

ACYCLIC CLUSTER ALGEBRAS OF FINITE TYPE

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RÉSUMÉ. L'objectif de cet article est de donner une preuve courte et élémentaire de la classification des algèbres amassées de type fini, due à Fomin et Zelevinsky, dans le cas particulier d'algèbres amassées acycliques.

ABSTRACT. The objective of this paper is to give a short and elementary proof of the finite type classification of cluster algebras, due to Fomin and Zelevinsky, in the particular case of acyclic cluster algebras.

Cluster algebras were first introduced in the early 2000 by Fomin and Zelevinsky [FZ02] with the purpose of building a combinatorial framework for studying total positivity in algebraic groups and canonical bases in quantum groups. Since then, the study of cluster algebras was shown to be connected to several areas of mathematics, notably combinatorics, Lie theory, Poisson geometry, Teichmüller theory, mathematical physics and representation theory of algebras.

A cluster algebra is a commutative algebra generated by overlapping sets of variables. The sets are called clusters and the variables are called cluster variables. One starts with the data consisting of a finite quiver Q without cycles of length one or two, having n points, and a so-called initial cluster of n variables associated to the vertices of Q . Then one generates new quivers and clusters recursively using a combinatorial process known as mutation. The cluster algebra $\mathcal{A}_Q$ is then generated by all cluster variables obtained in this way. Consequently, the cluster algebra $\mathcal{A}_Q$ depends not on the quiver Q itself but rather on its mutation class, that is, the class of all quivers which can be reached from Q by iterated mutations. A cluster algebra $\mathcal{A}_Q$ is said to be acyclic if the mutation class of Q contains a finite, connected and acyclic quiver. We are interested in acyclic cluster algebras which are of *finite type*, that is, which have only finitely many cluster variables, see [FZ03]. We prove the following theorem.

Theorem. *An acyclic cluster algebra $\mathcal{A}_Q$ is of finite type if and only if the mutation class of Q contains a quiver whose underlying graph is a Dynkin diagram.*

This is a particular case of [FZ03], where the authors prove this equivalence without the acyclicity hypothesis. The main tool in our proof is the geometric interpretation of cluster algebras of type $\mathbb{A}$ and $\mathbb{D}$, see [FST08] (see also [CCS06] and [Sch08]).

The paper is organized as follows. After a short introductory section, devoted to fixing notation and recalling the necessary notions, we study in the second section the cluster algebras of type $\mathbb{A}_n$ as arising from triangulations of the $(n+3)$ -gon. In the third section, we study likewise the cluster algebras of type $\mathbb{D}_n$ as arising from triangulations

of the once punctured n -gon. In both cases, we give an explicit formula associating a cluster variable to an arc of the triangulated polygon. Finally, in the fourth section, we prove our main result in Theorem 4.8.

1. Preliminaries

1.1. Quivers

A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of two sets Q_0 and Q_1 , and two maps $s, t : Q_1 \rightarrow Q_0$. The elements of Q_0 are called *points* and, those of Q_1 , *arrows*. For $\alpha \in Q_1$, $s(\alpha)$ is called its *source*, and $t(\alpha)$ its *target*. To a point $i \in Q_0$, we associate the sets of arrows

$$i^+ = \{\alpha \in Q_1 \mid s(\alpha) = i\} \quad \text{and} \quad i^- = \{\alpha \in Q_1 \mid t(\alpha) = i\}.$$

A quiver $Q' = (Q'_0, Q'_1, s', t')$ is said to be a *subquiver* of Q as defined above if $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and, if $\alpha \in Q'_1$ then $s(\alpha), t(\alpha) \in Q'_0$, $s'(\alpha) = s(\alpha)$ and $t'(\alpha) = t(\alpha)$. A subquiver Q' of Q is said to be *full* if, for all $x, y \in Q'_0$ and all $\alpha \in Q_1$ such that $s(\alpha) = x$ and $t(\alpha) = y$, we have $\alpha \in Q'_1$.

Let Q be a quiver without loops nor 2-cycles (that is, oriented cycles of length 2), and $k \in Q_0$. The *mutation* in k transforms Q into a new quiver $\mu_k Q$ having the same points as Q and:

- (a) For each path $i \leftarrow k \leftarrow j$ in Q having k as midpoint, we insert a new arrow $i \leftarrow j$ in $\mu_k Q$;
- (b) All arrows having k as source or target are inverted;
- (c) All 2-cycles created in the two previous steps are deleted.

Clearly, if k is a source or a sink, i.e., if $k^- = \emptyset$ or $k^+ = \emptyset$, then $\mu_k Q$ is obtained from Q simply by inverting all arrows incident to k . We observe that quiver mutation is involutive.

Two quivers Q and Q' are called *mutation equivalent* if there exists a sequence of mutations $\mu_1, \dots, \mu_t$ such that $\mu_t \cdots \mu_1 Q = Q'$. The following property of quivers whose underlying graphs are trees is well known (see, for instance, [ASS06, (VII.5.2)]). We give the proof for convenience of the reader.

Lemma 1.1. *Let Q and Q' be two quivers having the same tree as underlying graph. Then there exists a sequence $i_1, \dots, i_t \in Q_0$ such that $\mu_{i_t} \cdots \mu_{i_1}(Q) = Q'$ (thus, Q and Q' are mutation-equivalent).*

Proof. It suffices to prove the statement in the case where Q and Q' differ only in the orientation of only one arrow. Let $\alpha : i \leftarrow j$ be the only arrow in Q which has the opposite orientation in Q' . Let Q'' be the quiver such that $Q''_0 = Q_0$ and $Q''_1 = Q_1 \setminus \{\alpha\}$. Because Q is a tree, i and j belong to two different connected components of Q'' , which we denote by Q^i and Q^j respectively. Obviously, Q^j is a tree. Therefore, it is possible to number the points of Q^j as $\{1, 2, \dots, t\}$, so that if there is an arrow $k \leftarrow l$, then $k > l$. Then $\mu_t \cdots \mu_1(Q) = Q'$. $\square$

1.2. Cluster algebras

Let Q be a quiver without loops nor 2-cycles. Let $\{1, 2, \dots, n\}$ denote the points of Q and consider a set of n variables $\underline{x} = \{x_1, \dots, x_n\}$, where we agree that the point i corresponds to the variable x_i . We define the mutation process on the pair $(Q, \underline{x})$. Let $k \in Q_0$. Then $\mu_k(Q, \underline{x}) = (Q', \underline{x}')$ is defined as follows. First, $Q' = \mu_k Q$. Next, $\underline{x}' = (\underline{x} \setminus \{x_k\}) \cup \{x'_k\}$, where $x'_k \in \mathbb{Q}(x_1, \dots, x_n)$ is defined by the so-called *exchange relation*

$$x_k x'_k = \prod_{\alpha \in k^+} x_{t(\alpha)} + \prod_{\alpha \in k^-} x_{s(\alpha)},$$

where an empty product is defined to be 1. Let $\mathcal{X}$ be the union of all possible sets of variables obtained from $\underline{x}$ by successive mutations. The *cluster algebra* $\mathcal{A}_Q$ is the $\mathbb{Z}$ -subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by $\mathcal{X}$, see [FZ02]. Each pair $(\tilde{Q}, \tilde{\underline{x}})$ obtained from $(Q, \underline{x})$ by successive mutations is a *seed*, and $\tilde{\underline{x}}$ is a *cluster*, whose elements are *cluster variables*. The pair $(Q, \underline{x})$ is the *initial seed*, and $\underline{x}$ is the *initial cluster*.

We say that $\mathcal{A}_Q$ is of *acyclic type* if the mutation class of Q contains an acyclic quiver (that is a quiver without oriented cycles), see [FZ03].

1.3. Cluster algebras from surfaces

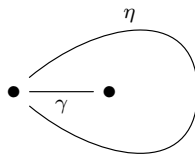
Let S be an oriented surface with or without boundary, and $M \subseteq S$ be a finite set containing at least one point from each connected component of the boundary. The points of M are called *marked points*, and those which are in the interior of S are called *punctures*. The pair (S, M) is called simply a *surface*. An *arc* γ in (S, M) is the isotopy class of a curve such that:

- (a) The endpoints of the curve are in M ;
- (b) The curve does not cross itself, except that its endpoints may coincide;
- (c) The curve is disjoint from M and the boundary of S except for its endpoints;
- (d) The curve does not cut out an unpunctured monogon $\bullet \bigcirc$.

An arc is *internal* if it is not (isotopic to) an arc lying on the boundary of S . Two arcs are *compatible* if they do not intersect, except perhaps at their endpoints. A *triangulation* is a maximal collection of pairwise compatible arcs.

A *flip* in a triangulation T replaces one particular arc γ by the unique arc $\gamma' \neq \gamma$ which, together with the remaining arcs of T , forms a new triangulation. The resulting triangulation is denoted by $\phi_\gamma(T)$. Because flips are clearly involutive, we call two triangulations *flip-equivalent* if they are connected by a sequence of flips.

Now, whenever (S, M) has punctures, a triangulation may contain a self-folded triangle. The arc inside such a triangle cannot be flipped, hence the more general notion of tagged arcs. For example, γ cannot be flipped in the following self-folded triangle formed by η cutting out a once-punctured monogon.



A *tagged arc* is obtained by taking an arc which does not cut out a once-punctured monogon and "tagging" each of its ends as *plain* or *notched*, so that:

- (a) each end connected to a point on the boundary is plain,
- (b) if both ends connect to the same point, then they have the same tagging.

Two tagged arcs α and β are *compatible* if the arcs α^0 and β^0 obtained by forgetting the taggings are compatible and:

- (a) if $\alpha^0 = \beta^0$, then at least one end of α must be tagged in the same way as the corresponding end of β ,
- (b) if $\alpha^0 \neq \beta^0$, but α^0 and β^0 share an endpoint a , then the taggings at a must be the same.

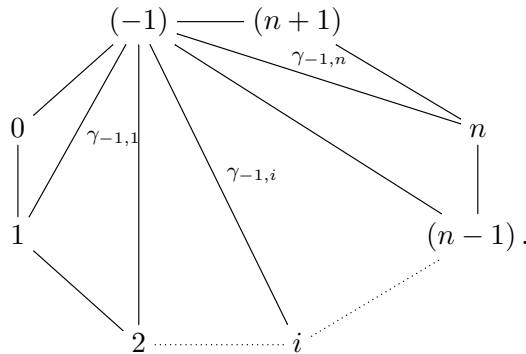
A *tagged triangulation* is a maximal collection of pairwise compatible tagged arcs. To a triangulation T , we associate a quiver Q_T as follows:

- (a) The points of Q_T are the internal arcs of T ;
- (b) If γ_1, γ_2 are two sides of a triangle in T such that γ_2 follows γ_1 with respect to the orientation of S around their common vertex, then there is an arrow $\gamma_1 \leftarrow \gamma_2$;
- (c) If two arcs, stripped of all taggings, are the same, then there is no arrow between them and they share predecessors and successors.

The cluster algebra $\mathcal{A}_{Q_T}$ given by the quiver Q_T is called the *cluster algebra associated to the surface* (S, M) , see [FST08]. Indeed, one can show that $\mathcal{A}_{Q_T}$ does not depend on the particular triangulation T we started from, see [FST08]. We shall not need this fact in all generality, but we rather need it in two particular cases, which we examine in the following two sections.

2. Cluster algebras of type $\mathbb{A}$

The objective of this section is to study the cluster algebras associated to triangulations of the regular $(n+3)$ -gon (with $n \geq 1$) having its vertices as marked points. We assume that the vertices are numbered as $-1, 0, 1, \dots, n+1$ and we denote by γ_{ij} the arc going from the vertex i to the vertex j , with $i < j$. Note that internal arcs in this case are just diagonals. One triangulation of particular interest to us is the so-called *fan triangulation*, which is as in the figure



In this case, the associated quiver as defined in Subsection 1.3 above is the linearly oriented quiver $\vec{\mathbb{A}}_n$

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow (n-1) \longrightarrow n.$$

For this reason, the fan triangulation will be denoted by $\vec{T}_{\mathbb{A}_n}$.

Lemma 2.1. *Let T be any triangulation of the $(n+3)$ -gon. Then there exists a sequence of flips $\phi_1, \dots, \phi_t$ such that $\phi_t \cdots \phi_1(T) = \vec{T}_{\mathbb{A}_n}$.*

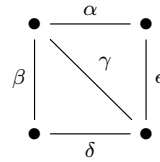
Proof. We proceed by induction. The result holds for $n = 1$, because there are exactly two triangulations obtained from each other by flipping the unique diagonal. Suppose that the statement holds true for all triangulations of a polygon with less than $n+3$ vertices. Let T be an arbitrary triangulation of the $(n+3)$ -gon. Consider the vertex -1 . If there exists an internal arc $\gamma_{-1,i}$ of the triangulation T incident to -1 , then the two triangulated polygons obtained by cutting along that arc are flip-equivalent to those where all arcs are incident to -1 , because of the induction hypothesis.

If T contains no arc of the form $\gamma_{-1,i}$, then the arc $\gamma_{0,n+1}$ belongs to the triangulation. Consequently we may flip on $\gamma_{0,n+1}$ and get an arc of the form $\gamma_{-1,i}$, returning to the former case and completing the proof. $\square$

It follows from Lemma 2.1 that any two triangulations T and T' of the $(n+3)$ -gon are flip-equivalent. We now show that flips correspond to mutations. Given an internal arc γ of a triangulation T , we denote by ϕ_γ the corresponding flip and by μ_γ the corresponding mutation of Q_T (recall from Subsection 1.3 that γ is a point of Q_T).

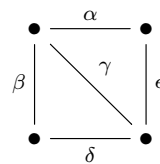
Lemma 2.2. (a) *Let T be any triangulation of the $(n+3)$ -gon and γ be an internal arc of T . Then $\mu_\gamma(Q_T) = Q_{\phi_\gamma(T)}$.*

(b) *The exchange relation arising from flipping γ in the quadrilateral*

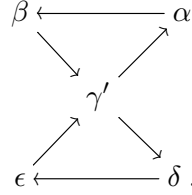


is the Ptolemy relation $x_\gamma x_{\gamma'} = x_\alpha x_\delta + x_\beta x_\epsilon$ (if some of the sides are boundary arcs, the corresponding variables are replaced by 1).

Proof. (a) The internal arc γ is the diagonal of some quadrilateral



where we may, for simplicity, assume that the four sides $\alpha, \beta, \delta, \epsilon$ are internal arcs. Then it is easily seen that $Q_{\phi_\gamma(T)}$ and $\mu_\gamma(Q_T)$ are both equal to the quiver



(b) It is a simple verification that the Ptolemy relation connecting the variables x_γ and $x_{\gamma'}$ is exactly the same as the exchange relation in Subsection 1.2. $\square$

As a direct consequence, given any two triangulations T and T' of the $(n+3)$ -gon, the quivers Q_T and $Q_{T'}$ are mutation-equivalent. This shows in the case of the $(n+3)$ -gon that the cluster algebra $\mathcal{A}_{Q_T}$ does not depend on the particular triangulation we started from. We may thus assume that $T = \vec{T}_{\vec{\mathbb{A}}_n}$, and then $Q_T = \vec{\mathbb{A}}_n$. Let the variables in the initial cluster $\underline{x} = \{x_1, \dots, x_n\}$ be in bijection with the points of $\vec{\mathbb{A}}_n$, thus with the internal arcs $\{\gamma_{-1,1}, \dots, \gamma_{-1,n}\}$ of the triangulation $\vec{T}_{\vec{\mathbb{A}}_n}$. We now want to demonstrate that the algebra associated with a quiver of type $\mathbb{A}_n$ is of finite type.

To do so, we give an explicit formula for the cluster variables in $\mathcal{A}_{\vec{\mathbb{A}}_n}$. For this purpose, we define a function x from the set of arcs of the $(n+3)$ -gon to the ring of Laurent polynomials in the initial cluster variables $x_1, \dots, x_n$. Let, as before, γ_{ij} stand for the arc from the vertex i to the vertex j (where $i < j$). Then we set

$$x_{ij} = x(\gamma_{ij}) = x_i x_j \sum_{k=i}^{j-1} \frac{1}{x_k x_{k+1}}$$

with the convention that once the expression on the right-hand side is reduced to a common denominator and simplified, then the variables x_0 and x_{n+1} are replaced by 1 and x_{-1} by 0.

The reader may wish to compare our result to the general one given in [MSW11]. Our formula is simpler due to the fact that it applies to specific surfaces.

We first show that the function x applied to one of the initial arcs gives the corresponding initial cluster variable and x applied to a boundary arc gives 1.

Lemma 2.3. *Let i be a vertex of the $(n+3)$ -gon. Then,*

- (a) $x_{-1,i} = x_i$,
- (b) $x_{i,i+1} = 1$.

Proof. (a) We have

$$x_{-1,i} = x_{-1} x_i \sum_{k=-1}^{i-1} \frac{1}{x_k x_{k+1}} = \frac{x_{-1} x_i}{x_{-1} x_0} = x_i,$$

because, in the sum, all terms not having x_{-1} as factor of the denominator will cancel when x_{-1} is replaced by 0, and the only remaining denominator is $x_{-1} x_0$.

(b) This is trivial. $\square$

We recall the classical Ptolemy theorem from Euclidean geometry: a quadrilateral is inscribable in a circle if and only if the product of the diagonals equals to the sum of the products of the opposite sides. The following property of the function x derives its name from this theorem.

Lemma 2.4. *The function x respects the Ptolemy relation; that is, if*

$$\begin{array}{ccc} i & \text{---} & l \\ | & & | \\ j & \text{---} & k \end{array}$$

is a quadrilateral in the $(n+3)$ -gon, then

$$x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}.$$

Proof. Without loss of generality, we may assume $-1 \leq i < j < k < l \leq n+1$. Then,

$$\begin{aligned} x_{ij}x_{kl} + x_{il}x_{jk} &= \left(\sum_{a=i}^{j-1} \frac{x_i x_j}{x_a x_{a+1}} \right) \left(\sum_{b=k}^{l-1} \frac{x_k x_l}{x_b x_{b+1}} \right) + \left(\sum_{c=i}^{l-1} \frac{x_i x_l}{x_c x_{c+1}} \right) \left(\sum_{d=j}^{k-1} \frac{x_j x_k}{x_d x_{d+1}} \right) \\ &= x_i x_j x_k x_l \left(\sum_{a=i}^{j-1} \sum_{b=k}^{l-1} \frac{1}{x_a x_{a+1} x_b x_{b+1}} + \sum_{c=i}^{l-1} \sum_{d=j}^{k-1} \frac{1}{x_c x_{c+1} x_d x_{d+1}} \right) \\ &= x_i x_j x_k x_l \left(\sum_{a=i}^{j-1} \sum_{b=k}^{l-1} \frac{1}{x_a x_{a+1} x_b x_{b+1}} + \sum_{c=i}^{j-1} \sum_{d=j}^{k-1} \frac{1}{x_c x_{c+1} x_d x_{d+1}} \right. \\ &\quad \left. + \sum_{c=j}^{k-1} \sum_{d=j}^{k-1} \frac{1}{x_c x_{c+1} x_d x_{d+1}} + \sum_{c=k}^{l-1} \sum_{d=j}^{k-1} \frac{1}{x_c x_{c+1} x_d x_{d+1}} \right) \\ &= x_i x_j x_k x_l \left(\sum_{a=i}^{j-1} \sum_{b=j}^{l-1} \frac{1}{x_a x_{a+1} x_b x_{b+1}} + \sum_{d=j}^{k-1} \sum_{c=j}^{l-1} \frac{1}{x_d x_{d+1} x_c x_{c+1}} \right) \\ &= \left(\sum_{a=j}^{l-1} \frac{x_j x_l}{x_a x_{a+1}} \right) \left(\sum_{b=i}^{k-1} \frac{x_i x_k}{x_b x_{b+1}} \right) \\ &= x_{ik}x_{jl}. \end{aligned}$$

$\square$

Corollary 2.5. *The cluster algebra $\mathcal{A}_{\mathbb{A}_n}$ is of finite type and has $\frac{n(n+3)}{2}$ cluster variables.*

Proof. By Lemma 2.3, the function x associates 1 to boundary arcs, and an initial cluster variable to every initial arc of $\vec{T}_{\mathbb{A}_n}$. By Lemma 2.4, the cluster variable resulting from a flip is also given by x , and therefore that variable does not depend on the quadrilateral used for the flip.

It follows that the function x gives all cluster variables of $\mathcal{A}_{\mathbb{A}_n}$. Thus, $\mathcal{A}_{\mathbb{A}_n}$ has as many cluster variables as there are internal arcs (diagonals) in the $(n+3)$ -gon, that is, $\frac{n(n+3)}{2}$. $\square$

Corollary 2.6. *Let Q be a quiver which is mutation-equivalent to a quiver whose underlying graph is the Dynkin diagram $\mathbb{A}_n$. Then $\mathcal{A}_Q$ is of finite type and has $\frac{n(n+3)}{2}$ cluster variables.*

Proof. This follows from Corollary 2.5 and Lemma 1.1. $\square$

Proposition 2.7. *The cluster variables of $\mathcal{A}_{\mathbb{A}_n}$ are*

$$x_{ij} = x_i x_j \sum_{k=i}^{j-1} \frac{1}{x_k x_{k+1}},$$

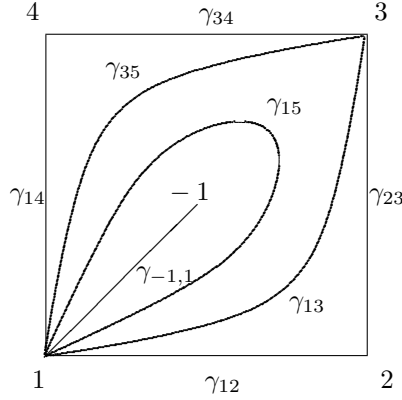
where $-1 \leq i < j \leq n+1$ and $j \neq i+1$.

Proof. This follows from Lemmata 2.2, 2.3 and 2.4. $\square$

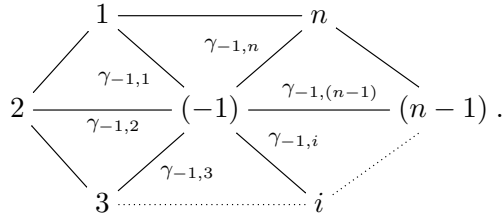
3. Cluster algebras of type $\mathbb{D}$

We now study cluster algebras associated to triangulations of the (once) punctured regular n -gon (with $n \geq 4$), having its vertices and the puncture as marked points. We assume the vertices numbered $1, \dots, n$ and arranged in the counter-clockwise order, and the puncture is labelled with -1 . We use the following notation for arcs, in which $i < j$. Boundary arcs from i to j are always denoted by γ_{ij} . Internal arcs going from the puncture to vertex j are denoted by $\gamma_{-1,j}$. An internal arc from i to j is denoted by γ_{ij} if it goes from i to j in the counter-clockwise direction around the puncture -1 , and by $\gamma_{j,i+n}$ if it goes from i to j in the clockwise direction around the puncture -1 . Loops around the puncture are denoted by $\gamma_{i,i+n}$.

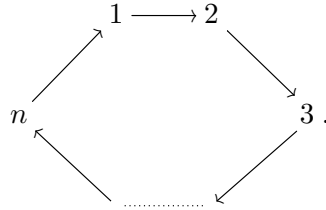
Example 3.1. The following example of a triangulation of the punctured square illustrates the notation introduced above.



We shall be particularly interested in the following triangulation of the punctured n -gon, which we call again the *fan triangulation*



The quiver associated to this triangulation as defined in Subsection 1.3 is the oriented cycle $\vec{\mathbb{D}}_n$



The notation $\vec{\mathbb{D}}_n$ is justified because the quiver $\mu_{n-2} \cdots \mu_2 \mu_1(\vec{\mathbb{D}}_n)$ has $\mathbb{D}_n$ as underlying graph. The fan triangulation will be denoted by $\vec{T}_{\vec{\mathbb{D}}_n}$.

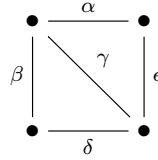
Lemma 3.2. *Let T be any tagged triangulation of the punctured n -gon. Then there exists a sequence of flips $\phi_1, \dots, \phi_t$ such that $\phi_t \cdots \phi_1(T) = \vec{T}_{\vec{\mathbb{D}}_n}$.*

Proof. We first observe that, given a tagged triangulation of the punctured n -gon T , one can flip on every arc tagged notched to get another triangulation where all the arcs are tagged plain. Then as there are no self-folded triangles in a tagged triangulation, there exist at least two plain-tagged internal arcs incident to the puncture, which we denote by $\gamma_{-1,i}$ and $\gamma_{-1,j}$, with $i \neq j$. This indeed follows from the maximality of T . But then the path $i \xrightarrow{\gamma_{(-1,i)}} (-1) \xrightarrow{\gamma_{(-1,j)}} j$ cuts the polygon into two unpunctured triangulated polygons. By Lemma 2.1, these polygons are flip-equivalent to unpunctured polygons with all arcs incident to the vertex -1 . $\square$

This lemma shows that any two triangulations of the punctured n -gon are flip-equivalent. As in Section 2, given an internal (tagged) arc γ of the triangulation T , we denote by ϕ_γ the corresponding flip and by μ_γ the corresponding mutation of Q_T .

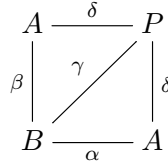
Lemma 3.3. (a) *Let T be any triangulation of the punctured n -gon, and γ be an interior arc of T . Then $\mu_\gamma(Q_T) = Q_{\phi_\gamma(T)}$.*

(b) *The exchange relation arising from flipping γ in the quadrilateral*

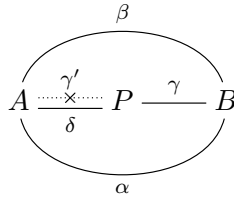


is the Ptolemy relation $x_\gamma x_{\gamma'} = x_\alpha x_\delta + x_\beta x_\epsilon$ if γ' is tagged plain (if some of the sides are boundary arcs, the corresponding variables are replaced by 1).

(c) *The exchange relation arising from flipping γ in the generalised quadrilateral (with puncture P)*

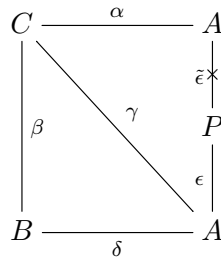


associated locally to the triangulation

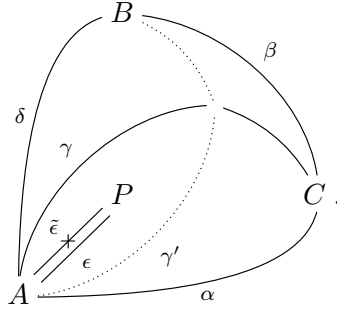


is the generalised Ptolemy relation $x_\gamma x_{\gamma'} = x_\alpha + x_\beta$ (if some of the sides are boundary arcs, the corresponding variables are again replaced by 1).

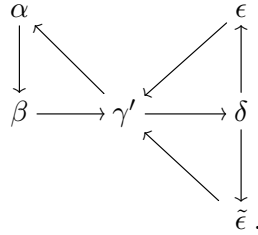
Proof. (a) Again, the arc γ is the diagonal of some quadrilateral. If the four sides of the quadrilateral are tagged plain, then we proceed exactly as in proof of Lemma 2.2(a), whether γ is plain or notched. Suppose one of the sides (adjacent to the puncture P) is notched, as in the following quadrilateral



associated locally to the triangulation



Assuming, for simplicity, that the sides α , β and δ are internal arcs, we see that both $Q_{\phi_\gamma(T)}$ and $\mu_\gamma(Q_T)$ are equal to the following quiver



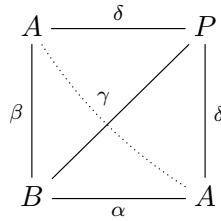
(b) The proof is identical to that of Lemma 2.2(b).

(c) It is again a simple verification that the generalised Ptolemy relation connecting the variables x_γ and $x_{\gamma'}$ is exactly the same as the exchange relation from Subsection 1.2. $\square$

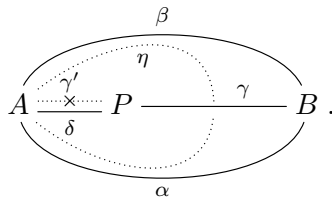
Remark 3.4. The generalised Ptolemy relation can be rewritten as

$$x_\gamma x_{\gamma'} = \frac{x_\alpha x_\delta + x_\beta x_\delta}{x_\delta},$$

which implies that $x_\gamma x_{\gamma'} = \frac{\tilde{x}_\eta}{x_\delta}$, where $\tilde{x}_\eta$ is the right-hand side polynomial obtained by the regular Ptolemy relation corresponding to flipping γ with respect to the quadrilateral



associated locally to the triangulation



We can thus associate $\tilde{x}_\eta$ to the loop η enclosing a puncture. The arcs γ' and η are both shown in the same triangulation for convenience of the reader.

Therefore, given any two triangulations of the punctured n -gon, the corresponding quivers are mutation-equivalent and the cluster algebra does not depend on the particular triangulation. We may thus assume this triangulation to be $T = \vec{T}_{\vec{\mathbb{D}}_n}$ and then $Q_T = \vec{\mathbb{D}}_n$. Let again the variables in the initial cluster $\underline{x} = \{x_1, \dots, x_n\}$ be in bijection with the points of $\vec{\mathbb{D}}_n$, thus with the internal arcs $\{\gamma_{-1,1}, \dots, \gamma_{-1,n}\}$ of the triangulation $\vec{T}_{\vec{\mathbb{D}}_n}$. We now want to demonstrate the finiteness of the algebra associated with a quiver of type $\mathbb{D}_n$.

To do so, we use the same method as in Section 2. We give an explicit formula for the cluster variables in $\mathcal{A}_{\vec{\mathbb{D}}_n}$ by defining a function x from the set of arcs of the punctured n -gon (here we replace all arcs tagged notched by the corresponding loops) to the ring of Laurent polynomials in the initial cluster variables $x_1, \dots, x_n$. Let γ_{ij} be an arc defined as before. Then we set

$$x_{ij} = x(\gamma_{ij}) = x_i x_j \sum_{k=i}^{j-1} \frac{1}{x_k x_{k+1}}.$$

Again, we agree that once the right-hand side is reduced to a common denominator and simplified, we replace the variables x_0 and x_{-1} by 1 and 0, respectively, and every variable of the form x_{i+n} by x_i . Note that the function x associates $\tilde{x}_\eta$ to the loop η , so this function does not always give cluster variables (see Proposition 3.9). We first show that the function x applied to the initial arcs of $\vec{T}_{\vec{\mathbb{D}}_n}$ gives the corresponding cluster variable, and x applied to boundary arcs gives 1.

Lemma 3.5. *Let i be a vertex of the punctured n -gon. Then,*

- (a) $x_{-1,i} = x_i$,
- (b) $x_{i,i+1} = 1$.

Proof. Identical to the proof of Lemma 2.3 □

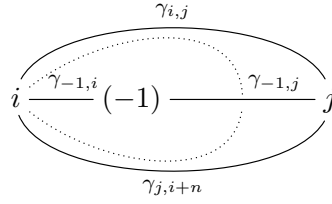
Lemma 3.6. *The function x respects the (regular) Ptolemy relation; that is, if*

$$\begin{array}{ccc} i & \text{---} & l \\ | & & | \\ j & \text{---} & k \end{array}$$

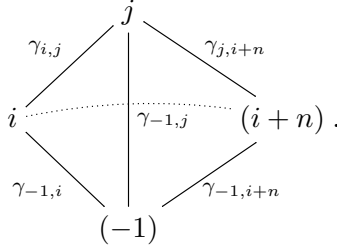
is a quadrilateral inside the punctured n -gon, then

$$x_{ik}x_{jl} = x_{ij}x_{kl} + x_{il}x_{jk}.$$

Proof. Again, this is identical to the proof of Lemma 2.4 once we represent the quadrilateral used for the flip of $\gamma_{-1,j}$ in the figure



as follows by artificially creating the vertex $i + n$:



This representation coincides with that from Lemma 3.3(c), due to our convention that $x(\gamma_{-1,i+n}) = x_{-1,i+n} = x_{i+n} = x_i = x_{-1,i} = x(\gamma_{-1,i})$ (see Lemma 3.5(a)). Now, the same calculation from the proof of Lemma 2.4 proves this lemma. $\square$

Corollary 3.7. *The cluster algebra $\mathcal{A}_{\vec{\mathbb{D}}_n}$ is of finite type and has n^2 cluster variables.*

Proof. By Lemma 3.5, the function x associates 1 to boundary arcs, and an initial cluster variable to every initial arc of $\vec{T}_{\mathbb{D}_n}$. By Lemma 3.6, the Laurent polynomial resulting from a flip is also given by x , and that polynomial does not depend on the quadrilateral used for the flip. For internal arcs tagged plain, the polynomial obtained in this way gives the cluster variable associated to the arc. For internal arcs forming a loop enclosing a puncture, the polynomial given by x is not a cluster variable, however we obtain a unique cluster variable associated to every notched arc by dividing that polynomial by the associated initial cluster variable; see Remark 3.4.

It follows that $\mathcal{A}_{\vec{\mathbb{D}}_n}$ has as many cluster variables as there are internal arcs (diagonals) in the punctured n -gon, that is, n^2 . $\square$

Corollary 3.8. *Let Q be a quiver mutation-equivalent to a quiver whose underlying graph is the Dynkin diagram $\mathbb{D}_n$. Then $\mathcal{A}_Q$ is of finite type and has n^2 cluster variables.*

Proof. This follows from Corollary 3.7 and Lemma 1.1. $\square$

Proposition 3.9. *The cluster variables of $\mathcal{A}_{\vec{\mathbb{D}}_n}$ are*

$$x_{ij} = x_i x_j \sum_{k=i}^{j-1} \frac{1}{x_k x_{k+1}},$$

where $-1 \leq i < j < i + n, j \neq i + 1$, and

$$\frac{x_{l,l+n}}{x_l} = x_l \sum_{k=l}^{l+n-1} \frac{1}{x_k x_{k+1}},$$

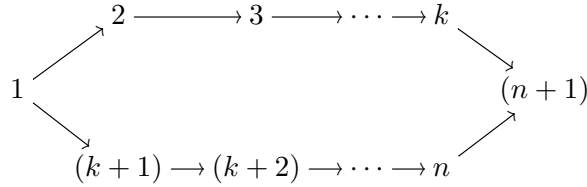
where $1 \leq l \leq n$.

Proof. This follows from Lemmata 3.3, 3.5 and 3.6 and Remark 3.4. $\square$

4. Algebras of infinite type

Throughout this section, we let Q be a finite, connected, acyclic quiver and $\overline{Q}$ be its underlying graph. Our first lemma is well known, see for instance [ASS06]. We give its proof for convenience of the reader.

Lemma 4.1. *Any orientation of $\tilde{\mathbb{A}}_n$ is mutation equivalent to*



for some k such that $1 \leq k \leq n$.

Proof. Let Q be an orientation of $\tilde{\mathbb{A}}_n$ (with $n > 1$). Let S_Q be the set of vertices that are sinks or sources in Q (note that $|S_Q| = 2l$ for $l > 0$). We prove, for $l > 1$, that we can always reduce S_Q to $2(l - 1)$ vertices by a sequence of mutations. Let Q' be a full subquiver of Q . In the following diagram representing Q' , we mark the elements of S_Q by boxes.

$$\rightarrow \boxed{1} \leftarrow 2 \leftarrow \cdots \leftarrow (i-1) \leftarrow \boxed{i} \rightarrow (i+1) \rightarrow \cdots \rightarrow (j-1) \rightarrow \boxed{j} \leftarrow .$$

Applying mutations, we get that $\mu_2 \cdots \mu_{(i-1)} \mu_i(Q')$ is equal to

$$\rightarrow 1 \rightarrow \boxed{2} \leftarrow \cdots \leftarrow (i-1) \leftarrow i \leftarrow \boxed{(i+1)} \rightarrow \cdots \rightarrow (j-1) \rightarrow \boxed{j} \leftarrow .$$

We can repeat such mutations moving the two boxes to the right until the leftmost is in the position $j - i + 1$. At this moment, the other two boxes disappear and we get

$$\rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow \boxed{(j-i+1)} \leftarrow \cdots \leftarrow j \leftarrow .$$

A simple induction then yields the proof. $\square$

Lemma 4.2. *Let Q' be a full subquiver of Q and $a_1, a_2, \dots, a_k$ be some points in Q' . Then $\mu_{a_k} \cdots \mu_{a_1}(Q')$ is a full subquiver of $\mu_{a_k} \cdots \mu_{a_1}(Q)$.*

Proof. Without loss of generality, we may assume that $k = 1$. Considering the definition of mutation (see Subsection 1.1) yields the proof. Indeed,

(a) If there is a path $i \leftarrow k \leftarrow j$ and $i, j \in Q'_0$, then the resulting arrow $i \leftarrow j$ exists both in $\mu_k(Q')$ and $\mu_k(Q)$. Otherwise, if i or j does not lie in Q' , then the arrow $i \leftarrow j$ does not lie in $\mu_k(Q')$;

(b) All arrows having k as source or target are inverted both in $\mu_k(Q')$ and $\mu_k(Q)$;

(c) Any 2-cycle created in $\mu_k(Q')$ or in $\mu_k(Q)$ are deleted. $\square$

Lemma 4.3. *Assume that $m \geq 2$ and $1 < a < b$. Then $1 + b^m > ab$.*

Proof. Since $a < b$, it suffices to show that $1 + b^m > b^2$ and this results from an immediate induction. $\square$

Lemma 4.4. Assume that Q contains a full subquiver consisting of two points 1 and 2, and m arrows from 1 to 2 (with $m \geq 2$). Then $\mathcal{A}_Q$ is of infinite type.

Proof. Let $\{x_1, x_2, \dots, x_n\}$ be the initial cluster, where we assume that x_1 and x_2 correspond to the points 1 and 2 respectively. Because of Lemma 4.2, for any $k \geq 1$, the quiver $1 \xrightarrow{m} 2$ is a full subquiver of $(\mu_2\mu_1)^k(Q)$, while $1 \xleftarrow{m} 2$ is a full subquiver of $\mu_1(\mu_2\mu_1)^k(Q)$ (where we denote by $1 \xrightarrow{m} 2$ a subquiver consisting of m arrows $\alpha_1, \dots, \alpha_m$ from 1 to 2). Applying the sequences of mutations $(\mu_2\mu_1)^k$ and $\mu_1(\mu_2\mu_1)^k$ yields two sequences of cluster variables:

$$\left\{ \begin{array}{l} x_1^{(k)} = \frac{\prod_{\gamma \in Q_1 \cap 1^-} x_s(\gamma) + \left(\prod_{\substack{\beta \in \{Q_1\} \cap 1^+ \\ \beta \neq \alpha_2}} x_t(\beta) \right) (x_2^{(k-1)})^m}{x_1^{(k-1)}}; \\ x_2^{(k)} = \frac{\prod_{\gamma \in Q_1 \cap 2^+} x_t(\gamma) + \left(\prod_{\substack{\beta \in \{Q_1\} \cap 2^- \\ \beta \neq \alpha_1}} x_s(\beta) \right) (x_1^{(k)})^m}{x_2^{(k-1)}}. \end{array} \right.$$

Denote by $\mathcal{E}(x_1^{(k)})$ and $\mathcal{E}(x_2^{(k)})$ the evaluations of the above sequences $x_1^{(k)}$ and $x_2^{(k)}$ when setting all cluster variables equal to 1. We prove that these two sequences of variables are infinite by proving that the numeric sequences $\mathcal{E}(x_1^{(k)})$ and $\mathcal{E}(x_2^{(k)})$ are infinite. To this end, we show that, for all $k > 0$, we have

$$\mathcal{E}(x_2^{(k)}) > \mathcal{E}(x_1^{(k)}) > \mathcal{E}(x_2^{(k-1)}) > \mathcal{E}(x_1^{(k-1)}).$$

We first observe that $\mathcal{E}(x_2^{(1)}) = 1 + 2^m > \mathcal{E}(x_1^{(1)}) = 2$ for any $m \geq 2$. Next, assume $\mathcal{E}(x_2^{(k)}) > \mathcal{E}(x_1^{(k)})$. Then

$$\mathcal{E}(x_1^{(k+1)}) = \frac{1 + (\mathcal{E}(x_2^{(k)}))^m}{\mathcal{E}(x_1^{(k)})} > \mathcal{E}(x_2^{(k)})$$

and also

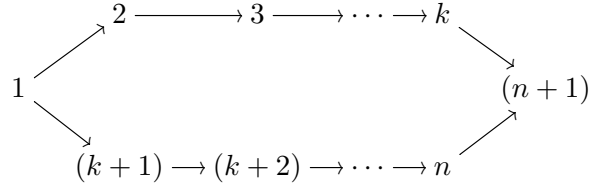
$$\mathcal{E}(x_2^{(k+1)}) = \frac{1 + (\mathcal{E}(x_1^{(k+1)}))^m}{\mathcal{E}(x_2^{(k)})} > \mathcal{E}(x_1^{(k+1)})$$

because of Lemma 4.3. This completes the proof. $\square$

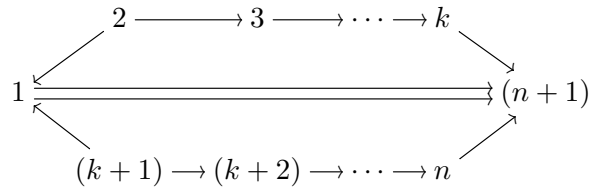
Lemma 4.5. If Q contains a Euclidean quiver $(\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8)$ as a full subquiver, then $\mathcal{A}_Q$ is of infinite type.

Proof. Because of Lemmata 1.1 and 4.1, we may, without loss of generality, choose the following orientations for the Euclidean full subquiver Q' of Q .

(a) If Q' is of type $\tilde{A}_n$, then we give it the orientation



and then $\mu_2\mu_3\cdots\mu_n(Q')$ is the quiver

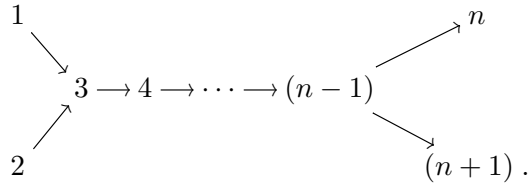


and it is a full subquiver of $\mu_2\mu_3\cdots\mu_n(Q)$. Therefore the Kronecker quiver

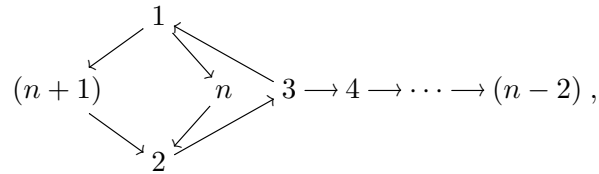
$$1 \rightrightarrows (n+1)$$

is a full subquiver of $\mu_2\mu_3\cdots\mu_n(Q)$. Because of Lemma 4.4, $\mathcal{A}_Q$ is of infinite type.

(b) If Q' is of type $\tilde{D}_n$, then we give it the orientation

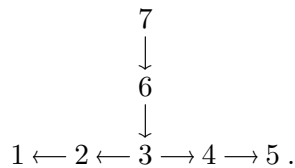


Then, $\mu_2\mu_3\cdots\mu_{(n-1)}(Q)$ is the quiver

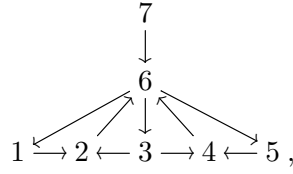


which contains $\tilde{A}_3$ as a full subquiver. Reasoning as in (a), we obtain a Kronecker quiver which is a full subquiver of a quiver mutation-equivalent to Q . Then $\mathcal{A}_Q$ is of infinite type due to Lemma 4.4.

(c) If Q' is of type $\tilde{E}_6$, then we give it the orientation

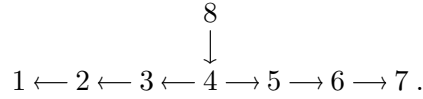


Then $\mu_4\mu_2\mu_3(Q')$ is the quiver

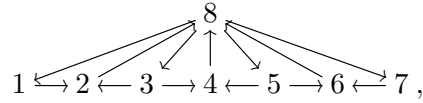


which contains a full subquiver of type $\tilde{\mathbb{D}}_4$. Applying the argument used in (b) and (a), we get that $\mathcal{A}_Q$ is of infinite type.

(d) If Q' is of type $\tilde{\mathbb{E}}_7$, then we give it the orientation

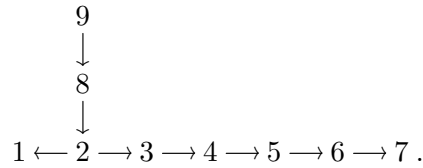


Then, $\mu_4\mu_6\mu_5\mu_2\mu_3\mu_4(Q')$ is the quiver

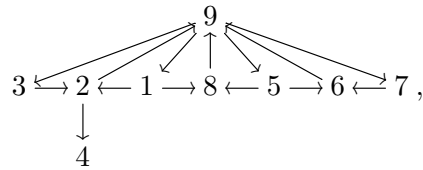


which contains $\tilde{\mathbb{D}}_4$ as a full subquiver. Hence $\mathcal{A}_Q$ is of infinite type.

(e) If Q' is of type $\tilde{\mathbb{E}}_8$, then we give it the orientation



Then, $\mu_8\mu_6\mu_5\mu_2\mu_1\mu_8\mu_4\mu_3\mu_2(Q')$ is the quiver



which contains $\tilde{\mathbb{D}}_4$ as a full subquiver. Hence $\mathcal{A}_Q$ is of infinite type. $\square$

The next lemma is a slightly modified version of [ASS06, (VII.2.1)]. We include its simple proof.

Lemma 4.6. *Let Q be a finite, connected and acyclic quiver. If Q does not contain as a full subquiver $i \xrightarrow{m} j$ (with $m \geq 2$) or a Euclidean quiver, then Q is a Dynkin quiver.*

Proof. The exclusion of $i \xrightarrow{m} j$ implies that between two points of Q there is at most one arrow. The exclusion of $\tilde{\mathbb{A}}$ implies that Q is a tree. The exclusion of $\tilde{\mathbb{D}}_4$ implies that no point in Q has more than 3 neighbours and the exclusion of $\tilde{\mathbb{D}}_n$, with

$n > 4$, that at most one point has 3 neighbours. Hence the underlying graph $\overline{Q}$ of Q is of the following form

$$\begin{array}{c} a_1 \\ | \\ a_2 \\ | \\ \vdots \\ | \\ b_1 - b_2 - \cdots - b_s - a_r - c_1 - c_2 - \cdots - c_t, \end{array}$$

where we may assume without loss of generality that $1 \leq r \leq s \leq t$. The exclusion of $\tilde{\mathbb{E}}_6$ gives $1 \leq r \leq 2$. If $r = 1$, then $\overline{Q} = \mathbb{A}_{s+t+1}$. If $r = 2$, then the exclusion of $\tilde{\mathbb{E}}_7$ gives $1 \leq s \leq 2$. If $s = 1$, then $\overline{Q} = \mathbb{D}_{t+3}$. Finally, if $s = 2$, then the exclusion of $\tilde{\mathbb{E}}_8$ gives $2 \leq t \leq 4$, so that $\overline{Q}$ is equal to $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$. $\square$

Remark 4.7. In order to prove our main theorem, it remains to show that if Q is a quiver of Dynkin type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, then the cluster algebra $\mathcal{A}_Q$ is of finite type. This cannot be done using the method of Sections 2 and 3: indeed, this method rests on the construction of the cluster algebras of types $\mathbb{A}$ and $\mathbb{D}$ as arising from marked surfaces, and cluster algebras of type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ cannot be constructed in this way. However, using Lemma 1.1 to choose, without loss of generality, an initial orientation of these quivers, the statement may be proved using a (long and tedious) computer check: one can compute all possible sequences of mutations as well as the corresponding cluster variables, thus proving that the cluster algebras associated to the mutation classes of $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ are of finite type.

We are now in a position to prove our main result.

Theorem 4.8. *Let Q be a finite, connected and acyclic quiver. Then $\mathcal{A}_Q$ is of finite type if and only if Q is mutation-equivalent to a Dynkin quiver.*

Proof. Assume that Q is mutation-equivalent to a Dynkin quiver. Then $\mathcal{A}_Q$ is of finite type, because of Corollaries 2.5, 3.6 and Remark 4.7. Conversely, if Q is not mutation-equivalent to a Dynkin quiver, then, because of Lemma 4.6, it contains $i \xrightarrow{m} j$ with $m \geq 2$ or a Euclidean quiver as a full subquiver. But then Lemmata 4.4 and 4.5 complete the proof. $\square$

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