

REAL QUADRATIC FIELDS IN WHICH EVERY NON-MAXIMAL ORDER HAS RELATIVE IDEAL CLASS NUMBER GREATER THAN ONE

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RÉSUMÉ. H. Cohn s’est demandé si pour chaque corps quadratique réel $\mathbb{Q}(\sqrt{m})$, où m est la partie sans facteur carré du discriminant d_0 du corps quadratique $\mathbb{Q}(\sqrt{m})$, il existe un ordre non maximal correspondant à $f > 1$ et ayant la propriété que le nombre relatif de classes d’idéaux $h(d_0 f^2)/h(d_0)$ vaut un. Nous prouvons que pour $\mathbb{Q}(\sqrt{46})$ il n’y a pas de tel ordre. Nous prouvons aussi le résultat analogue correspondant au nombre relatif de classes d’idéaux au sens strict et au nombre relatif de classes de formes quadratiques.

ABSTRACT. H. Cohn asked if for every real quadratic field $\mathbb{Q}(\sqrt{m})$, with m being the squarefree part of the field discriminant d_0 of $\mathbb{Q}(\sqrt{m})$, there exists a non-maximal order corresponding to $f > 1$ such that the relative ideal class number $h(d_0 f^2)/h(d_0)$ is one. We prove that for $\mathbb{Q}(\sqrt{46})$ there is no such order. We also prove the analogous result for relative strict ideal class numbers and relative quadratic form class numbers.

1. Introduction

In his 1801 *Disquisitiones arithmeticae* [6], Gauss established the theory of integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ under an equivalence relation he called proper equivalence. (This is an equivalence under transformations of variables in $SL(2, \mathbb{Z})$.) Gauss developed his theory only for quadratic forms of even middle coefficient $b = 2b'$ and he listed quadratic forms by values of their determinant $D = (b')^2 - ac$. In this paper, it will be convenient to use the discriminant $d = 4D = b^2 - 4ac$, rather than the determinant D . A discriminant d is called a *fundamental discriminant* or a *field discriminant* if it is not divisible by another discriminant with the quotient being a perfect square. Thus an arbitrary discriminant can be written $d = d_0 f^2$ where d_0 is a field discriminant. The extension of Gauss’s theory to allow an odd middle coefficient was done later. Gauss called a quadratic form $Q(x, y)$ *properly primitive* if $\gcd(a, b, c) = 1$. He developed a theory of composition of quadratic forms that puts a group structure on the set of equivalence classes of properly primitive quadratic forms of a given discriminant d . We let $H(d)$ denote the number of such equivalence classes and call $H(d)$ the *quadratic form class number*. Gauss also divided classes of quadratic forms into genera, whose number is always a power of 2, depending on the prime factorization of D , which implies that $H(d) = H(4D)$ is divisible by

this power of 2. Gauss computed tables of the number of quadratic form classes for different D , and for definite quadratic forms (namely $D < 0$) he observed that the number of D with one class per genus appears to be bounded, with the largest value found being $D = -1848$. For indefinite quadratic forms (namely $D > 0$) he observed (in Article 304) that there are many positive non-square D which have one class per genus, and he commented that one could not doubt there are infinitely many such positive D .

In 1858, Dirichlet [4] addressed the comment of Gauss about indefinite quadratic forms. He introduced the notion of *relative quadratic form class number* $H_{d_0}(f)$, defined by

$$H_{d_0}(f) := \frac{H(d_0 f^2)}{H(d_0)},$$

and noted that $H_{d_0}(f)$ is an integer. He gave a formula for it, and proved that there exist field discriminants d_0 for which there are infinitely many f such that the relative quadratic form class number is 1. He also stated that one can furthermore find such values d_0 whose quadratic forms have one class per genus, and concluded by varying f that there do exist infinitely many positive non-square d for which the set of properly primitive quadratic forms of discriminant d have one class per genus. Thus he rigorously justified Gauss's comment. Dirichlet did not give an explicit construction of a particular d_0 . In 1962, Cohn [2, p. 130] computed some explicit cases, and in particular his results imply that for $d_0 = 5$ and $f = 5^n$, for $n \geq 1$, one has $H(5) = 1$, $H_5(5^n) = 1$ and so $H(5^{2n+1}) = 1$.

It is natural to ask for which d_0 does Dirichlet's conclusion hold: For which d_0 do there exist infinitely many f with relative quadratic form class number $H_{d_0}(f) = 1$? In 1962, Cohn [3, p. 219] raised a closely related question concerning relative ideal class numbers $h_{d_0}(f)$ of orders in the real quadratic field $\mathbb{Q}(\sqrt{d_0})$ (defined below). Cohn said that it is not known whether for each positive fundamental discriminant d_0 there always exists some $f > 1$ such that the relative ideal class number $h_{d_0}(f)$ is 1. This question was incorrectly attributed to Dirichlet in [5]. The object of this paper is to answer Cohn's question, showing that there are some integers d_0 for which this cannot be done. We shall simultaneously show that for the same d_0 the relative quadratic form class number $H_{d_0}(f)$ is greater than 1 for all $f > 1$.

We make use of the relation of classes of quadratic forms to strict invertible ideal classes in an order $\mathcal{O}_f$ of the quadratic field $\mathbb{Q}(\sqrt{d_0})$, which we review in Section 2. For the quadratic field $\mathbb{Q}(\sqrt{d_0})$ and $f \geq 1$, the order $\mathcal{O}_f = \mathbb{Z} \left[1, f \left(\frac{d_0 + \sqrt{d_0}}{2} \right) \right]$ is a subring of the maximal order $\mathcal{O}_1$ and is a Dedekind domain if and only if $f = 1$. We let $h(d_0 f^2)$ denote the size of its group of invertible ideal classes under equivalence of ideals by principal ideals, and $h_+(d_0 f^2)$ be the order of its group under strict equivalence of ideals, which uses principal ideals of totally positive elements. Here $h_+(d_0 f^2) = e h(d_0 f^2)$, where $e \in \{1, 2\}$. Parallel to relative quadratic form class numbers, we define the *relative ideal class number*

$$h_{d_0}(f) := \frac{h(d_0 f^2)}{h(d_0)}$$

and the *relative strict ideal class number*

$$h_{d_0}^+(f) := \frac{h_+(d_0 f^2)}{h_+(d_0)}.$$

Our result will answer Cohn's question for all three of these quantities.

Theorem 1.1. *For each $f > 1$, the three relative class numbers described above satisfy*

$$h_{184}(f) = \frac{h(184f^2)}{h(184)} = \frac{h_+(184f^2)}{h_+(184)} = \frac{H(184f^2)}{H(184)} > 1.$$

2. Background

In this section we give background on the three types of class numbers described in the introduction, namely $H(d)$, $h_+(d)$ and $h(d)$, as well as connections between them. All results are taken from [3].

Most well known is the case where $d = d_0$ is a field discriminant. In this case there is a one-to-one correspondence between properly primitive quadratic form classes and ideal classes of the quadratic field $\mathbb{Q}(\sqrt{d_0})$ under strict equivalence (also called narrow equivalence) of ideals in the full ring of integers $\mathcal{O}_1 = \mathbb{Z} \left[1, \frac{d_0 + \sqrt{d_0}}{2} \right]$. Here two ideals A, B are *strictly equivalent* if $(\lambda)A = (\mu)B$ for two principal ideals (λ) and (μ) with $\frac{\lambda}{\mu}$ a totally positive element (meaning that itself and its algebraic conjugates are positive) or equivalently $N(\lambda\mu) > 0$. Here $N(\cdot)$ is the norm from the (real) quadratic field $K = \mathbb{Q}(\sqrt{d_0})$ to $\mathbb{Q}$. This should be compared with ordinary (or wide) equivalence where there is no requirement of total positivity. Following Cohn [3, Chap. XII.3] we denote the number of strict classes by $h_+(d_0)$ which we call the *strict ideal class number*, and we denote the number of ordinary classes by $h(d_0)$ which we call the *ideal class number*. The correspondence between properly primitive quadratic form classes and strict ideal classes described in [3, Chap. XII.6, Theorems 6 and 7] gives

$$H(d_0) = h_+(d_0).$$

Also note that $h_+(d_0)$ equals either $h(d_0)$ or $2h(d_0)$ as stated in [3, Chapter XII.3, Theorem 3].

Less well known is the case where $d = d_0 f^2$ is a non-fundamental discriminant. The correspondence is now with invertible ideal classes with respect to the non-maximal order $\mathcal{O}_f = \mathbb{Z} \left[1, f \left(\frac{d_0 + \sqrt{d_0}}{2} \right) \right]$ of $\mathbb{Q}(\sqrt{d_0})$. This ring is not a Dedekind domain, but it does contain a group of invertible ideals, which are those ideals having norm relatively prime to f . One defines strict invertible ideal classes in this ring the same way as for the full ring of integers except we require all the ideals A, B, μ, λ to be invertible in $\mathcal{O}_f$. Similarly for ordinary equivalence. We define the *strict invertible ideal class number* $h_+(d_0 f^2)$ to be the number of such strict invertible classes and the *invertible ideal class number* $h(d_0 f^2)$ to be the number of ordinary invertible classes. These definitions are detailed in [3, Chap. XIII.2]. Results analogous

to those previously stated for fundamental discriminants are still true when we look at non-fundamental discriminants.

Theorem 2.1. *For $d = d_0 f^2$, with d_0 a field discriminant, the quadratic form class number $H(d)$ equals the strict ideal class number $h_+(d)$.*

Proof. The proof is sketched in Cohn's book [3]. The case $f = 1$ is explicitly treated in Theorems 6 and 7 in Chap. XII.6. The general case of $f > 1$ is stated in Chap. XIII.2, p. 219, but the details are omitted. $\square$

Since $H(d_0 f^2) = h_+(d_0 f^2)$ for $f \geq 1$, we see that the relative quadratic form class number $H_{d_0}(f)$ is equal to the relative strict ideal class number $h_{d_0}^+(f)$. Because of Theorem 2.4 below, we have a simple way to calculate $h_{d_0}(f)$, and so we want to relate $h_{d_0}^+(f)$ and $h_{d_0}(f)$. The following theorem will be helpful in showing that connection.

Theorem 2.2. *For a real quadratic field with field discriminant d_0 , we have*

$$h_+(d_0 f^2) = \begin{cases} h(d_0 f^2) & \text{if } N(\varepsilon_{d_0 f^2}) = -1, \\ 2h(d_0 f^2) & \text{if } N(\varepsilon_{d_0 f^2}) = 1, \end{cases}$$

where $\varepsilon_{d_0 f^2}$ is the fundamental unit in the order $\mathcal{O}_f$ of $\mathbb{Q}(\sqrt{d_0})$.

Proof. The case of $f = 1$ is given in Theorem 3 of [3, p. 198]. The details for $f > 1$ are sketched in Chap. XIII.2. $\square$

An immediate consequence is a condition for equality of the relative strict ideal class number and the relative ideal class number.

Corollary 2.3. *The relative class numbers for real quadratic fields satisfy*

$$h_{d_0}^+(f) = 2h_{d_0}(f)$$

if and only if $N(\varepsilon_{d_0}) = -1$ and $N(\varepsilon_{d_0 f^2}) = 1$. In all other cases $h_{d_0}^+(f) = h_{d_0}(f)$.

Proof. Certainly we will have equality if $N(\varepsilon_{d_0}) = N(\varepsilon_{d_0 f^2})$. Since $\varepsilon_{d_0 f^2}$ is also a unit in $\mathcal{O}_1$, we know that $\varepsilon_{d_0 f^2} = \varepsilon_{d_0}^k$ for some k that depends on d_0 and f . This means that the case $N(\varepsilon_{d_0}) = 1$ and $N(\varepsilon_{d_0 f^2}) = -1$ never occurs. The only other possibility is $N(\varepsilon_{d_0}) = -1$ and $N(\varepsilon_{d_0 f^2}) = 1$ (i.e., the k above is even). This case gives $h_{d_0}^+(f) = 2h_{d_0}(f)$ by Theorem 2.2. $\square$

If d_0 is a field discriminant, we let m denote its squarefree part, so that

$$m = \begin{cases} d_0 & \text{if } m \equiv 1 \pmod{4}, \\ \frac{d_0}{4} & \text{if } m \equiv 2, 3 \pmod{4}. \end{cases}$$

We will use the following result from [2], [3, p. 217], which gives a formula for the relative class number $h_{d_0}(f)$.

Theorem 2.4. Let d_0 be a field discriminant, with m its squarefree part, so that $K = \mathbb{Q}(\sqrt{m})$. Let ε_m be the fundamental unit of K written as

$$\varepsilon_m = \frac{x + y\sqrt{m}}{c} \text{ where } c = \begin{cases} 2 & \text{if } m \equiv 1 \pmod{4}, \\ 1 & \text{if } m \equiv 2, 3 \pmod{4}. \end{cases}$$

Define

$$\psi(f) = f \prod_{q|f} \left(1 - \left(\frac{d_0}{q} \right) \frac{1}{q} \right),$$

where $\left(\frac{d_0}{q} \right)$ is the Legendre symbol and q is prime. Define $\phi(f)$ to be the smallest positive integer such that $(\varepsilon_m)^{\phi(f)}$ is in a non-maximal order $\mathcal{O}_f$, (namely that $\phi(f)$ is the smallest integer such that $(\varepsilon_m)^{\phi(f)} = \frac{a + b\sqrt{m}}{c}$, where $b \equiv 0 \pmod{f}$). Then

$$h_{d_0}(f) = \frac{\psi(f)}{\phi(f)}.$$

Please note that because we write ε_m in the form where we divide by c , the “ y -coordinate” of the fundamental unit will always be an integer, and so it makes sense to ask if f divides y .

3. Results

A consequence of Theorem 2.4 is the following, which gives a criterion for when the relative ideal class number will be equal to one.

Theorem 3.1. Let $K = \mathbb{Q}(\sqrt{m})$ be a real quadratic field, with m being the square-free part of the field discriminant d_0 of K . If the fundamental unit $\varepsilon_m = \frac{x+y\sqrt{m}}{c}$ (using the notation of Theorem 2.4) has the property that m does not divide y , then there is some prime $f > 1$ such that $h_{d_0}(f) = 1$.

Proof. If m does not divide y , then there is a prime f that divides m but not y (because m is squarefree). Choose such an f . On the one hand, since $f|m$ and $m|d_0$, we know that the Legendre symbol is 0, and hence $\psi(f) = f$. On the other hand, since $h_d(f)$ is an integer, $\phi(f) = 1$ or $\phi(f) = f$. And since we chose f not dividing y , we know that $(\varepsilon_m)^1 \notin \mathcal{O}_f$. So $\phi(f) = f$ and $h_{d_0}(f) = 1$. $\square$

In the case of $m = 46$ it is impossible to find such an f since the fundamental unit is $\varepsilon_{46} = 24335 + 3588\sqrt{46}$ and $46|3588$. This is the first fundamental unit with the property that m divides y , and so it is a natural candidate to consider more carefully.

We will now prove Theorem 1.1.

Proof of Theorem 1.1. For all positive fundamental discriminants d_0 with square-free part $1 < m < 46$ there is some relative ideal class number $h_{d_0}(f) = 1$ by the criterion of Theorem 3.1. We now treat the case $m = 46$.

Since $m \equiv 2 \pmod{4}$, we have $d_0 = 4m = 184$ and $c = 1$. We will start by considering primes f which do not divide 46. Because f is prime,

$$\psi(f) = f - \left(\frac{184}{f}\right) = f - \left(\frac{46}{f}\right)$$

is even.

We also know that the fundamental unit is $\varepsilon_{46} = 24335 + 3588\sqrt{46}$. This is the smallest solution to the Pell equation $x^2 - 46y^2 = 1$. If we write $\varepsilon_{46} = a_1 + b_1\sqrt{46}$, then other solutions are of the form $(a_n + b_n\sqrt{46}) = (a_1 + b_1\sqrt{46})^n$. These obey the recurrence relations

$$(3.1) \quad \begin{cases} a_{n+1} &= a_1 a_n + 46 b_1 b_n, \\ b_{n+1} &= a_1 b_n + b_1 a_n. \end{cases}$$

We can write this in matrix notation as

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}$$

or

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^n \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}$$

or

$$(3.2) \quad \begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{n-k} \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

It will be helpful to recognize that all powers of this matrix have determinant 1 (since $a_1^2 - 46b_1^2 = 1$) and have a nice “almost diagonal” form

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^t = \begin{pmatrix} A & 46B \\ B & A \end{pmatrix}$$

for some integers A, B .

We start by examining $(\varepsilon_{46})^{\psi(f)}$. We know that $\phi(f)$ (the minimum exponent such that $(\varepsilon_{46})^{\psi(f)} \in \mathcal{O}_f$) must divide $\psi(f)$ because $h_d(f) \in \mathbb{Z}$. Therefore, $(\varepsilon_{46})^{\psi(f)} \in \mathcal{O}_f$. In other words, for

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)-1} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_{\psi(f)} \\ b_{\psi(f)} \end{pmatrix},$$

we know that $f|b_{\psi(f)}$. Then our goal will be to show that $f|b_n$ for some $n < \psi(f)$.

Let us have a look modulo f . Menezes, in [7, p. 59], discusses this in a slightly different way. He considers C to be the set of all solutions $(x, y) \in F_q \times F_q$ to the Pell equation $x^2 - Dy^2 = 1$, where q is a power of an odd prime. He defines a group structure by

$$(3.3) \quad (x_1, y_1) \oplus (x_2, y_2) = (x_1 x_2 + D y_1 y_2, x_1 y_2 + x_2 y_1).$$

Then his Lemma 4.4 states that $(C, \oplus)$ is an abelian group with identity $(1, 0)$ and his Theorem 4.5 states that $(C, \oplus)$ is a cyclic group of order $q - \left(\frac{D}{q}\right)$, (which is our $\psi(f)$ when $D = 46$ and $q = f$).

Notice that the group structure given by (3.3) is exactly the structure given by relations (3.1) and the subsequent matrix multiplications. Therefore,

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{f}.$$

It turns out that $\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)/2} \pmod{f}$ is diagonal. To see this, we write

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)/2} = \begin{pmatrix} A & 46B \\ B & A \end{pmatrix}.$$

Then

$$\begin{aligned} \begin{pmatrix} A & 46B \\ B & A \end{pmatrix} \begin{pmatrix} A & 46B \\ B & A \end{pmatrix} &= \begin{pmatrix} A^2 + 46B^2 & 2 \cdot 46AB \\ 2AB & A^2 + 46B^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{f}. \end{aligned}$$

Since f does not divide 46, f divides either A or B . If $f|A$, then $46B^2 \equiv 1 \pmod{f}$. But at the same time, this matrix should have determinant $A^2 - 46B^2 = 1$, which means $-46B^2 \equiv 1 \pmod{f}$, which gives a contradiction. Therefore $f|B$ and we see that

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)/2} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \pmod{f}.$$

Setting $n = \psi(f)$ and $k = \psi(f)/2$ in equation (3.2) above, we obtain the equation

$$\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)/2} \begin{pmatrix} a_{\psi(f)/2} \\ b_{\psi(f)/2} \end{pmatrix} = \begin{pmatrix} a_{\psi(f)} \\ b_{\psi(f)} \end{pmatrix}.$$

Multiplying on the left by $\begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)/2}$ gives the following equations:

$$\begin{cases} \begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)} \begin{pmatrix} a_{\psi(f)/2} \\ b_{\psi(f)/2} \end{pmatrix} = \begin{pmatrix} a_1 & 46b_1 \\ b_1 & a_1 \end{pmatrix}^{\psi(f)/2} \begin{pmatrix} a_{\psi(f)} \\ b_{\psi(f)} \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{\psi(f)/2} \\ b_{\psi(f)/2} \end{pmatrix} = \begin{pmatrix} a_{\psi(f)/2} \\ b_{\psi(f)/2} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} a_{\psi(f)} \\ b_{\psi(f)} \end{pmatrix} \pmod{f}. \end{cases}$$

Therefore $b_{\psi(f)/2} \equiv Ab_{\psi(f)} \pmod{f}$, and since we already know that $f|b_{\psi(f)}$ we have $(\varepsilon_{46})^{\psi(f)/2} \in \mathcal{O}_f$. Therefore the relative ideal class number satisfies $h_{d_0}(f) \geq 2$, answering the question of Cohn.

A computation shows that when $m = 46$ we have $h_{d_0}(23) = 23$ and $h_{d_0}(2) = 2$, because in both these cases, $\varepsilon_{46} \in \mathcal{O}_f$. Therefore, for all primes f , we have $h_{d_0}(f) > 1$. And since $f|g$ implies $h_{d_0}(f)|h_{d_0}(g)$, this proves that $h_{d_0}(f) > 1$ for all $f > 1$.

To show the connection with the other relative class numbers we only need to notice that $N(\varepsilon_{184}) = 1$. Then by the analysis in Corollary 2.3, we see that for all $f > 1$ we

have $N(\varepsilon_{184f^2}) = 1$ and so

$$h_{d_0}^+(f) = \frac{h_+(d_0f^2)}{h_+(d_0)} = \frac{2h(d_0f^2)}{2h(d_0)} = h_{d_0}(f),$$

as desired. Since $H_{d_0}(f) = h_{d_0}^+(f)$, by Theorem 2.1 we see that all three relative class numbers, $H_{d_0}(f)$, $h_{d_0}^+(f)$, and $h_{d_0}(f)$, are all equal and are all greater than 1 for all $f > 1$. $\square$

One might ask how many other quadratic fields $\mathbb{Q}(\sqrt{m})$ satisfy the property that every non-maximal order gives a relative ideal class number greater than 1. By Theorem 3.1, it suffices to only consider cases where m divides the y coordinate of ε_m . Quadratic fields with this property were studied in [9] while researching powerful numbers. They tested all $\mathbb{Q}(\sqrt{m})$ with $m < 10^7$ and found only 8 fields such that m divides y . They are $m = 46, 430, 1817, 58254, 209991, 1752299, 3124318$ and 4099215 . Hence for all other quadratic fields $\mathbb{Q}(\sqrt{m})$ with m squarefree and less than 10^7 one can find a prime f that divides m but not y , and so by Theorem 3.1 we have $h_{d_0}(f) = 1$. While our proof of Theorem 1.1 is specific to $m = 46$, the arguments can be adapted to show that the above seven other quadratic fields where m divides the y coordinate of ε_m will never have a (non-maximal) relative ideal class number equal to 1.

There appear to be few general results for when m divides (or does not divide) y . Ankeny, Artin and Chowla in [1] asked if p is a prime with $p \equiv 1 \pmod{4}$ and if $\frac{x + y\sqrt{p}}{2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{p})$, then does $p \nmid y$? Mordell in [8] conjectured the same is true for $p \equiv 3 \pmod{4}$. Neither of these have been proven, though both have been checked for large primes, see [9] and [10]. It seems to be an open problem whether there exist infinitely many squarefree m such that the fundamental unit

$$\varepsilon_m = \frac{x + y\sqrt{m}}{c}$$

of $\mathbb{Q}(\sqrt{m})$ given as in Theorem 2.4 satisfies $m \mid y$.

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