2\nu & 0 & \cdots & 0 & 0 \\ 1 & -\tau & 4\nu & \cdots & 0 & 0 \\ 0 & 1 & -\tau & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\tau & 2(n-1)\nu \\ 0 & 0 & 0 & \cdots & 1 & -\tau \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{n-3} \\ q_{n-2} \\ q_{n-1} \end{bmatrix} + \begin{bmatrix} P_\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-2} \\ p_{n-1} \\ p_n \end{bmatrix}.$$

Listing the bottom n equations from bottom-to-top yields the n quotient values q_k , for $k = 0, 1, \dots, n-1$, and substitution of these values into the top equation yields $q_{-1} = P_\nu(\tau)$. $\square$

Applied to the parametrized Hermite polynomials, ν -synthetic division yields the next corollary.

Corollary 3.7. *Let ν be real and τ be complex. There exist polynomials*

$$B_n^\nu(z, \tau) = \sum_{k=0}^{n-1} b_k^n H_k^\nu(z)$$

such that

$$B_n^\nu(z, \tau) = \frac{H_n^\nu(z) - H_n^\nu(\tau)}{z - \tau},$$

with

$$\begin{cases} b_n^n = 0 \text{ and } b_{n-1}^n = 1 & \text{for } n = 1, 2, \dots, \\ b_{n-k}^n = \tau b_{n-k+1}^n - 2(n-k+2)\nu b_{n-k+2}^n & \text{for } k = 2, 3, \dots, n. \end{cases}$$

The polynomials in Corollary 3.7 are clearly symmetric, the first few being

$$\begin{cases} B_1^\nu(z, \tau) = 1, \\ B_2^\nu(z, \tau) = z + \tau, \\ B_3^\nu(z, \tau) = z^2 + \tau z + \tau^2 - 6\nu, \\ B_4^\nu(z, \tau) = z^3 + \tau z^2 + \tau^2 z + \tau^3 - 12\nu z - 12\nu\tau, \\ B_5^\nu(z, \tau) = z^4 + \tau z^3 + \tau^2 z^2 + \tau^3 z + \tau^4 - 20\nu z^2 - 20\nu\tau z + 60\nu^2. \end{cases}$$

Performing elementary column operations on the matrix $\mathcal{F}_n^\nu(\tau)$ produces a reduced matrix with the same range. To this end, let $\mathcal{G}_n^\nu(\tau)$ be any matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$, so that $\mathcal{G}_n^\nu(\tau) = \mathcal{F}_n^\nu(\tau)M$, where M is

an $n \times n$ nonsingular matrix. The next theorem is a generalization of the Remainder and Factor Theorems, and displays the independence of basis polynomials in the formulation of the quotient.

Theorem 3.8. *Let ν be real and τ be complex. Let $P(z)$ be an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. Let $\mathcal{G}_n^\nu(\tau)$ be any matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$. There exist unique polynomials $G(z)$ and $G_\nu(z)$, having degree $n - 1$, such that*

$$\begin{aligned} P_\nu(z) = (z - \tau)G_\nu(z) + P_\nu(\tau) &\iff P(z) = [z - \tau, G(z)]_\nu + P_\nu(\tau) \\ &\iff P = \mathcal{G}_n^\nu(\tau)G + R, \end{aligned}$$

where $G \in \mathbb{C}^n$ is the quotient vector with respect to $\mathcal{G}_n^\nu(\tau)$, and $G_\nu(z) = e^{-\nu D^2} G(z)$. Further, G is the coefficient vector for basis polynomials $B(z) = [B_1(z) \dots B_n(z)]$, so that $G(z) = B(z)G$, where $B(z) = A(z)M$, and $A(z) = [1 \ z \ \dots \ z^{n-1}]$. Furthermore, if $P_\nu(\tau) = 0$, then

$$P = \mathcal{G}_n^\nu(\tau)G \iff P(z) = [z - \tau, G(z)]_\nu \iff P_\nu(z) = (z - \tau)G_\nu(z).$$

Proof. By Theorem 3.4, there are unique polynomials $Q(z)$ and $Q_\nu(z)$, having degree $n - 1$, and vector $Q \in \mathbb{C}^n$ such that

$$P_\nu(z) = (z - \tau)Q_\nu(z) + P_\nu(\tau), \quad P(z) = [z - \tau, Q(z)]_\nu + P_\nu(\tau) \quad \text{and} \quad P = \mathcal{F}_n^\nu(\tau)Q + R,$$

where $R = [P_\nu(\tau) \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{C}^{n+1}$, with $Q_\nu(z) = e^{-\nu D^2} Q(z)$. Thus, it is seen that

$$\mathcal{F}_n^\nu(\tau)Q = P = \mathcal{G}_n^\nu(\tau)M^{-1}Q = \mathcal{G}_n^\nu(\tau)G,$$

so that $Q = MG$. Then,

$$Q(z) = A(z)Q = A(z)MG = B(z)G,$$

and thus $B(z) = A(z)M$. □

It is shown that elementary column operations applied to $\mathcal{F}_n^\nu(\tau)$ yield the reduced $(n + 1) \times n$ matrix

$$\mathcal{E}_n^\nu(\tau) = \begin{bmatrix} -H_1^\nu(\tau) & -H_2^\nu(\tau) & -H_3^\nu(\tau) & \cdots & -H_{n-1}^\nu(\tau) & -H_n^\nu(\tau) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Indeed, let $[\gamma c_j + c_k \rightarrow c_k]$ denote the elementary matrix which replaces column k by adding column j multiplied by γ . Define the sequence of products of such elementary

matrices, by putting

$$\begin{cases} M_2 = [\tau c_1 + c_2 \rightarrow c_2], \\ M_3 = M_2[-4\nu c_1 + c_3 \rightarrow c_3][\tau c_2 + c_3 \rightarrow c_3], \\ M_4 = M_3[-6\nu c_2 + c_4 \rightarrow c_4][\tau c_3 + c_4 \rightarrow c_4], \end{cases}$$

and recursively

$$M_{n+1} = M_n[-2n\nu c_{n-1} + c_{n+1} \rightarrow c_{n+1}][\tau c_n + c_{n+1} \rightarrow c_{n+1}], \quad \text{for } n = 2, 3, \dots$$

A routine induction establishes the reduced form $\mathcal{E}_n^\nu(\tau) = \mathcal{F}_n^\nu(\tau)M_n$, for $n = 2, 3, \dots$

Putting $M_1 = [1]$, it is seen that M_n is the $n \times n$ upper triangular matrix

$$M_n = \begin{bmatrix} b_0^1 & b_0^2 & b_0^3 & b_0^4 & \cdots & b_0^n \\ 0 & b_1^2 & b_1^3 & b_1^4 & \cdots & b_1^n \\ 0 & 0 & b_2^3 & b_2^4 & \cdots & b_2^n \\ 0 & 0 & 0 & b_3^4 & \cdots & b_3^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1}^n \end{bmatrix},$$

where the values $\{b_k^n\}_{k=0}^{n-1}$ are given in Corollary 3.7, for $n = 1, 2, \dots$. The next corollary gives the form for the various quotients corresponding to matrix $\mathcal{E}_n^\nu(\tau)$.

Corollary 3.9. *Let ν be real and τ be complex, and let $\mathcal{E}_n^\nu(\tau) = \mathcal{F}_n^\nu(\tau)M_n$ for $n = 1, 2, \dots$, be given as above. Let $P(z)$ be an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2}P(z)$. There exist unique polynomials $E(z)$ and $E_\nu(z)$, having degree $n-1$, such that*

$$\begin{aligned} P_\nu(z) = (z - \tau)E_\nu(z) + P_\nu(\tau) &\iff P(z) = [z - \tau, E(z)]_\nu + P_\nu(\tau) \\ &\iff P = \mathcal{E}_n^\nu(\tau)E + R, \end{aligned}$$

where $E \in \mathbb{C}^n$ is the quotient vector with respect to $\mathcal{E}_n^\nu(\tau)$, and $E_\nu(z) = e^{-\nu D^2}E(z)$. Moreover, E is the coefficient vector for basis polynomials $B(z) = [B_1(z) \dots B_n(z)]$, with

$$B_m(z) = \sum_{k=0}^{m-1} b_k^m z^k,$$

for $m = 1, 2, \dots, n$, where

$$E(z) = \sum_{k=1}^n p_k B_k(z) \quad \text{and} \quad E_\nu(z) = \sum_{k=1}^n p_k B_k^\nu(z, \tau).$$

Proof. In view of Theorem 3.8, it remains to show that the three quotients have the stated forms. To determine the quotient vector put $E = [e_0 \ e_1 \ e_2 \ \dots \ e_{n-1}]^T$. Then,

the formulation $P = \mathcal{E}_n^\nu(\tau)E + R$ is given by

$$\begin{bmatrix} -H_1^\nu(\tau) & -H_2^\nu(\tau) & \cdots & -H_n^\nu(\tau) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{n-2} \\ e_{n-1} \end{bmatrix} + \begin{bmatrix} P_\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}.$$

Listing the equations from bottom-to-top yields the n quotient values $e_j = p_{j+1}$ for $j = 0, 1, \dots, n-1$. Then substitution into the top equation yields the remainder,

$$P_\nu(\tau) = \sum_{k=0}^n p_k H_k^\nu(\tau).$$

It follows that $E = [p_1 \ p_2 \ p_3 \ \dots \ p_n]^\top$, and thus

$$E(z) = \sum_{k=1}^n p_k B_k(z)$$

with basis $B(z) = A(z)M_n$, by Theorem 3.8. Alternately, the quotient is given by

$$E_\nu(z) = \frac{P_\nu(z) - P_\nu(\tau)}{z - \tau} = \sum_{k=1}^n p_k \frac{H_k^\nu(z) - H_k^\nu(\tau)}{z - \tau},$$

which gives, $e^{-\nu D^2} B_k(z) = B_k^\nu(z, \tau)$, for $k = 1, 2, \dots, n$, by Corollary 3.7. $\square$

Successive applications of the quotient-remainder forms given in Theorem 3.4 generalizes these forms, as given in the corollary below. To this end, a lemma is needed.

Lemma 3.10. *If ν is real and τ is complex, then*

$$(D - \tau)_\nu^k = e^{-\tau D} D_\nu^k e^{\tau D},$$

for $k = 0, 1, \dots$

Proof. Let $Q(z)$ be a polynomial. Combining the operational property for expressing $(D - \tau)_\nu^k$ as composition product and the translation property of the composition product from the first section gives

$$e^{\tau D} (D - \tau)_\nu^k Q(z) = e^{\tau D} [(z - \tau)^k, Q(z)]_\nu = [z^k, e^{\tau D} Q(z)]_\nu = D_\nu^k e^{\tau D} Q(z),$$

and thus $(D - \tau)_\nu^k Q(z) = e^{-\tau D} D_\nu^k e^{\tau D} Q(z)$. $\square$

Let τ be complex and P be an n -th degree polynomial. Let Q_0 be defined by $P(z) = [z - \tau, Q_0(z)]_\nu + R_0$, where $R_0 = P_\nu(\tau)$. Recursively, let

$$Q_{m-1}(z) = [z - \tau, Q_m(z)]_\nu + R_m,$$

where $R_m = (Q_{m-1})_\nu(\tau)$, so that

$$Q_{m-1}(z) = (D - \tau)_\nu Q_m(z) + R_m,$$

for $m = 1, 2, \dots, n-1$. Substitution of these quotients and remainders yields

$$\begin{aligned} P(z) &= (D - \tau)_\nu Q_0(z) + R_0 \\ &= (D - \tau)_\nu^2 Q_1(z) + (D - \tau)_\nu R_1 + R_0 \\ &\quad \vdots \\ &= (D - \tau)_\nu^{m+1} Q_m(z) + (D - \tau)_\nu^m R_m + \dots + (D - \tau)_\nu R_1 + R_0 \\ &= (D - \tau)_\nu^{m+1} Q(z) + R(z), \end{aligned}$$

where $Q(z) = Q_m(z)$, and by Lemma 3.10,

$$R(z) = \sum_{k=0}^m R_k (D - \tau)_\nu^k 1 = \sum_{k=0}^m R_k e^{-\tau D} D_\nu^k e^{\tau D} 1 = \sum_{k=0}^m R_k H_k^{-\nu}(z - \tau),$$

are the generalized ν -quotient and remainder, having degrees $n - m - 1$ and m , respectively, for $m = 0, 1, \dots, n-1$. The generalized quotient-remainder form above translates to a generalized quotient-remainder form for the polynomial $P_\nu(z)$, since

$$P_\nu(z) = e^{-\nu D^2} [(D - \tau)_\nu^{m+1} Q(z)] + e^{-\nu D^2} R(z) = (z - \tau)^{m+1} Q_\nu(z) + R_\nu(z),$$

where

$$Q_\nu(z) = e^{-\nu D^2} Q(z) \quad \text{and} \quad R_\nu(z) = \sum_{k=0}^m R_k (z - \tau)^k$$

are the generalized quotient and remainder, having degrees $n - m - 1$ and m , respectively. Thus, it is seen that a formula for the coefficients in the expansion of the generalized remainder is given by

$$R_k = \frac{D^k P_\nu(\tau)}{k!},$$

for $k = 0, 1, \dots, m$ and $m = 0, 1, \dots, n-1$. In particular, if $m = n-1$, the quotient $Q_{n-1}(z)$ is constant, given by the remainder formula above for $m = n$, so that we have $R_n = (Q_{n-1})_\nu(\tau)$. In this case, $P_\nu(z) = e^{-\nu D^2} P(z)$ is just a Taylor series

$$P_\nu(z) = \sum_{k=0}^n R_k (z - \tau)^k,$$

and so

$$P(z) = \sum_{k=0}^n R_k H_k^{-\nu}(z - \tau),$$

with R_k , $k = 0, 1, \dots, n$, given above. If $P_\nu(\tau) = 0$ and m_τ is the order of the zero $z = \tau$, then $1 \leq m_\tau \leq n$ and $D^k P_\nu(\tau) = 0$ if and only if $k < m_\tau$. It follows that

$$P(z) = (D - \tau)_\nu^{k+1} Q(z) \iff P_\nu(z) = (z - \tau)^{k+1} Q_\nu(z),$$

whenever $k < m_\tau$, and conversely.

Consider the matrix $[\mathcal{F}_n^\nu(\tau)]^0 = I_{n+1}$ and the matrix products

$$[\mathcal{F}_n^\nu(\tau)]^m = \prod_{k=0}^m \mathcal{F}_{n-k}^\nu(\tau),$$

for $m = 0, 1, \dots, n$, where $\mathcal{F}_0^\nu(\tau) = 1$, in case $m = n$. The generalized quotient-remainder form given above for $P(z)$ translates to the generalized quotient-remainder vector form for vector P given by

$$P = [\mathcal{F}_n^\nu(\tau)]^{m+1}Q + R,$$

where $R = \sum_{k=0}^m [\mathcal{F}_n^\nu(\tau)]^k R_k$, $Q = Q_m \in \mathbb{C}^{n-m}$ is the generalized quotient vector, and by the notational convention above, R_k is the generalized remainder-vector $[R_k \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{C}^{n-k+1}$, for $k = 0, 1, \dots, m$, and $m = 0, 1, \dots, n-1$.

These observations are summarized in the next corollary.

Corollary 3.11. *Let ν be real and τ be complex. If $P(z)$ is an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ is the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$, then*

$$\begin{aligned} P = [\mathcal{F}_n^\nu(\tau)]^{m+1}Q + R &\iff P(z) = (D - \tau)_\nu^{m+1}Q(z) + R(z) \\ &\iff P_\nu(z) = (z - \tau)^{m+1}Q_\nu(z) + R_\nu(z), \end{aligned}$$

where the quotients $Q(z)$ and $Q_\nu(z)$ are unique polynomials having degree $n - m - 1$, with $Q_\nu(z) = e^{-\nu D^2} Q(z)$, where $Q = Q_m \in \mathbb{C}^{n-m}$ is the quotient vector, and where the remainders $R(z)$ and $R_\nu(z)$ are unique polynomials having degree m , with $R_\nu(z) = e^{-\nu D^2} R(z)$, and remainder vector $R \in \mathbb{C}^{n+1}$ given above. Moreover, if $P_\nu(\tau) = 0$, then

$$P = [\mathcal{F}_n^\nu(\tau)]^{m+1}Q \iff P(z) = (D - \tau)_\nu^{m+1}Q(z) \iff P_\nu(z) = (z - \tau)^{m+1}Q_\nu(z),$$

whenever $m < m_\tau$, and conversely, with $1 \leq m_\tau \leq n$.

4. Geometric formulation of ν -divisibility

Denote the ordinary inner-product of vectors U and V in $\mathbb{C}^{n+1}$ by $(U, V) = V^*U$, where V^* is the usual conjugate transpose of V . For positive constants $\delta_0, \delta_1, \dots, \delta_n$, denote the $(n+1) \times (n+1)$ diagonal matrix

$$J = \begin{bmatrix} \delta_0 & 0 & 0 & \cdots & 0 \\ 0 & \delta_1 & 0 & \cdots & 0 \\ 0 & 0 & \delta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \delta_n \end{bmatrix},$$

and define the J -inner product of vectors U and V in $\mathbb{C}^{n+1}$ by $\langle U, V \rangle = V^*JU$. The length of a vector V is distinguished in these two inner products by the notation $\|V\|^2 = (V, V)$ and $\|V\|_J^2 = \langle V, V \rangle$. The transpose $M^\dagger$ of a matrix M on $\mathbb{C}^{n+1}$, with respect to the J -inner product, is given by $M^\dagger = M^*J$, where M^* is the usual complex conjugate of M . Let $\mathcal{G}_n^\nu(\tau)$ be any matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$, defined at the beginning of the previous section. With respect to the J -inner product, the *geometric formulation* defines the quotient as the projection of the polynomial vector $P \in \mathbb{C}^{n+1}$ onto the *quotient subspace* $\mathcal{Q}_n^\nu(\tau) \subset \mathbb{C}^{n+1}$, where the

dimension of $\mathcal{Q}_n^\nu(\tau)$ is $n = \text{Rank}(\mathcal{G}_n^\nu(\tau))$. The *remainder subspace* is the orthogonal complement $\mathcal{R}_n^\nu(\tau) = [\mathcal{Q}_n^\nu(\tau)]^\perp$, so that $\mathbb{C}^{n+1} = \mathcal{Q}_n^\nu(\tau) \oplus \mathcal{R}_n^\nu(\tau)$, and thus the remainder subspace has dimension 1. To consolidate notation below, it is convenient to put

$$\hat{H}_k^\nu(z) = \frac{H_k^\nu(z)}{\delta_k},$$

for $k = 0, 1, \dots, n$, and for ν real.

Lemma 4.1. *If ν is real, τ is complex, and*

$$Z_n^\nu(\tau) = [\hat{H}_0^\nu(\tau) \ \hat{H}_1^\nu(\tau) \ \hat{H}_2^\nu(\tau) \ \dots \ \hat{H}_n^\nu(\tau)]^*,$$

then $\mathcal{R}_n^\nu(\tau) = \text{Span}\{Z_n^\nu(\tau)\}$, for $n = 2, 3, \dots$

Proof. In view of the relation

$$\mathcal{R}_n^\nu(\tau) = \text{Rng}[\mathcal{F}_n^\nu(\tau)]^\perp = \text{Rng}[\mathcal{E}_n^\nu(\tau)]^\perp = \text{Ker}[\mathcal{E}_n^\nu(\tau)]^\dagger,$$

the solution of the homogeneous system

$$[\mathcal{E}_n^\nu(\tau)]^\dagger X = 0,$$

where $X = [x_0, x_1, \dots, x_n] \in \mathbb{C}^{n+1}$, is given by the system

$$\begin{bmatrix} -H_1^\nu(\bar{\tau}) & 1 & 0 & 0 & \cdots & 0 & 0 \\ -H_2^\nu(\bar{\tau}) & 0 & 1 & 0 & \cdots & 0 & 0 \\ -H_3^\nu(\bar{\tau}) & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -H_{n-1}^\nu(\bar{\tau}) & 0 & 0 & 0 & \cdots & 1 & 0 \\ -H_n^\nu(\bar{\tau}) & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \delta_0 \\ x_1 \delta_1 \\ x_2 \delta_2 \\ \vdots \\ x_{n-1} \delta_{n-1} \\ x_n \delta_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

This gives $x_i = x_0 \delta_0 \hat{H}_i^\nu(\bar{\tau})$, for $i = 1, 2, \dots, n$, giving the solution $X = x_0 \delta_0 Z_n^\nu(\tau)$. $\square$

The polynomial vector $P = [p_0 \ p_1 \ \dots \ p_n]^\top \in \mathbb{C}^{n+1}$ is uniquely expressed as a linear combination

$$P = P_q^\nu(\tau) + P_r^\nu(\tau),$$

where $P_q^\nu(\tau) \in \mathcal{Q}_n^\nu(\tau)$ and $P_r^\nu(\tau) \in \mathcal{R}_n^\nu(\tau)$ are the projections onto the quotient-remainder subspace decomposition of $\mathbb{C}^{n+1}$. These component vectors are given by the system $P = \mathcal{G}_n^\nu(\tau)G' + R'$, where $P_q^\nu(\tau) = \mathcal{G}_n^\nu(\tau)G'$ and $R' = P_r^\nu(\tau)$, with

$$R' = \frac{\langle P, Z_n^\nu(\tau) \rangle}{\|Z_n^\nu(\tau)\|_J^2} Z_n^\nu(\tau) = \frac{P_\nu(\tau)}{\|Z_n^\nu(\tau)\|_J^2} Z_n^\nu(\tau),$$

so that $R' = 0$ if and only if $P_\nu(\tau) = 0$. Denoting the column vectors of $\mathcal{E}_n^\nu(\tau)$ by $\mathcal{E}_n^\nu(\tau) = [E_0^\nu(\tau) E_1^\nu(\tau) \ \dots \ E_{n-1}^\nu(\tau)]$, for $n = 2, 3, \dots$, there is a unique vector $X = [x_0 \ x_1 \ x_2 \ \dots \ x_n]^\top$, such that

$$P_r^\nu(\tau) = x_0 Z_n^\nu(\tau) \quad \text{and} \quad P_q^\nu(\tau) = \sum_{k=1}^n x_k E_k^\nu(\tau).$$

If $E' = [x_1 \ x_2 \ x_3 \ \dots \ x_n]^T$, then $P = \mathcal{E}_n^\nu(\tau)E' + R'$ is given by

$$\begin{bmatrix} \hat{H}_0^\nu(\bar{\tau}) & -H_1^\nu(\tau) & -H_2^\nu(\tau) & \cdots & -H_n^\nu(\tau) \\ \hat{H}_1^\nu(\bar{\tau}) & 1 & 0 & \cdots & 0 \\ \hat{H}_2^\nu(\bar{\tau}) & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{H}_{n-1}^\nu(\bar{\tau}) & 0 & 0 & \cdots & 0 \\ \hat{H}_n^\nu(\bar{\tau}) & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix}.$$

In particular, solving this system for the constant polynomial vector $[p_0 \ 0 \ 0 \ \dots \ 0]^T$ yields

$$X = \frac{p_0}{\|Z_n^\nu(\tau)\|_J^2} [1 - \hat{H}_1^\nu(\tau) - \hat{H}_2^\nu(\tau) - \hat{H}_3^\nu(\tau) - \dots - \hat{H}_n^\nu(\tau)]^*,$$

which shows that a nonzero constant polynomial has nonzero components in both subspaces. The algebraic and geometric formulations, given by $P = \mathcal{G}_n^\nu(\tau)G + R$ and $P = \mathcal{G}_n^\nu(\tau)G' + R'$, respectively, are distinguished by their remainder terms R and R' . Whereas the algebraic and geometric quotient terms, given by $\mathcal{G}_n^\nu(\tau)G$ and $\mathcal{G}_n^\nu(\tau)G'$, are both in the quotient subspace, the geometric remainder is in the remainder subspace, and by the remarks above, the algebraic remainder has nonzero components in both subspaces. Moreover, the algebraic and geometric remainders have length

$$\|R\|_J = |\delta_0| |P_\nu(\tau)| \quad \text{and} \quad \|R'\|_J = \frac{|P_\nu(\tau)|}{\|Z_n^\nu(\tau)\|_J},$$

so that

$$G' = G \iff R' = R = 0 \iff P_\nu(\tau) = 0.$$

Denote the columns of the reduced matrix so that

$$\mathcal{G}_n^\nu(\tau) = [G_0^\nu(\tau) G_1^\nu(\tau) \ \dots \ G_{n-1}^\nu(\tau)].$$

Then $\mathcal{F}_n^\nu(\tau)M = \mathcal{G}_n^\nu(\tau)$, where M is an $n \times n$ nonsingular matrix. The *Gram matrix* [11] associated with $\mathcal{G}_n^\nu(\tau)$ is defined by $\Gamma(\mathcal{G}_n^\nu(\tau)) = [\mathcal{G}_n^\nu(\tau)]^\dagger \mathcal{G}_n^\nu(\tau)$, so that

$$\Gamma(\mathcal{G}_n^\nu(\tau)) = \begin{bmatrix} \langle G_0^\nu, G_0^\nu \rangle & \langle G_1^\nu, G_0^\nu \rangle & \langle G_2^\nu, G_0^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_0^\nu \rangle \\ \langle G_0^\nu, G_1^\nu \rangle & \langle G_1^\nu, G_1^\nu \rangle & \langle G_2^\nu, G_1^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_1^\nu \rangle \\ \langle G_0^\nu, G_2^\nu \rangle & \langle G_1^\nu, G_2^\nu \rangle & \langle G_2^\nu, G_2^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_2^\nu \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle G_0^\nu, G_{n-1}^\nu \rangle & \langle G_1^\nu, G_{n-1}^\nu \rangle & \langle G_2^\nu, G_{n-1}^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_{n-1}^\nu \rangle \end{bmatrix},$$

with the abridged notation $G_k^\nu = G_k^\nu(\tau)$, for $k = 1, 2, \dots, n-1$. The Gram determinant is denoted by $\Delta(\mathcal{G}_n^\nu(\tau)) = \det \Gamma(\mathcal{G}_n^\nu(\tau))$, for $n = 2, 3, \dots$. Since the columns of $\mathcal{G}_n^\nu(\tau)$ are linearly independent, the associated Gram matrix is positive definite, and thus nonsingular, and the corresponding Gram determinant is positive. The next theorem details the geometric formulation; see also [8].

Theorem 4.2. Let ν be real, τ be complex and let $\mathcal{G}_n^\nu(\tau)$ be any reduced matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$. Let $P(z)$ be an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector. Finally, let $P_\nu(z) = e^{-\nu D^2} P(z)$, $R = [P_\nu(\tau) \ 0 \ 0 \ \dots \ 0]^T \in \mathbb{C}^{n+1}$ and

$$\hat{P} = [\mathcal{G}_n^\nu(\tau)]^\dagger P = [\langle P, G_0^\nu \rangle \ \langle P, G_1^\nu \rangle \ \dots \ \langle P, G_{n-1}^\nu \rangle]^T.$$

Then, the geometric formulation $P = \mathcal{G}_n^\nu(\tau)G' + R'$ is given by

$$G' = [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P} \quad \text{and} \quad R' = \frac{P_\nu(\tau)}{\|Z_n^\nu(\tau)\|_J^2} Z_n^\nu(\tau),$$

where $P_r^\nu(\tau) = R'$, and where $P_q^\nu(\tau) = \mathcal{G}_n^\nu(\tau)G'$ is given by the expression

$$-\frac{1}{\Delta(\mathcal{G}_n^\nu(\tau))} \det \begin{bmatrix} \langle G_0^\nu, G_0^\nu \rangle & \langle G_1^\nu, G_0^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_0^\nu \rangle & \langle P, G_0^\nu \rangle \\ \langle G_0^\nu, G_1^\nu \rangle & \langle G_1^\nu, G_1^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_1^\nu \rangle & \langle P, G_1^\nu \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle G_0^\nu, G_{n-1}^\nu \rangle & \langle G_1^\nu, G_{n-1}^\nu \rangle & \cdots & \langle G_{n-1}^\nu, G_{n-1}^\nu \rangle & \langle P, G_{n-1}^\nu \rangle \\ G_0^\nu & G_1^\nu & \cdots & G_{n-1}^\nu & 0 \end{bmatrix}.$$

Moreover, the correspondence between the algebraic formulation $P = \mathcal{G}_n^\nu(\tau)G + R$, and the geometric formulation is given by

$$G' - G = [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{R},$$

where $\hat{R} = [\mathcal{G}_n^\nu(\tau)]^\dagger R$.

Proof. By definition of the geometric formulation, it follows that $P_r^\nu(\tau) = R'$ and $P_q^\nu(\tau) = \mathcal{G}_n^\nu(\tau)G'$. The formula for the remainder term follows by projection of P onto the remainder subspace, as given above. As the vector $R' = P - \mathcal{G}_n^\nu(\tau)G'$ is orthogonal to the quotient subspace, taking inner products gives

$$0 = [\mathcal{G}_n^\nu(\tau)]^\dagger R' = [\mathcal{G}_n^\nu(\tau)]^\dagger P - [\mathcal{G}_n^\nu(\tau)]^\dagger \mathcal{G}_n^\nu(\tau)G',$$

which gives the system $\Gamma(\mathcal{G}_n^\nu(\tau))G' = \hat{P}$. Cramer's rule yields G' , which gives the determinantal formula for $P_q^\nu(\tau)$, and which is verified by expanding about the last row. The correspondence between the two divisibility formulations follows from taking inner products of $\mathcal{G}_n^\nu(\tau)(G' - G) = R - R'$, giving that

$$\Gamma(\mathcal{G}_n^\nu(\tau))(G' - G) = [\mathcal{G}_n^\nu(\tau)]^\dagger (R - R') = \hat{R},$$

since $[\mathcal{G}_n^\nu(\tau)]^\dagger R' = 0$. □

Corollary 4.3. Let ν be real and τ be complex. Let $\mathcal{G}_n^\nu(\tau)$ be any reduced matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$. If $P(z)$ is an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ the associated polynomial vector, then the quotient and remainder formulae are given by

$$\|P_q^\nu(\tau)\|_J^2 = \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P} \quad \text{and} \quad \|P_r^\nu(\tau)\|_J^2 = \|P\|_J^2 - \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P},$$

where $\hat{P} = [\mathcal{G}_n^\nu(\tau)]^\dagger P$.

Proof. Since $P_r^\nu(\tau) \perp P_q^\nu(\tau)$ it follows that

$$\begin{aligned}\|P_r^\nu(\tau)\|_J^2 &= \langle P - P_q^\nu(\tau), P_r^\nu(\tau) \rangle \\ &= \langle P, P_r^\nu(\tau) \rangle \\ &= \langle P, P - P_q^\nu(\tau) \rangle \\ &= \|P\|_J^2 - \langle P, P_q^\nu(\tau) \rangle.\end{aligned}$$

By Theorem 4.1, the quotient vector is given by $P_q^\nu(\tau) = \mathcal{G}_n^\nu(\tau) [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P}$, and thus

$$\begin{aligned}\langle P, P_q^\nu(\tau) \rangle &= [\mathcal{G}_n^\nu(\tau) [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P}]^* J P \\ &= \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} [\mathcal{G}_n^\nu(\tau)]^* J P \\ &= \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^* \hat{P} \\ &= \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P},\end{aligned}$$

since the Gram matrix is Hermitian. The expression for the square of the length of the quotient vector now follows directly from the Pythagorean formula. $\square$

The remainder formula in Corollary 4.3 is given by Schur's formula [17] for the determinant of a partitioned matrix

$$[A; B; C] = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

where A is an $n \times n$ nonsingular matrix, B is an $n \times 1$ vector, and C is a 1×1 matrix, is given by

$$\det[A; B; C] = \det A \left[\frac{[A; B; C]}{A} \right],$$

where the *Schur complement* A in $[A; B; C]$ is defined by

$$\frac{[A; B; C]}{A} = C - B^* A^{-1} B.$$

Substituting $A = \Gamma(\mathcal{G}_n^\nu(\tau))$, $B = \hat{P}$, and $C = \|P\|_J^2$ gives the $(n+1) \times (n+1)$ partitioned matrix

$$[\Gamma(\mathcal{G}_n^\nu(\tau)); \hat{P}; \|P\|_J^2] = \begin{bmatrix} \Gamma(\mathcal{G}_n^\nu(\tau)) & \hat{P} \\ \hat{P}^* & \|P\|_J^2 \end{bmatrix},$$

where $\hat{P} = [\mathcal{G}_n^\nu(\tau)]^\dagger P$. The Schur complement of $\Gamma(\mathcal{G}_n^\nu(\tau))$ in $[\Gamma(\mathcal{G}_n^\nu(\tau)); \hat{P}; \|P\|_J^2]$, yields the remainder formula in Corollary 4.3, since

$$\frac{[\Gamma(\mathcal{G}_n^\nu(\tau)); \hat{P}; \|P\|_J^2]}{\Gamma(\mathcal{G}_n^\nu(\tau))} = \|P\|_J^2 - \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P} = \|P_r^\nu(\tau)\|_J^2.$$

It is seen that

$$[\Gamma(\mathcal{G}_n^\nu(\tau)); \hat{P}; \|P\|_J^2] = [\mathcal{G}_n^\nu(\tau) P]^\dagger [\mathcal{G}_n^\nu(\tau) P],$$

so that $[\Gamma(\mathcal{G}_n^\nu(\tau)); \hat{P}; \|P\|_J^2]$ is also a Gram matrix.

Lemma 4.4. Let ν be real and τ be complex. Let $\mathcal{G}_n^\nu(\tau)$ be any reduced matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$. Let U be a $n \times n$ unitary matrix, and put $\mathcal{U}_n^\nu(\tau) = \mathcal{G}_n^\nu(\tau)U$. If $P(z)$ is an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ the associated polynomial vector, then

$$\frac{[\Gamma(\mathcal{U}_n^\nu(\tau)); \tilde{P}; \|P\|_J^2]}{\Gamma(\mathcal{U}_n^\nu(\tau))} = \frac{[\Gamma(\mathcal{G}_n^\nu(\tau)); \hat{P}; \|P\|_J^2]}{\Gamma(\mathcal{G}_n^\nu(\tau))},$$

where $\hat{P} = [\mathcal{G}_n^\nu(\tau)]^\dagger P$ and $\tilde{P} = [\mathcal{U}_n^\nu(\tau)]^\dagger P$.

Proof. Since $U^{-1} = U^*$, it follows that $\Gamma(\mathcal{U}_n^\nu(\tau)) = U^* \Gamma(\mathcal{G}_n^\nu(\tau)) U$ and

$$\begin{aligned} \tilde{P}^* [\Gamma(\mathcal{U}_n^\nu(\tau))]^{-1} \tilde{P} &= ([\mathcal{U}_n^\nu(\tau)]^\dagger P)^* [U^* \Gamma(\mathcal{G}_n^\nu(\tau)) U]^{-1} ([\mathcal{U}_n^\nu(\tau)]^\dagger P) \\ &= ([\mathcal{G}_n^\nu(\tau)]^\dagger P)^* U [U^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} U] U^* ([\mathcal{G}_n^\nu(\tau)]^\dagger P) \\ &= \hat{P}^* [\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1} \hat{P}, \end{aligned}$$

which yields the result. $\square$

Since the Gram matrix $\Gamma(\mathcal{G}_n^\nu(\tau))$ is nonsingular and Hermitian, there exists an orthonormal system of positive eigenvalues and orthonormal eigenvectors corresponding to $\Gamma(\mathcal{G}_n^\nu(\tau))$, respectively λ_k and Λ_k , for $k = 1, 2, \dots, n$. It follows that there exists a unique unitary matrix $U = [\Lambda_1 \ \Lambda_2 \ \dots \ \Lambda_n]$ such that $\Gamma(\mathcal{U}_n^\nu(\tau)) = U^* \Gamma(\mathcal{G}_n^\nu(\tau)) U$ is a diagonal matrix, given by

$$\Gamma(\mathcal{U}_n^\nu(\tau)) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\Gamma(\mathcal{U}_n^\nu(\tau)) = [\mathcal{U}_n^\nu(\tau)]^* \mathcal{U}_n^\nu(\tau)$, with $\mathcal{U}_n^\nu(\tau) = \mathcal{G}_n^\nu(\tau)U$, and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. For $p \geq 1$, the Schatten p -norm [1] of $[\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1}$ is given by

$$\|[\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1}\|_p = \sqrt[p]{\sum_{k=1}^n \frac{1}{\lambda_k^p}}.$$

The value $p = \infty$ corresponds to the spectral norm and equals

$$\|[\Gamma(\mathcal{G}_n^\nu(\tau))]^{-1}\|_\infty = 1/\lambda_1.$$

Corollary 4.5. Let ν be real and τ be complex. Let $\mathcal{G}_n^\nu(\tau)$ be any reduced matrix obtained by performing elementary column operations on $\mathcal{F}_n^\nu(\tau)$. Let the orthonormal system of positive eigenvalues and orthonormal eigenvectors corresponding to $\Gamma(\mathcal{G}_n^\nu(\tau))$ be given as above by λ_k and Λ_k , for $k = 1, 2, \dots, n$, respectively. If $P(z)$ is

an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ the associated polynomial vector, then

$$\|P_r^\nu(\tau)\|_J^2 = \|P\|_J^2 - \sum_{k=1}^n \frac{1}{\lambda_k} |(\hat{P}, \Lambda_k)|^2,$$

where $\hat{P} = [\mathcal{G}_n^\nu(\tau)]^\dagger P$. The Schatten lower bounds are

$$\|P_r^\nu(\tau)\|_J^2 \geq \|P\|_J^2 - \|\left[\Gamma(\mathcal{G}_n^\nu(\tau))\right]^{-1}\|_p \|\hat{P}\|^2,$$

where $p \geq 1$.

Proof. Putting $U = [\Lambda_1 \ \Lambda_2 \ \dots \ \Lambda_n]$ and $\mathcal{U}_n^\nu(\tau) = \mathcal{G}_n^\nu(\tau)U$, Lemma 4.4 gives

$$\|P_r^\nu(\tau)\|_J^2 = \frac{[\Gamma(\mathcal{U}_n^\nu(\tau)); \tilde{P}; \|P\|_J^2]}{\Gamma(\mathcal{U}_n^\nu(\tau))} = \|P\|_J^2 - \tilde{P}^* [\Gamma(\mathcal{U}_n^\nu(\tau))]^{-1} \tilde{P},$$

where $\tilde{P} = [\mathcal{U}_n^\nu(\tau)]^\dagger P = U^* \hat{P} = [(\hat{P}, \Lambda_1) \ (\hat{P}, \Lambda_2) \ \dots \ (\hat{P}, \Lambda_n)]^T$. Therefore

$$\tilde{P}^* [\Gamma(\mathcal{U}_n^\nu(\tau))]^{-1} \tilde{P} = \sum_{k=1}^n \lambda_k^{-1} |(\hat{P}, \Lambda_k)|^2,$$

giving the remainder formula. Since the eigenvectors are an orthonormal basis, it follows that

$$\sum_{k=1}^n \frac{1}{\lambda_k} |(\hat{P}, \Lambda_k)|^2 \leq \frac{1}{\lambda_1} \sum_{k=1}^n |(\hat{P}, \Lambda_k)|^2 = \frac{1}{\lambda_1} \|\hat{P}\|^2,$$

which yields the spectral bound. If $q \geq 1$, with $1/p + 1/q = 1$, then by Hölder's inequality [16], we have

$$\sum_{k=1}^n \frac{1}{\lambda_k} |(\hat{P}, \Lambda_k)|^2 \leq \sqrt[p]{\sum_{k=1}^n \frac{1}{\lambda_k^p}} \sqrt[q]{\sum_{k=1}^n |(\hat{P}, \Lambda_k)|^{2q}} \leq \|\left[\Gamma(\mathcal{G}_n^\nu(\tau))\right]^{-1}\|_p \|\hat{P}\|^2,$$

which yields the Schatten bounds. $\square$

The results below utilize the geometric formulation $P = \mathcal{E}_n^\nu(\tau)E' + R'$, where E' and R' are formulated in Theorem 4.2. Putting $\mathcal{E}_n^\nu(\tau) = [E_0^\nu(\tau) E_1^\nu(\tau) \ \dots \ E_{n-1}^\nu(\tau)]$, we have

$$\Gamma(\mathcal{E}_n^\nu(\tau)) = \begin{bmatrix} \langle E_0^\nu, E_0^\nu \rangle & \langle E_1^\nu, E_0^\nu \rangle & \langle E_2^\nu, E_0^\nu \rangle & \cdots & \langle E_{n-1}^\nu, E_0^\nu \rangle \\ \langle E_0^\nu, E_1^\nu \rangle & \langle E_1^\nu, E_1^\nu \rangle & \langle E_2^\nu, E_1^\nu \rangle & \cdots & \langle E_{n-1}^\nu, E_1^\nu \rangle \\ \langle E_0^\nu, E_2^\nu \rangle & \langle E_1^\nu, E_2^\nu \rangle & \langle E_2^\nu, E_2^\nu \rangle & \cdots & \langle E_{n-1}^\nu, E_2^\nu \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle E_0^\nu, E_{n-1}^\nu \rangle & \langle E_1^\nu, E_{n-1}^\nu \rangle & \langle E_2^\nu, E_{n-1}^\nu \rangle & \cdots & \langle E_{n-1}^\nu, E_{n-1}^\nu \rangle \end{bmatrix},$$

with the notation $E_k^\nu = E_k^\nu(\tau)$, for $k = 0, 1, \dots, n-1$. In the next lemma, we also let

$$X_n^\nu(\tau) = [H_1^\nu(\tau) \ H_2^\nu(\tau) \ \dots \ H_n^\nu(\tau)]^*,$$

for $n = 2, 3, \dots$, and L is the $n \times n$ diagonal matrix

$$L = \begin{bmatrix} \delta_1 & 0 & 0 & \cdots & 0 \\ 0 & \delta_2 & 0 & \cdots & 0 \\ 0 & 0 & \delta_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \delta_n \end{bmatrix},$$

which is distinct from matrix J , defined at the beginning of this section.

Lemma 4.6. *If ν is real and τ is complex, then*

$$[\Gamma(\mathcal{E}_n^\nu(\tau))]^{-1} = [I_n - \mathcal{X}_n^\nu(\tau)] L^{-1},$$

where

$$\mathcal{X}_n^\nu(\tau) = \frac{1}{\|Z_n^\nu(\tau)\|_J^2} L^{-1} X_n^\nu(\tau) [X_n^\nu(\tau)]^*,$$

for $n = 2, 3, \dots$

Proof. The Sherman-Morrison formula [18] for the inverse of a matrix $A + BC^*$, where A is an $n \times n$ nonsingular matrix, and B and C are $n \times 1$ vectors, is the formula

$$[A + BC^*]^{-1} = \left[I_n - \frac{1}{1 + C^* A^{-1} B} A^{-1} B C^* \right] A^{-1},$$

provided that $1 + C^* A^{-1} B \neq 0$. If $A = L$ and $B = C = \sqrt{\delta_0} X_n^\nu(\tau)$, then $\Gamma(\mathcal{E}_n^\nu(\tau))$ is equal to $A + BC^*$, and the Sherman-Morrison formula gives

$$[\Gamma(\mathcal{E}_n^\nu(\tau))]^{-1} = \left[I_n - \frac{1}{\|Z_n^\nu(\tau)\|_J^2} L^{-1} X_n^\nu(\tau) [X_n^\nu(\tau)]^* \right] L^{-1} = [I_n - \mathcal{X}_n^\nu(\tau)] L^{-1},$$

for $n = 2, 3, \dots$ □

A few remarks are in order. If ν is real, τ is complex, $P(z)$ is an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ the associated polynomial vector, then Theorem 4.2, Lemma 4.6 and Corollary 3.9 yield relationships between the algebraic and geometric formulations $P = \mathcal{E}_n^\nu(\tau)E + R$ and $P = \mathcal{E}_n^\nu(\tau)E' + R'$, respectively, given by

$$E' = [I_n - \mathcal{X}_n^\nu(\tau)] L^{-1} \hat{P} \quad \text{and} \quad E' - E = [I_n - \mathcal{X}_n^\nu(\tau)] L^{-1} \hat{R},$$

where $\hat{P} = [\mathcal{E}_n^\nu(\tau)]^\dagger P$ and $\hat{R} = [\mathcal{E}_n^\nu(\tau)]^\dagger R$, for $n = 2, 3, \dots$

The matrix $L^{-1} X_n^\nu(\tau) [X_n^\nu(\tau)]^*$ in Lemma 4.6 has eigenvalue $\lambda_1 = \|Z_n^\nu(\tau)\|_J^2 - \frac{1}{\delta_0}$ with eigenvector $\Lambda_1 = L^{-1} X_n^\nu(\tau)$, and eigenvalue $\lambda_2 = 0$ with multiplicity $n - 1$ and

nonorthogonal eigenvectors

$$\Lambda_2^\nu = \begin{bmatrix} -H_n^\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ H_1^\nu(\tau) \end{bmatrix}, \quad \Lambda_3^\nu = \begin{bmatrix} 0 \\ -H_n^\nu(\tau) \\ 0 \\ \vdots \\ 0 \\ 0 \\ H_2^\nu(\tau) \end{bmatrix}, \quad \dots, \quad \Lambda_n^\nu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -H_n^\nu(\tau) \\ H_{n-1}^\nu(\tau) \end{bmatrix}.$$

Thus, $\mathcal{X}_n^\nu(\tau)$ has eigenvalue λ_1' given by

$$0 < \lambda_1' = \frac{\|Z_n^\nu(\tau)\|_J^2 - 1/\delta_0}{\|Z_n^\nu(\tau)\|_J^2} < 1,$$

with eigenvector Λ_1 ; and eigenvalue $\lambda_2 = 0$ with multiplicity $n - 1$, and eigenvectors $\Lambda_2^\nu, \Lambda_3^\nu, \dots, \Lambda_n^\nu$. A routine calculation gives the spectral norm $\|\mathcal{X}_n^\nu(\tau)\|_2 = \lambda_1' < 1$, and thus

$$[I_n - \mathcal{X}_n^\nu(\tau)]^{-1} = \sum_{k=0}^{\infty} [\mathcal{X}_n^\nu(\tau)]^k,$$

by the geometric series for $[I_n - \mathcal{X}_n^\nu(\tau)]^{-1}$, which yields $\Gamma(\mathcal{E}_n^\nu(\tau)) = L \sum_{k=0}^{\infty} [\mathcal{X}_n^\nu(\tau)]^k$ for $n = 2, 3, \dots$. Moreover, the estimates

$$\|\Gamma(\mathcal{E}_n^\nu(\tau))\|_2 \leq \|L\|_2 \sum_{k=0}^{\infty} \lambda_1'^k = \|L\|_2 [1 - \lambda_1']^{-1}$$

and

$$\|[\Gamma(\mathcal{E}_n^\nu(\tau))]^{-1}\|_2 \leq [1 + \lambda_1'] \|L^{-1}\|_2$$

follow directly.

As a final remark, Lemma 4.4 gives the remainder formula by the Schur complement

$$\|P_r^\nu(\tau)\|_J^2 = \frac{[\Gamma(\mathcal{E}_n^\nu(\tau)); \widehat{P}; \|P\|_J^2]}{\Gamma(\mathcal{E}_n^\nu(\tau))} = \frac{\det[\Gamma(\mathcal{E}_n^\nu(\tau)); \widehat{P}; \|P\|_J^2]}{\det[\Gamma(\mathcal{E}_n^\nu(\tau))]},$$

where $\widehat{P} = [\mathcal{E}_n^\nu(\tau)]^\dagger P$. As noted above, it is seen that

$$[\Gamma(\mathcal{E}_n^\nu(\tau)); \widehat{P}; \|P\|_J^2] = [\mathcal{E}_n^\nu(\tau) \ P]^\dagger [\mathcal{E}_n^\nu(\tau) \ P] = [\mathcal{E}_n^\nu(\tau) \ P]^* J [\mathcal{E}_n^\nu(\tau) \ P],$$

so that

$$\det[\Gamma(\mathcal{E}_n^\nu(\tau)); \widehat{P}; \|P\|_J^2] = |\det[\mathcal{E}_n^\nu(\tau) \ P]|^2 \det[J].$$

Routine calculations yield determinants

$$|\det[\mathcal{E}_n^\nu(\tau) \ P]|^2 = |P_\nu(\tau)|^2 \quad \text{and} \quad \det[\Gamma(\mathcal{E}_n^\nu(\tau))] = \|Z_n^\nu(\tau)\|_J^2 \det[J].$$

Substitution of these determinants above verifies the geometric remainder formula, as given in Theorem 4.2. The next two lemmas combine to re-express the remainder formula.

Lemma 4.7. *Let ν be real and τ be complex. If $P(z)$ is an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ is the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$, then the remainder formula is given by*

$$\|P_r^\nu(\tau)\|_j^2 = \|P\|_j^2 - \hat{P}^* L^{-1} \hat{P} + \hat{P}^* \mathcal{X}_n^\nu(\tau) L^{-1} \hat{P},$$

where $\hat{P} = [\mathcal{E}_n^\nu(\tau)]^\dagger P$, and where the last term is given by

$$\hat{P}^* \mathcal{X}_n^\nu(\tau) L^{-1} \hat{P} = \frac{|P_\nu(\tau) - \delta_0 p_0 \|Z_n^\nu(\tau)\|_j^2|^2}{\|Z_n^\nu(\tau)\|_j^2},$$

and the middle term is given by

$$\begin{aligned} \hat{P}^* L^{-1} \hat{P} &= \frac{|P_\nu(\tau) - \delta_0 p_0 [\|Z_n^\nu\|^2 - 1/\delta_0^2 + 1/\delta_0]|^2}{\|Z_n^\nu\|^2 - 1/\delta_0^2} + \sum_{j=2}^n |(JP, \Upsilon_j^\nu)|^2 \\ &\quad + \delta_0^2 |p_0|^2 \|Z_n^\nu(\tau)\|_j^2 - \delta_0^2 |p_0|^2 \|Z_n^\nu(\tau)\|^2 \\ &\quad + \|P\|_j^2 - \|JP\|^2 + |p_0|^2 (\delta_0 - 1)^2, \end{aligned}$$

for orthonormal vectors $\Upsilon_j^\nu \in \mathbb{C}^{n+1}$, $j = 2, 3, \dots, n$, given below in (*), and for $n = 2, 3, \dots$

Proof. By Corollary 4.3,

$$\|P_r^\nu(\tau)\|_j^2 = \|P\|_j^2 - \hat{P}^* [\Gamma(\mathcal{E}_n^\nu(\tau))]^{-1} \hat{P},$$

giving the remainder formula, by Lemma 4.6. The Hermitian matrix

$$L^{-1} X_n^\nu(\tau) [X_n^\nu(\tau)]^* L^{-1}$$

has eigenvalue $\lambda_1 = \|L^{-1} X_n^\nu(\tau)\|^2$ with eigenvector $\Lambda_1^\nu = L^{-1} X_n^\nu(\tau)$, and has eigenvalue $\lambda_2 = 0$ with multiplicity $n - 1$ and nonorthogonal eigenvectors

$$\Lambda_2^\nu = \begin{bmatrix} -\hat{H}_n^\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hat{H}_1^\nu(\tau) \end{bmatrix}, \quad \Lambda_3^\nu = \begin{bmatrix} 0 \\ -\hat{H}_n^\nu(\tau) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \hat{H}_2^\nu(\tau) \end{bmatrix}, \quad \dots, \quad \Lambda_n^\nu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\hat{H}_n^\nu(\tau) \\ \hat{H}_{n-1}^\nu(\tau) \end{bmatrix}.$$

The Gram-Schmidt process on vectors $\Lambda_2^\nu, \Lambda_3^\nu, \dots, \Lambda_n^\nu$, yields orthonormal vectors $\Upsilon_2^\nu, \Upsilon_3^\nu, \dots, \Upsilon_n^\nu$. Thus, an orthonormal system of eigenvalues and eigenvectors for $L^{-1} X_n^\nu(\tau) [X_n^\nu(\tau)]^* L^{-1}$ is given by

$$\lambda_1 = \|L^{-1} X_n^\nu(\tau)\|^2 \quad \text{with} \quad \Upsilon_1^\nu = \frac{L^{-1} X_n^\nu(\tau)}{\|L^{-1} X_n^\nu(\tau)\|},$$

and $\lambda_k = 0$ with Υ_k^ν , for $k = 2, 3, \dots, n$. Putting $U = [\Upsilon_1^\nu \ \Upsilon_2^\nu \ \dots \ \Upsilon_n^\nu]$ gives the diagonal matrix

$$U^* L^{-1} X_n^\nu(\tau) [X_n^\nu(\tau)]^* L^{-1} U = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix},$$

and it follows that

$$\hat{P}^* \mathcal{X}_n^\nu(\tau) L^{-1} \hat{P} = \frac{1}{\|Z_n^\nu(\tau)\|_J^2} \sum_{k=1}^n \lambda_k |(\hat{P}, \Upsilon_k^\nu)|^2 = \frac{1}{\|Z_n^\nu(\tau)\|_J^2} \lambda_1 |(\hat{P}, \Upsilon_1^\nu)|^2,$$

where it is seen that

$$\lambda_1 |(\hat{P}, \Upsilon_1^\nu)|^2 = |(\hat{P}, L^{-1} X_n^\nu(\tau))|^2 = |(P, J \mathcal{E}_n^\nu(\tau) L^{-1} X_n^\nu(\tau))|^2,$$

and where

$$J \mathcal{E}_n^\nu(\tau) L^{-1} X_n^\nu(\tau) = [1 - \delta_0 \|Z_n^\nu(\tau)\|_J^2 \ H_1^\nu(\tau) \ H_2^\nu(\tau) \ H_3^\nu(\tau) \ \dots \ H_n^\nu(\tau)]^*,$$

from which the last term follows. The middle term is computed by expanding

$$\hat{P}^* L^{-1} \hat{P} = (JP)^* [\mathcal{E}_n^\nu(\tau) L^{-1} [\mathcal{E}_n^\nu(\tau)]^*] (JP),$$

and putting $\mathcal{E}_n^\nu(\tau) L^{-1} [\mathcal{E}_n^\nu(\tau)]^* = T + S$, where

$$T = \begin{bmatrix} \Delta_1^\nu(\tau) - \Delta_2^\nu(\tau) & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{\delta_1} - 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \frac{1}{\delta_2} - 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\delta_{n-1}} - 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\delta_n} - 1 \end{bmatrix}$$

and

$$S = \begin{bmatrix} \Delta_2^\nu(\tau) & -\hat{H}_1^\nu(\tau) & -\hat{H}_2^\nu(\tau) & \dots & -\hat{H}_{n-1}^\nu(\tau) & -\hat{H}_n^\nu(\tau) \\ -\hat{H}_1^\nu(\bar{\tau}) & 1 & 0 & \dots & 0 & 0 \\ -\hat{H}_2^\nu(\bar{\tau}) & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\hat{H}_{n-1}^\nu(\bar{\tau}) & 0 & 0 & \dots & 1 & 0 \\ -\hat{H}_n^\nu(\bar{\tau}) & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

and where $\Delta_1^\nu(\tau) = \|Z_n^\nu(\tau)\|_J^2 - 1/\delta_0$ and $\Delta_2^\nu(\tau) = \|Z_n^\nu(\tau)\|^2 - 1/\delta_0^2$. The matrix S has eigenvalue $\lambda_0 = 0$ with eigenvector $\Lambda_0^\nu = Z_n^\nu(\tau)$, eigenvalue $\lambda_1 = \Delta_2^\nu(\tau) + 1$

with eigenvector $\Lambda_1^\nu = [-\Delta_2^\nu(\tau) \hat{H}_1^\nu(\tau) \hat{H}_2^\nu(\tau) \dots \hat{H}_n^\nu(\tau)]^*$, and eigenvalue $\lambda_j = 1$ having multiplicity $n - 1$ with nonorthogonal eigenvectors given by

$$\Lambda_2^\nu = \begin{bmatrix} 0 \\ \hat{H}_k^\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\hat{H}_1^\nu(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \Lambda_3^\nu = \begin{bmatrix} 0 \\ 0 \\ \hat{H}_k^\nu(\tau) \\ 0 \\ \vdots \\ 0 \\ 0 \\ -\hat{H}_2^\nu(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \Lambda_k^\nu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hat{H}_k^\nu(\tau) \\ -\hat{H}_{k-1}^\nu(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$\Lambda_{k+2}^\nu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\hat{H}_{k+1}^\nu(\tau) \\ \hat{H}_k^\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \Lambda_{k+3}^\nu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\hat{H}_{k+2}^\nu(\tau) \\ 0 \\ \hat{H}_k^\nu(\tau) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad \Lambda_{n+1}^\nu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\hat{H}_n^\nu(\tau) \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hat{H}_k^\nu(\tau) \end{bmatrix},$$

where the entries

$$-\hat{H}_1^\nu(\tau), -\hat{H}_2^\nu(\tau), \dots, -\hat{H}_{k-1}^\nu(\tau), -\hat{H}_{k+1}^\nu(\tau), \dots, -\hat{H}_n^\nu(\tau)$$

are in the $(k + 1)$ -th position of the eigenvectors $\Lambda_2^\nu, \Lambda_3^\nu, \dots, \Lambda_k^\nu, \Lambda_{k+2}^\nu, \dots, \Lambda_{n+1}^\nu$, respectively, and where $k = 1, 2, \dots, n$ is fixed. The Gram-Schmidt process applied to vectors $\Lambda_2^\nu, \dots, \Lambda_k^\nu, \Lambda_{k+2}^\nu, \dots, \Lambda_{n+1}^\nu$, yields orthonormal vectors $\Upsilon_2^\nu, \Upsilon_3^\nu, \dots, \Upsilon_n^\nu$, so that an orthonormal system of eigenvalues and eigenvectors for S is given by

$$(*) \quad \begin{cases} \lambda_0 = 0, & \Upsilon_0^\nu = \Lambda_0^\nu / \|\Lambda_0^\nu\|, \\ \lambda_1 = \Delta_2^\nu + 1, & \Upsilon_1^\nu = \Lambda_1^\nu / \|\Lambda_1^\nu\|, \\ \lambda_j = 1, & \Upsilon_j^\nu, \quad j = 2, 3, \dots, n. \end{cases}$$

Thus, the unitary matrix $U = [\Upsilon_0^\nu \ \Upsilon_1^\nu \ \Upsilon_2^\nu \ \dots \ \Upsilon_n^\nu]$ gives the diagonal matrix

$$U^* S U = \begin{bmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

and it follows that

$$(JP)^* S (JP) = \sum_{j=0}^n \lambda_j |(JP, \Upsilon_j)|^2 = \lambda_1 |(JP, \Upsilon_1)|^2 + \sum_{j=2}^n |(JP, \Upsilon_j)|^2,$$

where it is seen that

$$\lambda_1 |(JP, \Upsilon_1)|^2 = \frac{1}{\Delta_2^\nu} |(JP, \Lambda_1)|^2 = \frac{|P_\nu(\tau) - \delta_0 p_0 [\Delta_2 + 1/\delta_0]|^2}{\Delta_2^\nu}.$$

A calculation gives that

$$\begin{aligned} (JP)^* T (JP) &= \delta_0^2 |p_0|^2 \|Z_n^\nu(\tau)\|_j^2 - \delta_0^2 |p_0|^2 \|Z_n^\nu(\tau)\|^2 + \|P\|_j^2 \\ &\quad - \|JP\|^2 + |p_0|^2 (\delta_0 - 1)^2, \end{aligned}$$

from which the result for $\widehat{P}^* L^{-1} \widehat{P}$ follows, for $n = 2, 3, \dots$ □

Lemma 4.8. *Let ν be real and τ be complex. If $P(z)$ is an n -th degree polynomial and $P \in \mathbb{C}^{n+1}$ the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$, then the remainder formula is equivalent to the proximity formula*

$$\frac{|P_\nu(\tau) - p_0|^2}{\|Z_n^\nu\|^2 - 1/\delta_0^2} = \left[\|JP\|^2 - \delta_0^2 |p_0|^2 \right] - \sum_{j=2}^n |(JP, \Upsilon_j^\nu)|^2,$$

for orthonormal vectors $\Upsilon_j^\nu \in \mathbb{C}^{n+1}$, with $j = 2, 3, \dots, n$, given above in (*), and for $n = 2, 3, \dots$

Proof. The expression

$$\frac{|P_\nu(\tau)|^2}{\|Z_n^\nu(\tau)\|_j^2} - \frac{|P_\nu(\tau) - \delta_0 p_0 \|Z_n^\nu(\tau)\|_j^2|^2}{\|Z_n^\nu(\tau)\|_j^2} + \frac{|P_\nu(\tau) - \delta_0 p_0 [\|Z_n^\nu\|^2 - 1/\delta_0^2 + 1/\delta_0]|^2}{\|Z_n^\nu\|^2 - 1/\delta_0^2}$$

is collected into the expression

$$\frac{|P_\nu(\tau) - p_0|^2}{\|Z_n^\nu\|^2 - 1/\delta_0^2} - \delta_0^2 |p_0|^2 \|Z_n^\nu(\tau)\|_j^2 + \delta_0^2 |p_0|^2 \|Z_n^\nu(\tau)\|^2 + |p_0|^2 (2\delta_0 - 1).$$

In view of this last expression, the proximity formula follows by combining the various terms in the remainder formula from Lemma 4.7. □

Denoting vectors by

$$Y_n^\nu(\tau) = [0 \ \hat{H}_1^\nu(\tau) \ \hat{H}_2^\nu(\tau) \ \dots \ \hat{H}_n^\nu(\tau)]^* \quad \text{and} \quad P' = [0 \ p_1 \ p_2 \ \dots \ p_n]^T,$$

both in $\mathbb{C}^{n+1}$, the proximity formula in Lemma 4.8 takes on a simpler form, given by

$$\frac{|P_\nu(\tau) - p_0|^2}{\|Y_n^\nu\|^2} = \|JP'\|^2 - \sum_{j=2}^n |(JP', \Upsilon_j^\nu)|^2,$$

for $n = 2, 3, \dots$. In particular, the right-hand side of this formula is the square of the distance of JP' from $\Upsilon_\nu^n(\tau) = \text{Span}\{\Upsilon_2^\nu, \Upsilon_3^\nu, \dots, \Upsilon_n^\nu\}$, since the summation term is the square of the length of the projection of JP' onto $\Upsilon_\nu^n(\tau)$. This notation is employed to detail a neighborhood for the proximity of $P_\nu(\tau)$, in the next corollary.

Corollary 4.9. *Let ν be real and τ be complex. Let $P(z)$ be an n -th degree polynomial, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. Denote the vector $V \in \mathbb{C}^{n+1}$ by $V = [0 \ v_1 \ v_2 \ \dots \ v_{k-1} \ 0 \ v_{k+1} \ \dots \ v_n]^T$, and put*

$$V(z) = v_1 \hat{H}_1^\nu(z) + v_2 \hat{H}_2^\nu(z) + \dots + v_{k-1} \hat{H}_{k-1}^\nu(z) + v_{k+1} \hat{H}_{k+1}^\nu(z) + \dots + v_n \hat{H}_n^\nu(z),$$

for some $k = 1, 2, \dots, n$, fixed. Denote the vector $\Upsilon \in \mathbb{C}^{n+1}$ by $\Upsilon = V/\|V\|$ if $V \neq 0$, and by $\Upsilon = 0$, if $V = 0$. If $V(\tau) = 0$, then the proximity inequality

$$\frac{|P_\nu(\tau) - p_0|^2}{\|Y_n^\nu(\tau)\|^2} \leq \|JP'\|^2 - |(JP', \Upsilon)|^2,$$

holds, where $P' = [0 \ p_1 \ p_2 \ \dots \ p_n]^T$, and a condition for $P_\nu(\tau) \neq 0$ is given by

$$\frac{|p_0|}{\|Y_n^\nu(\tau)\|} > \sqrt{\|JP'\|^2 - |(JP', \Upsilon)|^2}.$$

Proof. The results clearly hold when $V = 0$, so assume that $V \neq 0$. The vector

$$\begin{aligned} \Lambda^\nu(\tau) &= v_1 \Lambda_2^\nu(\tau) + v_2 \Lambda_3^\nu(\tau) + \dots + v_{k-1} \Lambda_k^\nu(\tau) + v_{k+1} \Lambda_{k+2}^\nu(\tau) + \dots \\ &\quad + v_n \Lambda_{n+1}^\nu(\tau) \\ &= \hat{H}_k^\nu(\tau) V \end{aligned}$$

is in $\Upsilon_\nu^n(\tau)$, so that $\Lambda^\nu(\tau) = \sum_{j=2}^n \lambda_j \Upsilon_j^\nu$, and so

$$\begin{aligned} |(JP', \Lambda^\nu(\tau))|^2 &= \left[\sum_{j=2}^n \lambda_j (JP', \Upsilon_j^\nu) \right]^2 \\ &\leq \left[\sum_{j=2}^n |\lambda_j|^2 \right] \left[\sum_{j=2}^n |(JP', \Upsilon_j^\nu)|^2 \right] \\ &= \|\Lambda^\nu(\tau)\|^2 \left[\sum_{j=2}^n |(JP', \Upsilon_j^\nu)|^2 \right]. \end{aligned}$$

It follows that

$$\frac{|(JP', V)|^2}{\|V\|^2} = \frac{|(JP', \Lambda^\nu(\tau))|^2}{\|\Lambda^\nu(\tau)\|^2} \leq \sum_{j=2}^n |(JP', \Upsilon_j^\nu(\tau))|^2,$$

and thus

$$\frac{|P_\nu(\tau) - p_0|^2}{\|Y_n^\nu(\tau)\|^2} = \|JP'\|^2 - \sum_{j=2}^n |(JP', \Upsilon_j^\nu(\tau))|^2 \leq \|JP'\|^2 - |(JP', \Upsilon)|^2,$$

which is the proximity inequality. Combining the proximity inequality with the inequality $|P_\nu(\tau)| \geq |p_0| - |P_\nu(\tau) - p_0|$ yields a condition for $|P_\nu(\tau)| > 0$, given by

$$|p_0| > \|Y_n^\nu(\tau)\| \sqrt{\|JP'\|^2 - |(JP', \Upsilon)|^2},$$

which is the condition for $P_\nu(\tau) \neq 0$ stated above. $\square$

The inequality $|P_\nu(\tau) - p_0| \leq \|JP'\| \|Y_n^\nu(\tau)\|$ is a special case in the proof of Corollary 4.9, by taking $V = [0 \ 0 \ 0 \ \dots]^T$. This inequality also follows directly from the Cauchy-Schwartz inequality [16]. It follows that a condition for $P_\nu(\tau) \neq 0$ is given by $\|JP'\| \|Y_n^\nu(\tau)\| < |p_0|$.

Lemma 4.10. *If ν and μ are real, and $\tau = x + iy$, then*

$$\sum_{k=0}^{\infty} \frac{\mu^k}{k!} |H_k^\nu(\tau)|^2 = \frac{1}{\sqrt{1 - 4\mu^2\nu^2}} e^{\mu x^2/(1+2\mu\nu) + \mu y^2/(1-2\mu\nu)},$$

whenever $2|\mu\nu| < 1$.

Proof. The case $\nu = 0$ is obvious. The generating function for the Hermite polynomials

$$e^{2t\tau - t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(\tau),$$

gives the generating function for the parametrized Hermite polynomials

$$e^{t\tau - \nu t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k^\nu(\tau).$$

For $\nu > 0$, it follows that the following sum is given as a Weierstrass transform

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} H_k^\nu(\tau) H_k^\nu(z) &= e^{-\nu D^2} e^{\mu z\tau - \nu(\mu z)^2} \\ &= \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} e^{i\mu t\tau + \nu(\mu t)^2} e^{-(iz+t)^2/4\nu} dt, \end{aligned}$$

whenever $0 < |\mu\nu| < 1/2$. A calculation gives

$$\frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} e^{i\mu t\tau + \nu(\mu t)^2} e^{-(iz+t)^2/4\nu} dt = \frac{1}{\sqrt{1 - 4\mu^2\nu^2}} e^{\frac{\mu\tau z - \nu\mu^2\tau^2 - \nu\mu^2z^2}{1 - 4\mu^2\nu^2}},$$

from which the result follows by putting $z = \bar{\tau}$. The replacement $\nu \rightarrow -\nu$ yields the case $\nu < 0$. $\square$

Corollary 4.11. *Let ν be real, and let $\mu > 0$, with $2\mu|\nu| < 1$. Let $P(z)$ be an n -th degree polynomial, with $P(0) \neq 0$, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. Let $V \in \mathbb{C}^{n+1}$, $\Upsilon \in \mathbb{C}^{n+1}$ be given in Corollary 4.9, and put*

$\Lambda = JP'/\|JP'\|$, where J has diagonal entries $\delta_k = \sqrt{k!/\mu^k}$, for $k = 0, 1, 2, \dots, n$. For $0 \leq |(\Lambda, \Upsilon)| < 1$, let

$$\rho_\mu(V) = \ln \left(\sqrt{1 - 4\mu^2\nu^2} \left[1 + \frac{|p_0|^2}{\|JP'\|^2} \left(\frac{1}{1 - |(\Lambda, \Upsilon)|^2} \right) \right] \right);$$

and, for $|(\Lambda, \Upsilon)| = 1$, put $\rho_\mu(V) = \infty$. If $V(\tau) = 0$ and

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{\rho_\mu(V)}{\mu},$$

where $\tau = x + iy$, then $P_\nu(\tau) \neq 0$. In particular, if

$$1 + \frac{|p_0|^2}{\|JP''\|^2} > \frac{1}{\sqrt{1 - 4\mu^2\nu^2}},$$

where $P'' = [0 \ 0 \ p_2 \ p_3 \ \dots \ p_n]^T$, then $P_\nu(0) \neq 0$.

Proof. The Cauchy-Schwartz inequality gives $|(\Lambda, \Upsilon)| \leq \|\Lambda\| \|\Upsilon\| \leq 1$, with $|(\Lambda, \Upsilon)| = 1$ equivalent to $JP' = v\Upsilon$ for some scalar v . If $|(\Lambda, \Upsilon)| = 1$, then $\rho_\mu(V) = \infty$, so clearly

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{\rho_\mu(V)}{\mu},$$

where $\tau = x + iy$. Since $JP' = v\Upsilon$, the proximity inequality in Corollary 4.9 gives $P_\nu(\tau) = p_0 \neq 0$. If $0 \leq |(\Lambda, \Upsilon)| < 1$, then a condition for $P_\nu(\tau) \neq 0$ is given by

$$\|Y_n^\nu(\tau)\|^2 < \frac{|p_0|^2}{\|JP'\|^2 - |(JP', \Upsilon)|^2},$$

where $\delta_k = \sqrt{k!/\mu^k}$, for $k = 0, 1, 2, \dots$, by Corollary 4.9. Lemma 4.10 gives

$$\|Y_n^\nu(\tau)\|^2 < \sum_{k=0}^{\infty} \frac{\mu^k}{k!} |H_k^\nu(\tau)|^2 - 1 = \frac{1}{\sqrt{1 - 4\mu^2\nu^2}} e^{\mu x^2/(1+2\mu\nu) + \mu y^2/(1-2\mu\nu)} - 1,$$

and thus, a condition for $P_\nu(\tau) \neq 0$ is given by

$$\frac{1}{\sqrt{1 - 4\mu^2\nu^2}} e^{\mu x^2/(1+2\mu\nu) + \mu y^2/(1-2\mu\nu)} - 1 < \frac{|p_0|^2}{\|JP'\|^2 - |(JP', \Upsilon)|^2},$$

which is equivalent to the inequality given in the statement of the corollary. The second part of the corollary follows from the first by taking $V = [0 \ 1 \ 0 \ \dots]^T$. $\square$

The trivial case $V = [0 \ 0 \ 0 \ \dots]^T$ is given in the next corollary.

Corollary 4.12. Let ν be real. Let $P(z)$ be an n -th degree polynomial, with $P(0) \neq 0$, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. Let $\mu > 0$, with $2\mu|\nu| < 1$, be such that

$$\rho_\mu(0) = \ln \left(\sqrt{1 - 4\mu^2\nu^2} \left(1 + \frac{|p_0|^2}{\|JP'\|^2} \right) \right) > 0,$$

where J has diagonal entries $\delta_k = \sqrt{k!/\mu^k}$, for $k = 0, 1, 2, \dots, n$. If membership $\tau \in T_\mu(0)$ is defined by

$$\frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{\rho_\mu(0)}{\mu},$$

where $\tau = x + iy$, then $P_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(0)$. Moreover, if $\rho_\mu(0) > 0$, then $T_\mu(0) \neq \emptyset$.

Proof. By the remark preceding Lemma 4.10, a condition for $P_\nu(\tau) \neq 0$ is given by $\|JP'\|^2 \|Y_n^\nu(\tau)\|^2 < |p_0|^2$. Putting $\delta_k = \sqrt{k!/\mu^k}$, for $k = 0, 1, 2, \dots$, it follows that

$$\|Y_n^\nu(\tau)\|^2 < \sum_{k=0}^{\infty} \frac{\mu^k}{k!} |H_k^\nu(\tau)|^2 - 1 = \frac{1}{\sqrt{1-4\mu^2\nu^2}} e^{\mu x^2/(1+2\mu\nu) + \mu y^2/(1-2\mu\nu)} - 1,$$

by Lemma 4.10. Thus, the condition for $P_\nu(\tau) \neq 0$ is given by

$$\frac{1}{\sqrt{1-4\mu^2\nu^2}} e^{\mu x^2/(1+2\mu\nu) + \mu y^2/(1-2\mu\nu)} - 1 < \frac{|p_0|^2}{\|JP'\|^2},$$

which yields the condition for membership in $T_\mu(0)$. If $\rho_\mu(0) > 0$, then $T_\mu(0)$ is the interior of an ellipse, so that $T_\mu(0) \neq \emptyset$. $\square$

The result from Corollary 4.12 is needed in the next corollary, of which it is a special case.

Corollary 4.13. Let ν be real, and let $\mu > 0$, with $2\mu|\nu| < 1$, and with $\rho_\mu(0) > 0$ as in Corollary 4.12. Let $P(z)$ be an n -th degree polynomial, with $P(0) \neq 0$, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. Let $V \in \mathbb{C}^{n+1}$ be given as in Corollary 4.9, and let $\rho_\mu(V)$ be given as in Corollary 4.11. If membership $\tau \in T_\mu(V)$ is defined by $V(\tau) = 0$ and

$$\frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{\rho_\mu(V)}{\mu},$$

where $\tau = x + iy$, then $T_\mu(0) \subseteq T_\mu(V)$ and $P_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(V)$.

Proof. Corollary 4.11 gives that $P_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(V)$. If $\rho_\mu(0) \leq 0$, then $T_\mu(0) = \emptyset$, so clearly $T_\mu(0) \subseteq T_\mu(V)$. Assume that $\rho_\mu(0) > 0$. If $|(\Lambda, \Upsilon)| = 1$, then $\rho_\mu(V) = \infty$, by Corollary 4.11. If $0 \leq |(\Lambda, \Upsilon)| < 1$, then

$$\begin{aligned} & 1 + \frac{|p_0|^2}{\|JP'\|^2} \left(\frac{1}{1 - |(\Lambda, \Upsilon)|^2} \right) \\ &= \left(1 + \frac{|p_0|^2}{\|JP'\|^2} \right) \left[1 + \frac{|p_0|^2}{\|JP\|^2} |(\Lambda, \Upsilon)|^2 + \frac{|p_0|^2}{\|JP\|^2} |(\Lambda, \Upsilon)|^4 + \dots \right] \\ &\geq \left(1 + \frac{|p_0|^2}{\|JP'\|^2} \right) \left[1 - \frac{|p_0|^2}{\|JP\|^2} |(\Lambda, \Upsilon)|^2 \right]^{-1}, \end{aligned}$$

which yields the lower bound

$$\rho_\mu(V) \geq \rho_\mu(0) - \ln\left(1 - \frac{|p_0|^2}{\|JP\|^2} |(\Lambda, \Upsilon)|^2\right) \geq \rho_\mu(0) + \frac{|p_0|^2}{\|JP\|^2} |(\Lambda, \Upsilon)|^2.$$

In either case $\rho_\mu(V) \geq \rho_\mu(0)$, so that $T_\mu(0) \subseteq T_\mu(V)$. $\square$

A few remarks are in order. The special case $\nu = 0$ corresponds to $P_0(z) = P(z)$, with $\rho_\mu(V)$ given by

$$\rho_\mu(V) = \ln\left(1 + \frac{|p_0|^2}{\|JP'\|^2} \left(\frac{1}{1 - |(\Lambda, \Upsilon)|^2}\right)\right),$$

for all $\mu > 0$. The trivial case $V = 0$ is given in Corollary 4.12, and a special case of Corollary 4.13. In this case, the identity $V(\tau) = 0$ holds for all τ , so that $T_\mu(0)$ is the interior of the ellipse

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{\rho_\mu(0)}{\mu},$$

where $\tau = x + iy$. Moreover, if $|\nu'| \leq |\nu|$, then $2\mu|\nu'| < 1$ and $\rho'_\mu(0) \geq \rho_\mu(0) > 0$, where

$$\rho'_\mu(0) = \ln\left(\sqrt{1 - 4\mu^2(\nu')^2} \left[1 + \frac{|p_0|^2}{\|JP'\|^2}\right]\right),$$

so that $T'_\mu(0)$ is the interior of the ellipse

$$\frac{x^2}{1 + 2\mu\nu'} + \frac{y^2}{1 - 2\mu\nu'} < \frac{\rho'_\mu(0)}{\mu},$$

and so $P_{\nu'}(\tau) \neq 0$ for all $\tau \in T'_\mu(0)$. Replacing $P(z)$ in Corollary 4.13 by

$$\hat{P}(z) = e^{\gamma D} P(z) = \sum_{k=0}^n \hat{p}_k z^k,$$

where γ is complex, gives a neighborhood $\tilde{T}_\mu(V)$ such that $\hat{P}_\nu(\tau) \neq 0$ for all τ in $\tilde{T}_\mu(V)$. In view of the relation $P_\nu(z) = e^{-\gamma D} \hat{P}_\nu(z)$, it follows that $P_\nu(\tau) \neq 0$ for all $\tau \in (\gamma + \tilde{T}_\mu(V))$.

An example of a τ -dependent nonzero vector $V \in \mathbb{C}^{n+1}$ so that $V(\tau) = 0$, for all τ , is given by

$$V = [0 \ \hat{H}_2^\nu(\tau) - \hat{H}_1^\nu(\tau) \ 0 \ 0 \ \dots \ 0]^T,$$

so that $V(z) = \hat{H}_2^\nu(\tau) \hat{H}_1^\nu(z) - \hat{H}_1^\nu(\tau) \hat{H}_2^\nu(z)$. Neighborhoods based on such identities, are given in the next corollary.

Corollary 4.14. *Let ν be real. Let $P(z)$ be an n -th degree polynomial, with $P(0) \neq 0$, $P \in \mathbb{C}^{n+1}$ be the associated polynomial vector, and $P_\nu(z) = e^{-\nu D^2} P(z)$. Let $\mu > 0$, with $2\mu|\nu| < 1$ and $\rho_\mu(0) > 0$, where $\rho_\mu(0)$ is given as in Corollary 4.12. Let $V \in \mathbb{C}^{n+1}$ be a τ -dependant vector given as in Corollary 4.9, with $V(\tau) = 0$ for all τ . Let $\rho_\mu(V)$ be given as in Corollary 4.11, and let*

$$\tilde{\rho}_\mu(V) = \rho_\mu(0) - \ln\left(1 - \frac{|p_0|^2}{\|JP\|^2} |(\Lambda, \Upsilon)|^2\right),$$

where $\Upsilon \in \mathbb{C}^{n+1}$ is given as in Corollary 4.9, $\Lambda = JP'/\|JP'\|$, and J has diagonal entries $\delta_k = \sqrt{k!/\mu^k}$, for $k = 0, 1, 2, \dots, n$. If membership $\tau \in \tilde{T}_\mu(V)$ is defined by

$$\frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{\tilde{\rho}_\mu(V)}{\mu},$$

where $\tau = x + iy$, then $\emptyset \neq T_\mu(0) \subseteq \tilde{T}_\mu(V) \subseteq T_\mu(V)$, and $P_\nu(\tau) \neq 0$ for all $\tau \in T_\mu(V)$.

Proof. By the identity $V(\tau) = 0$ for all τ , the proof is included in the proof of Corollary 4.13. $\square$

An advantage of the estimate $\tilde{\rho}_\mu(V)$, over $\rho_\mu(V)$, is that the value $|(\Lambda, \Upsilon)| = 1$ is not a separate case for defining membership in $\tilde{T}_\mu(V)$, as in Corollary 4.13. The condition $p_0 \neq 0$ is sufficient for $P_\nu(\tau) \neq 0$ in a neighborhood of the origin, provided ν is sufficiently small. The next corollary quantifies this observation, by putting upper bounds on the members of P .

Corollary 4.15. *Let ν be real and $\epsilon > 0$, with $2\epsilon^2|\nu| < 1$. Let*

$$P(z) = \sum_{k=0}^n p_k z^k$$

be an n -th degree polynomial, with $p_0 \neq 0$ and $k!|p_k|^2 \leq \delta^2 \epsilon^{2k}$, for $k = 1, 2, \dots, n$, where

$$0 < \delta < |p_0| \left(\frac{1}{\epsilon \sqrt{2|\nu|}} - 1 \right).$$

Then, $P_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1+2\mu'\nu} + \frac{y^2}{1-2\mu'\nu} < \frac{\rho_{\mu'}(0)}{\mu'}$$

for some $\epsilon^2 < \mu' < 1/(2|\nu|)$; in particular, for $\mu' = \epsilon^2/2(1 - \delta^2/|p_0|^2) + 1/(4|\nu|)$. If $|\nu'| \leq |\nu|$, then $P_{\nu'}(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1+2\mu'\nu'} + \frac{y^2}{1-2\mu'\nu'} < \frac{\rho'_{\mu'}(0)}{\mu'},$$

where μ' and $\rho'_{\mu'}(0)$ correspond to ν and ν' , respectively.

Proof. If $\rho_\mu(0) > 0$, then $P_\nu(\tau) \neq 0$ inside the ellipse given above, where

$$\rho_\mu(0) = \ln \left(\sqrt{1 - 4\mu^2\nu^2} \left[1 + \frac{|p_0|^2}{\|JP'\|^2} \right] \right),$$

by Corollary 4.12. Assuming $\epsilon^2 < \mu$ gives the estimate

$$\|JP'\|^2 = \sum_{k=1}^n \frac{k!}{\mu^k} |p_k|^2 \leq \delta^2 \sum_{k=1}^{\infty} \left(\frac{\epsilon^2}{\mu} \right)^k = \frac{\delta^2 \epsilon^2}{\mu - \epsilon^2},$$

so that

$$\sqrt{1 - 4\mu^2\nu^2} \left[1 + \frac{|p_0|^2}{\|JP'\|^2} \right] \geq (1 - 2\mu|\nu|) \left(1 + \frac{|p_0|^2}{\epsilon^2 \delta^2} (\mu - \epsilon^2) \right).$$

Putting $\kappa = |p_0|^2/\delta^2$, it follows that $\rho_\mu(0) > 0$ is implied by inequalities

$$(1 - 2\mu|\nu|)\left(1 + \frac{\kappa}{\epsilon^2}(\mu - \epsilon^2)\right) > 1 \iff \mu^2 - \left(\epsilon^2 + \frac{1}{2|\nu|} - \frac{\epsilon^2}{\kappa}\right)\mu + \frac{\epsilon^2}{2|\nu|} < 0,$$

where the parabola

$$\tau = \mu^2 - \left(\epsilon^2 + \frac{1}{2|\nu|} - \frac{\epsilon^2}{\kappa}\right)\mu + \frac{\epsilon^2}{2|\nu|}$$

has vertex (μ', τ') with

$$\mu' = \frac{1}{2}\left(\epsilon^2 + \frac{1}{2|\nu|} - \frac{\epsilon^2}{\kappa}\right) \leq \frac{1}{2}\left(\epsilon^2 + \frac{1}{2|\nu|}\right) \leq \frac{1}{2|\nu|}.$$

The upper bound on δ yields the bound

$$\delta < |p_0|\left(\frac{1}{2\epsilon^2|\nu|} - 1\right),$$

or equivalently $\epsilon^2 < \mu'$, so that $\epsilon^2 < \mu' < 1/(2|\nu|)$. The inequality $\tau' < 0$ is equivalent to inequalities

$$\begin{aligned} (\mu')^2 - \left(\epsilon^2 + \frac{1}{2|\nu|} - \frac{\epsilon^2}{\kappa}\right)\mu' + \frac{\epsilon^2}{2|\nu|} < 0 &\iff \mu' > \frac{\epsilon}{\sqrt{2|\nu|}} \\ &\iff \frac{\delta^2}{|p_0|^2} < \left(\frac{1}{\epsilon\sqrt{2|\nu|}} - 1\right)^2, \end{aligned}$$

which yields the first part of the corollary. The second part follows from the remark after Corollary 4.13. $\square$

The proof of the Corollary 4.15 produces the value $\mu' = \frac{\epsilon^2}{2(1 - \delta^2/|p_0|^2)} + \frac{1}{(4|\nu|)}$, whereas this value is used in both $\rho_{\mu'}(0)$ and $\rho'_{\mu'}(0)$, corresponding to values ν and ν' , respectively. Replacing $P(z)$ in Corollary 4.15 with

$$\hat{P}(z) = e^{(\alpha+i\beta)D}P(z) = \sum_{k=0}^n \hat{p}_k z^k,$$

where α and β are real, gives $\hat{P}_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu'\nu} + \frac{y^2}{1 - 2\mu'\nu} < \frac{\hat{\rho}_{\mu'}(0)}{\mu'},$$

where

$$\hat{\rho}_{\mu'}(0) = \ln \left(\sqrt{1 - 4(\mu')^2\nu^2} \left(1 + \frac{|\hat{p}_0|^2}{\|J\hat{P}'\|^2} \right) \right),$$

and J has diagonal entries $\delta_k = \sqrt{\frac{k!}{(\mu')^k}}$, for $k = 0, 1, \dots, n$, for some $\epsilon^2 < \mu' < \frac{1}{(2|\nu|)}$.

Thus, in view of the relation $P_\nu(z) = e^{-(\alpha+i\beta)D}\hat{P}_\nu(z)$, it follows that $P_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the translated ellipse

$$\frac{(x - \alpha)^2}{1 + 2\mu'\nu} + \frac{(y - \beta)^2}{1 - 2\mu'\nu} < \frac{\hat{\rho}_{\mu'}(0)}{\mu'}.$$

A simpler sufficient condition can be derived from the sufficient conditions of Corollary 4.15, as given in the next corollary.

Corollary 4.16. *Let ν be real. Let $P(z) = \sum_{k=0}^n p_k z^k$ be an n -th degree polynomial, with $p_0 \neq 0$, and put*

$$\Delta^2 = \max_{1 \leq k \leq n} \{(2|\nu|)^k k! |p_k|^2\}.$$

If $\epsilon > 0$ with $2\epsilon^2|\nu| < 1$, and

$$\frac{\Delta}{(\epsilon\sqrt{2|\nu|})^n} < |p_0| \left(\frac{1}{\epsilon\sqrt{2|\nu|}} - 1 \right),$$

then $P_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{\rho_\mu(0)}{\mu},$$

for some $\epsilon^2 < \mu < 1/(2|\nu|)$. If $|\nu'| \leq |\nu|$, then $P_{\nu'}(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu\nu'} + \frac{y^2}{1 - 2\mu\nu'} < \frac{\rho'_\mu(0)}{\mu},$$

where μ and $\rho'_\mu(0)$ correspond to ν and ν' , respectively.

Proof. If $\delta > 0$ satisfies

$$\frac{\Delta}{(\epsilon\sqrt{2|\nu|})^n} \leq \delta < |p_0| \left(\frac{1}{\epsilon\sqrt{2|\nu|}} - 1 \right),$$

then $\delta^2 \geq \Delta^2 / (\epsilon\sqrt{2|\nu|})^{2k}$, for $k = 1, 2, \dots, n$, since $2\epsilon^2|\nu| < 1$. It follows that

$$\delta^2 \geq k! |p_k|^2 / \epsilon^{2k},$$

for $k = 1, 2, \dots, n$, so that Corollary 4.15 yields the result. $\square$

Remarks similar to the two remarks following Corollary 4.15, on the range of Weierstrass parameter ν and translation of the elliptical neighborhood $T_\mu(0)$, also apply to Corollary 4.16.

5. Extensions to entire functions

The corollaries below extend certain results from the last section to entire functions. An entire function $\Phi(z)$ has *finite order* if $|\Phi(z)| \leq C_\varrho e^{|z|^\varrho}$, for some $\varrho > 0$, $C_\varrho > 0$, and all z . In this case, the value $\rho = \inf\{\varrho\}$ is the *order* of Φ . An entire function Φ with order ρ has *finite type* if $|\Phi(z)| \leq C_\varsigma e^{\varsigma|z|^\rho}$, for some $\varsigma > 0$, $C_\varsigma > 0$, and all z . In this case, the value $\sigma = \inf\{\varsigma\}$ is the *type* of Φ . If Φ has Taylor series $\Phi(z) = \sum_{k=0}^\infty p_k z^k$, then the criterion to be *entire*, and formulae for order and type are given by

$$\lim_{k \rightarrow \infty} \sqrt[k]{|p_k|} = 0, \quad \rho = \limsup_{k \rightarrow \infty} \frac{-\ln k}{\ln \sqrt[k]{|p_k|}} \quad \text{and} \quad \sigma = \frac{1}{e\rho} \limsup_{k \rightarrow \infty} k \sqrt[k]{|p_k|^\rho},$$

respectively [14].

Lemma 5.1. *Let the Taylor polynomials associated with $\Phi(z)$ be*

$$P_n(z) = \sum_{k=0}^n p_k z^k$$

for $n = 0, 1, \dots$. If Φ has order ρ , then for all $\varrho > \rho$, there is $C_\varrho > 0$ such that $|P_n(z)| \leq C_\varrho e^{|z|^\varrho}$, for $n = 0, 1, \dots$. If Φ has order ρ and type σ , then for all $\varsigma > \sigma$, there is $C_\varsigma > 0$ such that $|P_n(z)| \leq C_\varsigma e^{\varsigma|z|^\rho}$, for $n = 0, 1, \dots$.

Proof. If Φ has order ρ , then the entire function

$$\Phi^*(z) = \sum_{k=0}^{\infty} |p_k| z^k$$

has order ρ . The result now follows from the inequality

$$|P_n(z)| \leq \Phi^*(|z|),$$

for $n = 0, 1, \dots$. The argument for the case when Φ has order ρ and type σ is similar. $\square$

The Weierstrass operators defined on entire functions are formally identical to polynomials in the first section. For $\nu > 0$ and Φ entire, the Weierstrass operators $e^{\pm\nu D^2}\Phi(z)$ are defined by

$$\begin{cases} e^{\nu D^2}\Phi(z) = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} \Phi(t) e^{-(z-t)^2/4\nu} dt, \\ e^{-\nu D^2}\Phi(z) = \frac{1}{\sqrt{4\pi\nu}} \int_{-\infty}^{\infty} \Phi(it) e^{-(iz+t)^2/4\nu} dt, \end{cases}$$

whenever the integrals exist. If ν is real, then $\Phi_\nu := e^{-\nu D^2}\Phi$ exists as an entire function for all $|\nu| > 0$ whenever Φ has order $\rho < 2$, and for all $|\nu| < 1/4\sigma$, if $\rho = 2$ with $\sigma < \infty$ [19]. The next lemma, needed to extend certain corollaries from the last section to entire functions having order $\rho \leq 2$, uses the *Stirling's approximation* given by

$$\sqrt{2\pi k} \left(\frac{k}{e}\right)^k < k! < \sqrt{2\pi k} \left(\frac{k}{e}\right)^k (1 + \kappa(k)),$$

where $\kappa(k) = 1/(12k - 1)$ [13].

Lemma 5.2. *Let $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ have order $\rho < 2$ or $\rho = 2$ and type $\sigma < \infty$. If $\rho < \varrho < 2$, then*

$$k!|p_k|^2 = \mathcal{O}\left(e^{-k} k^{-\varepsilon k}\right) \quad \text{as } k \rightarrow \infty,$$

where $\varepsilon = 2/\varrho - 1 > 0$. Moreover, if $\rho = 2$ and $\sigma < \infty$, then $k!|p_k|^2 = \mathcal{O}\left((2\varsigma)^k\right)$ as $k \rightarrow \infty$, where $\varsigma > \sigma$.

Proof. The entire function $\Phi'(z) = \sum_{k=0}^{\infty} p'_k z^k$ with $p'_k = \sqrt[4]{k} p_k$, for $k = 0, 1, 2, \dots$, has the same order and type as Φ , so it suffices to replace Φ with Φ' . If $\rho < \varrho < 2$, then

$$\frac{-\ln k}{\ln \sqrt[k]{|p'_k|}} \leq \varrho \iff |p'_k|^2 \leq k^{-2k/\varrho} \iff |p_k|^2 \leq k^{-1/2-2k/\varrho},$$

for k large. Thus, Stirling's formula yields $k!|p_k|^2 \leq \sqrt{2\pi}(1 + \kappa(k))e^{-k}k^{-(2/\varrho-1)k}$, for k large. If $\sigma < \varsigma < \infty$, then

$$\frac{1}{2e}k \sqrt[k]{|p'_k|^2} \leq \varsigma \iff |p'_k|^2 \leq \left(\frac{2e\varsigma}{k}\right)^k \iff |p_k|^2 \leq (2e\varsigma)^k k^{-k-1/2},$$

for k large. Thus, Stirling's formula yields $k!|p_k|^2 \leq \sqrt{2\pi}(1 + \kappa(k))(2\varsigma)^k$, for k large. $\square$

For an entire function $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ having order $\rho < 2$ or $\rho = 2$ and type $\sigma < \infty$, consider the complex vectors $\Phi = [p_0 \ p_1 \ p_2 \ \dots]^T$, $\Phi' = [0 \ p_1 \ p_2 \ p_3 \ \dots]^T$, and $\Phi'' = [0 \ 0 \ p_2 \ p_3 \ p_4 \ \dots]^T$. For $\mu > 0$, consider the matrix

$$J = \begin{bmatrix} \delta_0 & 0 & 0 & \dots & 0 \\ 0 & \delta_1 & 0 & \dots & 0 \\ 0 & 0 & \delta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix},$$

with diagonal entries $\delta_k = \sqrt{k!/\mu^k}$, for $k = 0, 1, \dots$, so that $J\Phi = [\delta_0 p_0 \ \delta_1 p_1 \ \dots]^T$, $J\Phi' = [0 \ \delta_1 p_1 \ \delta_2 p_2 \ \dots]^T$ and $J\Phi'' = [0 \ 0 \ \delta_2 p_2 \ \delta_3 p_3 \ \dots]^T$. Let

$$\|J\Phi\|^2 = \sum_{k=0}^{\infty} \delta_k^2 |p_k|^2, \quad \|J\Phi'\|^2 = \sum_{k=1}^{\infty} \delta_k^2 |p_k|^2 \quad \text{and} \quad \|J\Phi''\|^2 = \sum_{k=2}^{\infty} \delta_k^2 |p_k|^2.$$

By Lemma 5.2, these sums converge for all $\mu > 0$, if $\rho < 2$; and, for all $\mu > 2\sigma$, if $\rho = 2$ and $\sigma < \infty$. The next lemma follows directly from these remarks.

Lemma 5.3. *Let ν be real. Let $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ be an entire function having order $\rho \leq 2$. Then, the value*

$$\rho_{\mu}(0) = \ln \left(\sqrt{1 - 4\mu^2 \nu^2} \left[1 + \frac{|p_0|^2}{\|J\Phi'\|^2} \right] \right),$$

exists for all $0 < \mu < 1/2|\nu|$, if $\rho < 2$; and, for all $2\sigma < \mu < 1/(2|\nu|)$, if $\rho = 2$ and $\sigma < \infty$.

The following corollary is the transcendental analogue of Corollary 4.9.

Corollary 5.4. *Let $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ be entire with $p_0 \neq 0$, and have order $\rho < 2$ or $\rho = 2$ and type $\sigma < \infty$. Let ν be real, and put $\Phi_{\nu}(z) = e^{-\nu D^2} \Phi(z)$. For $\rho < 2$, let $0 < \mu < 1/2|\nu|$, and for $\rho = 2$, let $2\sigma < \mu < 1/(2|\nu|)$. Denote the vector*

$$V = [0 \ v_1 \ v_2 \ \dots \ v_{k-1} \ 0 \ v_{k+1} \ \dots \ v_m \ 0 \ 0 \ \dots]^T,$$

for some $k = 1, 2, \dots, m$, where $m = 2, 3, \dots$, is fixed, and put

$$V(z) = v_1 \hat{H}_1^\nu(z) + v_2 \hat{H}_2^\nu(z) + \dots + v_{k-1} \hat{H}_{k-1}^\nu(z) + v_{k+1} \hat{H}_{k+1}^\nu(z) + \dots + v_m \hat{H}_m^\nu(z).$$

Let Υ be the vector $\Upsilon = V/\|V\|$ if $V \neq 0$, and $\Upsilon = 0$ if $V = 0$. Put $\Lambda = J\Phi'/\|J\Phi'\|$. For $0 \leq |(\Lambda, \Upsilon)| < 1$, let

$$\rho_\mu(V) = \ln \left(\sqrt{1 - 4\mu^2\nu^2} \left[1 + \frac{|p_0|^2}{\|J\Phi'\|^2} \left(\frac{1}{1 - |(\Lambda, \Upsilon)|^2} \right) \right] \right)$$

and, for $|(\Lambda, \Upsilon)| = 1$, put $\rho_\mu(V) = \infty$. If $V(\tau) = 0$ and

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{\rho_\mu(V)}{\mu},$$

where $\tau = x + iy$, then $\Phi_\nu(\tau) \neq 0$. In particular, if

$$1 + \frac{|p_0|^2}{\|J\Phi''\|^2} > \frac{1}{\sqrt{1 - 4\mu^2\nu^2}},$$

then $\Phi_\nu(0) \neq 0$.

Proof. If the P_n 's, for $n = 0, 1, \dots$, are the Taylor polynomials associated with Φ , then $P_n \rightarrow \Phi$ as $n \rightarrow \infty$, uniformly on compacta, with

$$\|JP'_0\| \leq \|JP'_1\| \leq \dots \leq \|J\Phi'\| < \infty$$

and

$$|(JP'_m, \Upsilon)| = |(JP'_{m+1}, \Upsilon)| = \dots = |(J\Phi', \Upsilon)|,$$

and so

$$|(\Lambda_m, \Upsilon)| \geq |(\Lambda_{m+1}, \Upsilon)| \geq \dots \geq |(\Lambda, \Upsilon)|,$$

where $\Lambda_n = JP'_n/\|JP'_n\|$, for $n = 1, 2, \dots$. The Cauchy-Schwartz inequality gives $|(\Lambda, \Upsilon)| \leq 1$, giving the following two cases. The proof of Corollary 4.9 uses the inequality

$$|(P_n)_\nu(\tau)| \geq |p_0| - \|Y_n^\nu(\tau)\| \sqrt{\|JP'_n\|^2 - |(JP'_n, \Upsilon)|^2},$$

which yields

$$|\Phi_\nu(\tau)| \geq |p_0| - \|Y^\nu(\tau)\| \sqrt{\|J\Phi'\|^2 - |(J\Phi', \Upsilon)|^2},$$

by letting $n \rightarrow \infty$, where $Y^\nu(\tau) = [0 \ \hat{H}_1^\nu(\tau) \ \hat{H}_2^\nu(\tau) \ \hat{H}_3^\nu(\tau) \ \dots]^*$, and where

$$\|Y^\nu(\tau)\|^2 = \sum_{k=1}^{\infty} \frac{\mu^k}{k!} |H_k^\nu(\tau)|^2 = \frac{1}{\sqrt{1 - 4\mu^2\nu^2}} e^{\mu x^2/(1+2\mu\nu) + \mu y^2/(1-2\mu\nu)} - 1,$$

by Lemma 4.10. In case $0 \leq |(\Lambda, \Upsilon)| < 1$, it follows that

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{\rho_\mu(V)}{\mu},$$

where $\tau = x + iy$, gives $|\Phi_\nu(\tau)| > 0$. This inequality also holds for $|(\Lambda, \Upsilon)| = 1$, since $\rho_\mu(V) = \infty$. Indeed,

$$|(\Lambda_m, \Upsilon)| = |(\Lambda_{m+1}, \Upsilon)| = \dots = |(\Lambda, \Upsilon)| = 1,$$

since $|(\Lambda_n, \Upsilon)| \leq \|\Lambda_n\| \|\Upsilon\| \leq 1$, for $n = 0, 1, \dots$. Thus, $JP'_n = v_n \Upsilon$, and thus $(P_n)_\nu(\tau) = p_0$, by the proximity inequality, for $n = m, m+1, \dots$, giving $\Phi_\nu(\tau) = p_0 \neq 0$. The second part of the corollary follows from the first by taking $V = [0 \ 1 \ 0 \ \dots]^T$. $\square$

The following corollary is the transcendental analogue of Corollary 4.12.

Corollary 5.5. *Let $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ be entire with $p_0 \neq 0$, and have order $\rho < 2$ or $\rho = 2$ and type $\sigma < \infty$. Let ν be real, and $\Phi_\nu(z) = e^{-\nu D^2} \Phi(z)$. For $\rho < 2$, let $0 < \mu < 1/2|\nu|$, and for $\rho = 2$, let $2\sigma < \mu < 1/(2|\nu|)$. Let $\rho_\mu(0) > 0$, where $\rho_\mu(0)$ is given as in Lemma 5.3. If membership $\tau \in T_\mu(0)$ is defined by*

$$\frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{\rho_\mu(0)}{\mu},$$

where $\tau = x + iy$, then $\Phi_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(0) \neq \emptyset$.

Proof. Since $\rho_\mu(0) > 0$, $T_\mu(0)$ is the interior of an ellipse, so that $T_\mu(0) \neq \emptyset$. If the P_n 's, for $n = 0, 1, \dots$, are the Taylor polynomials associated with Φ , then $P_n \rightarrow \Phi$ as $n \rightarrow \infty$, uniformly on compacta, with $\|JP'_0\| \leq \|JP'_1\| \leq \dots \leq \|J\Phi'\| < \infty$. Putting

$$(\rho_\mu)_n(0) = \ln \left(\sqrt{1 - 4\mu^2\nu^2} \left[1 + \frac{|p_0|^2}{\|JP'_n\|^2} \right] \right),$$

for $n = 1, 2, \dots$, it follows that $(\rho_\mu)_1(0) \geq (\rho_\mu)_2(0) \geq \dots \geq \rho_\mu(0) > 0$. By Corollary 4.11, $(P_n)_\nu(\tau) \neq 0$, for all $\tau \in (T_\mu)_n(0)$, where $\tau \in (T_\mu)_n(0)$ is defined by

$$\frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{(\rho_\mu)_n(0)}{\mu},$$

where $\tau = x + iy$, for $n = 1, 2, \dots$. It follows that $(P_n)_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(0)$, since $(T_\mu)_0(0) \supseteq (T_\mu)_1(0) \supseteq \dots \supseteq T_\mu(0) \neq \emptyset$, for $n = 1, 2, \dots$. By Lemma 5.1, $(P_n)_\nu \rightarrow \Phi_\nu$ as $n \rightarrow \infty$, uniformly on compacta, thus $\Phi_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(0)$, by Hurwitz's theorem. $\square$

The following corollary is the transcendental analogue of Corollary 4.14.

Corollary 5.6. *Let $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ be entire with $p_0 \neq 0$, and have order $\rho < 2$ or $\rho = 2$ and type $\sigma < \infty$. Let ν be real, and $\Phi_\nu(z) = e^{-\nu D^2} \Phi(z)$. For $\rho < 2$, let $0 < \mu < 1/2|\nu|$, and for $\rho = 2$, let $2\sigma < \mu < 1/(2|\nu|)$, satisfy $\rho_\mu(0) > 0$, where $\rho_\mu(0)$ is given as in Lemma 5.3. Let $V = [0 \ v_1 \ v_2 \ \dots \ v_{k-1} \ 0 \ v_{k+1} \ \dots \ v_m \ 0 \ 0 \ \dots]^T$, for some $k = 1, 2, \dots, m$, where $m = 2, 3, \dots$ is fixed, be a τ -dependant vector, with $V(\tau) = 0$ for all τ . Let vectors Υ and Λ , and value $\rho_\mu(V)$, be given as in Corollary 5.4. If membership $\tau \in T_\mu(V)$ and $\tau \in \tilde{T}_\mu(V)$ is defined by*

$$\frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{\rho_\mu(V)}{\mu} \quad \text{and} \quad \frac{x^2}{1+2\mu\nu} + \frac{y^2}{1-2\mu\nu} < \frac{\tilde{\rho}_\mu(V)}{\mu},$$

respectively, where $\tau = x + iy$ and where

$$\tilde{\rho}_\mu(V) = \rho_\mu(0) - \ln \left(1 - \frac{|p_0|^2}{\|J\Phi\|^2} |(\Lambda, \Upsilon)|^2 \right),$$

then $0 < \rho_\mu(0) \leq \tilde{\rho}_\mu(V) \leq \rho_\mu(V)$, $\emptyset \neq T_\mu(0) \subseteq \tilde{T}_\mu(V) \subseteq T_\mu(V)$, and $\Phi_\nu(\tau) \neq 0$ for all $\tau \in T_\mu(V)$.

Proof. If the P_n 's, for $n = 0, 1, \dots$, are the Taylor polynomials associated with Φ , then $P_n \rightarrow \Phi$ as $n \rightarrow \infty$, uniformly on compacta, with

$$\|JP_0\| \leq \|JP_1\| \leq \dots \leq \|J\Phi\| < \infty \quad \text{and} \quad \|JP'_0\| \leq \|JP'_1\| \leq \dots \leq \|J\Phi'\| < \infty.$$

Put

$$(\tilde{\rho}_\mu)_n(V) = (\rho_\mu)_n(0) - \ln\left(1 - \frac{|p_0|^2}{\|JP_n\|^2} |(\Lambda_n, \Upsilon)|^2\right),$$

with $(\rho_\mu)_n(0)$ defined as in the proof of Corollary 5.5, and $\Lambda_n = JP'_n / \|JP'_n\|$, for $n = 1, 2, \dots$. It follows that

$$(\tilde{\rho}_\mu)_m(V) \geq (\tilde{\rho}_\mu)_{m+1}(V) \geq \dots \geq \tilde{\rho}_\mu(V) > 0,$$

since

$$(\rho_\mu)_1(0) \geq (\rho_\mu)_2(0) \geq \dots \geq \rho_\mu(0) > 0$$

and

$$|(\Lambda_m, \Upsilon)| \geq |(\Lambda_{m+1}, \Upsilon)| \geq \dots \geq |(\Lambda, \Upsilon)|.$$

By Corollary 4.14, $(P_n)_\nu(\tau) \neq 0$, for all $\tau \in (\tilde{T}_\mu)_n(V)$, where $\tau \in (\tilde{T}_\mu)_n(V)$ is defined by

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{(\tilde{\rho}_\mu)_n(V)}{\mu},$$

where $\tau = x + iy$, for $n = 1, 2, \dots$. It follows that $(P_n)_\nu(\tau) \neq 0$, for all $\tau \in \tilde{T}_\mu(V)$, since $(\tilde{T}_\mu)_m(V) \supseteq (\tilde{T}_\mu)_{m+1}(V) \supseteq \dots \supseteq \tilde{T}_\mu(V) \supseteq T_\mu(0) \neq \emptyset$, for $n = m, m+1, \dots$. By Lemma 5.1, $(P_n)_\nu \rightarrow \Phi_\nu$ as $n \rightarrow \infty$, uniformly on compacta, and thus $\Phi_\nu(\tau) \neq 0$, for all $\tau \in \tilde{T}_\mu(V)$, by Hurwitz's theorem. Details provided above give that the values

$$(\rho_\mu)_n(V) = \ln\left(\sqrt{1 - 4\mu^2\nu^2} \left[1 + \frac{|p_0|^2}{\|JP'_n\|^2} \left(\frac{1}{1 - |(\Lambda_n, \Upsilon)|^2}\right)\right]\right),$$

for $n = 1, 2, \dots$, satisfy $(\rho_\mu)_m(V) \geq (\rho_\mu)_{m+1}(V) \geq \dots \geq \rho_\mu(V) > 0$, so that

$$(T_\mu)_m(V) \supseteq (T_\mu)_{m+1}(V) \supseteq \dots \supseteq T_\mu(V) \supseteq \tilde{T}_\mu(V) \neq \emptyset,$$

where membership $\tau \in (T_\mu)_n(V)$ is given by

$$\frac{x^2}{1 + 2\mu\nu} + \frac{y^2}{1 - 2\mu\nu} < \frac{(\rho_\mu)_n(V)}{\mu},$$

where $\tau = x + iy$, for $n = 1, 2, \dots$. Hurwitz's theorem gives $\Phi_\nu(\tau) \neq 0$, for all $\tau \in T_\mu(V)$, by an argument similar to one above. $\square$

A few remarks are in order. Putting $V = 0$ in Corollary 5.6 gives Corollary 5.5 as a special case. Replacing Φ in Corollary 5.6 by

$$\hat{\Phi}(z) = e^{\gamma D} \Phi(z) = \sum_{k=0}^{\infty} \hat{p}_k z^k,$$

where γ is complex, gives a neighborhood $\hat{T}_\mu(V)$ such that $\hat{\Phi}_\nu(\tau) \neq 0$ for all τ in $\hat{T}_\mu(V)$. In view of the relation $\Phi_\nu(z) = e^{-\gamma D} \hat{\Phi}_\nu(z)$, it follows that $\Phi_\nu(\tau) \neq 0$ for all

$\tau \in \gamma + \hat{T}_\mu(V)$. Expanding the argument in the proof of Corollary 5.6, it is seen that $(\rho_\mu)_n(V) \rightarrow \rho_\mu(V)$ as $n \rightarrow \infty$, and thus

$$T_\mu(V) = \bigcap_{m=0}^{\infty} (T_\mu)_m(V).$$

If $|\nu'| \leq |\nu|$, then $2\mu|\nu'| < 1$ and $\rho'_\mu(0) \geq \rho_\mu(0) > 0$, where

$$\rho'_\mu(0) = \ln \left(\sqrt{1 - 4\mu^2(\nu')^2} \left[1 + \frac{|p_0|^2}{\|JP'\|^2} \right] \right),$$

so that $T'_\mu(0)$ is the interior of the ellipse

$$\frac{x^2}{1 + 2\mu\nu'} + \frac{y^2}{1 - 2\mu\nu'} < \frac{\rho'_\mu(0)}{\mu},$$

and so $\Phi_{\nu'}(\tau) \neq 0$ for all $\tau \in T'_\mu(0)$.

The next lemma shows that the Weierstrass operators apply when the coefficients are bounded as in Corollary 4.15.

Lemma 5.7. *If the coefficients of the Taylor series $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ are bounded by $|p_k| \leq \delta \epsilon^k / \sqrt{k!}$, for $k = 1, 2, \dots$, where $\epsilon > 0$ and $\delta > 0$, then Φ is entire and has order $\rho \leq 2$ or $\rho = 2$ and type $\sigma \leq \epsilon^2/2$.*

Proof. The upper bounds on $|p_k|$ and the lower Stirling's approximation bound gives

$$\sqrt[k]{|p_k|} \leq \sqrt[k]{\frac{\delta \epsilon^k}{\sqrt{k!}}} < \sqrt[k]{\frac{\delta \epsilon^k}{\sqrt{\sqrt{2\pi k}(k/e)^k}}} = \epsilon \sqrt[k]{\frac{\delta}{\sqrt[4]{2\pi k}}} \sqrt{\frac{e}{k}},$$

for $k = 1, 2, \dots$, so that $\sqrt[k]{|p_k|} \rightarrow 0$ as $k \rightarrow \infty$, and so Φ is entire. The principal logarithm of this inequality gives

$$\ln \sqrt[k]{|p_k|} < \ln(\epsilon \sqrt{e}) + \frac{1}{k} \ln \delta - \frac{1}{4k} \ln(2\pi k) - \frac{1}{2} \ln k,$$

which yields $\rho \leq 2$. In case $\rho = 2$, the upper bound above gives

$$k \sqrt[k]{|p_k|^2} \leq e \epsilon^2 \sqrt[k]{\frac{\delta^2}{\sqrt{2\pi k}}}$$

which yields $\sigma \leq \epsilon^2/2$. □

The following corollary is the transcendental analogue of Corollary 4.15.

Corollary 5.8. *Let ν be real and $\epsilon > 0$, with $2\epsilon^2|\nu| < 1$. Let $\Phi(z) = \sum_{k=0}^{\infty} p_k z^k$ be entire with order $\rho < 2$ or $\rho = 2$ and type $\sigma < \infty$. Let $\Phi_\nu(z) = e^{-\nu D^2} \Phi(z)$. Let $p_0 \neq 0$, let $k!|p_k|^2 \leq \delta^2 \epsilon^{2k}$, for $k = 1, 2, \dots$, where*

$$0 < \delta < |p_0| \left(\frac{1}{\epsilon \sqrt{2|\nu|}} - 1 \right),$$

and let $\rho_\mu(0)$ be given as in Lemma 5.3. Then, $\Phi_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu'\nu} + \frac{y^2}{1 - 2\mu'\nu} < \frac{\rho_{\mu'}(0)}{\mu'},$$

for some $\epsilon^2 < \mu' < 1/(2|\nu|)$; in particular, for $\mu' = \epsilon^2/2(1 - \delta^2/|p_0|^2) + 1/(4|\nu|)$. If $|\nu'| \leq |\nu|$, then $\Phi_{\nu'}(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu'\nu'} + \frac{y^2}{1 - 2\mu'\nu'} < \frac{\rho'_{\mu'}(0)}{\mu'},$$

where μ' and $\rho'_{\mu'}(0)$ correspond to ν and ν' , respectively.

Proof. By Lemma 5.7, Φ has order $\rho < 2$ or $\rho = 2$ and type $\sigma \leq \epsilon^2/2$. It follows that $\Phi_\nu = e^{-\nu D^2} \Phi$ exists as an entire function for $\nu > 0$ in case $\rho < 2$, and for $|\nu| < 1/(4\sigma)$ in case $\rho = 2$. In either case, Φ_ν is entire, since $|\nu| < 1/(2\epsilon^2) \leq 1/(4\sigma)$. Moreover, the value $\rho_\mu(0)$ exists for all $2\sigma \leq \epsilon^2 < \mu < 1/(2|\nu|)$, by Lemma 5.3. If the P_n 's, for $n = 0, 1, \dots$, are the Taylor polynomials associated with Φ , then $P_n \rightarrow \Phi$ as $n \rightarrow \infty$, uniformly on compacta, with $\|JP'_0\| \leq \|JP'_1\| \leq \dots \leq \|J\Phi'\| < \infty$, and with $\|JP'_n\| \rightarrow \|J\Phi'\|$ as $n \rightarrow \infty$. These polynomials all satisfy the conditions of Corollary 4.15, and so $(P_n)_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu'\nu} + \frac{y^2}{1 - 2\mu'\nu} < \frac{(\rho_{\mu'})_n(0)}{\mu'},$$

for some $\epsilon^2 < \mu' < 1/(2|\nu|)$, where

$$(\rho_{\mu'})_n(0) = \ln \left(\sqrt{1 - 4(\mu')^2\nu^2} \left[1 + \frac{|p_0|^2}{\|JP'_n\|^2} \right] \right),$$

and where $\mu' = \epsilon^2/2(1 - \delta^2/|p_0|^2) + 1/(4|\nu|)$, for $n = 1, 2, \dots$. The proof of Corollary 4.15 gives the lower bound $(\rho_{\mu'})_n(0) \geq \Delta(\mu') > 0$, where

$$\Delta(\mu) = \ln \left((1 - 2\mu|\nu|) \left[1 + \frac{|p_0|^2}{\epsilon^2\delta^2}(\mu - \epsilon^2) \right] \right),$$

for $\epsilon^2 < \mu < 1/2|\nu|$ and $n = 1, 2, \dots$. Since $(\rho_{\mu'})_n(0) \rightarrow \rho_{\mu'}(0)$ as $n \rightarrow \infty$, and $(\rho_{\mu'})_1(0) \geq (\rho_{\mu'})_2(0) \geq \dots \geq \rho_{\mu'}(0)$, it follows that $\rho_{\mu'}(0) > 0$. Thus, $(P_n)_\nu(\tau) \neq 0$, for all $\tau \in T_{\mu'}(0)$, since $(T_{\mu'})_1(0) \supseteq (T_{\mu'})_2(0) \supseteq \dots \supseteq T_{\mu'}(0) \neq \emptyset$, for $n = 1, 2, \dots$. By Lemma 5.1, $(P_n)_\nu \rightarrow \Phi_\nu$ as $n \rightarrow \infty$, uniformly on compacta, and thus $\Phi_\nu(\tau) \neq 0$, for all $\tau \in T_{\mu'}(V)$, by Hurwitz's theorem, which yields the first part of the corollary. The second part follows from the remark following Corollary 5.6. $\square$

The proof of the last corollary displays the value

$$\mu' = \frac{\epsilon^2}{2(1 - \delta^2/|p_0|^2)} + \frac{1}{(4|\nu|)},$$

whereas this value is used in both $\rho_{\mu'}(0)$ and $\rho'_{\mu'}(0)$, corresponding to values ν and ν' , respectively. Replacing Φ in Corollary 5.8 by

$$\hat{\Phi}(z) = e^{(\alpha + i\beta)D} \Phi(z) = \sum_{k=0}^n \hat{p}_k z^k,$$

where α and β are real, gives $\hat{\Phi}_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the ellipse

$$\frac{x^2}{1 + 2\mu'\nu} + \frac{y^2}{1 - 2\mu'\nu} < \frac{\hat{\rho}_{\mu'}(0)}{\mu'},$$

where

$$\hat{\rho}_{\mu'}(0) = \ln \left(\sqrt{1 - 4(\mu')^2\nu^2} \left(1 + \frac{|\hat{p}_0|^2}{\|J\hat{\Phi}'\|^2} \right) \right),$$

for some $\epsilon^2 < \mu' < 1/(2|\nu|)$. Thus, in view of the relation $\Phi_\nu(z) = e^{-(\alpha+i\beta)D}\hat{\Phi}_\nu(z)$, it follows that $\Phi_\nu(\tau) \neq 0$ for $\tau = x + iy$ inside the translated ellipse

$$\frac{(x - \alpha)^2}{1 + 2\mu'\nu} + \frac{(y - \beta)^2}{1 - 2\mu'\nu} < \frac{\hat{\rho}_{\mu'}(0)}{\mu'}.$$

Acknowledgements. The author is grateful to his wife Esther and his son Joey for their encouragement while preparing the manuscript. The author is indebted to the referee for many detailed suggestions while correcting and clarifying the manuscript.

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