

THE DUALITY PROBLEM FOR THE CLASS OF ORDER ABSOLUTELY WEAKLY COMPACT OPERATORS

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RÉSUMÉ. Nous établissons certaines propriétés pour les opérateurs, sur les treillis de Banach, appartenant à la classe des opérateurs absolument faiblement compacts pour l'ordre. Nous étudions le problème de dualité pour cette classe. De plus, nous déduisons quelques conséquences sur les treillis de Banach discrets.

ABSTRACT. We establish some properties of the class of order absolutely weakly compact operators on Banach lattices and we study the duality problem for this class. Also, we give some consequences on discrete Banach lattices.

1. Introduction and notation

In [7] the class of order absolutely weakly compact operators was introduced and the compactness of AM-compact operators on Banach lattices was studied. This class appeared in the paper of Aliprantis and Burkhinshaw [2] when they studied the domination problem for the class of Dunford–Pettis operators on

A regular operator $T: E \longrightarrow F$ between two Banach lattices is said to be *order absolutely weakly compact* if it carries order bounded subsets of E onto precompact subsets of F for the weak absolute topology $|\sigma|(F, F')$. This definition is not far from that of order weakly compact operators introduced by Dodds in [10] and the definition of AM-compact operators introduced by Fremlin in [12]. It is clear that each regular AM-compact operator is order absolutely weakly compact; however, the converse is false in general. Indeed, the identity operator Id_c of the Banach lattice c of all convergent sequences is order absolutely weakly compact but it is not AM-compact.

The classes of order weakly compact operators and order absolutely weakly compact operators are not necessarily equal. For instance, Id_c is order absolutely weakly compact but it is not order weakly compact, while the identity operator of the Banach lattice $L^1[0, 1]$ is order weakly compact but it is not order absolutely weakly compact.

As it is the case for AM-compact operators [9] and [15], the class of order absolutely weakly compact operators does not satisfy the duality property, that is there exist order absolutely weakly compact operators whose adjoints are not order absolutely weakly compact. In fact, the identity operator Id_{l^1} of the Banach lattice l^1 is order

absolutely weakly compact, but its adjoint, the identity operator Id_{l^∞} of l^∞ , is not order absolutely weakly compact (see Remarks 3.8 below).

In [9, 15] (resp. [5]), the duality problem was studied for the class of AM-compact (resp. order weakly compact) operators. We now study the duality problem for the class of order absolutely weakly compact operators. More precisely, we prove that if T is a regular operator from a Banach lattice E to another Banach lattice F such that its adjoint $T': F' \rightarrow E'$ is order absolutely weakly compact, then T is necessarily order absolutely weakly compact. We also establish that the class of order absolutely weakly compact operator from a Banach lattice E into itself satisfy the duality property if and only if the norm of E' is order continuous.

2. Notations and definitions

To state our results, we need to fix some notations and recall some definitions. A *Banach lattice* is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property : for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E' , endowed with the dual norm, is also a Banach lattice. We refer to Zaanen [15] for any unexplained terminology on the theory of Banach lattices.

A vector lattice equipped with a vector topology is said to be a *locally convex solid lattice* if zero admits a fundamental system of convex and solid neighbourhoods. We write $x_\alpha \downarrow 0$ to mean that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. The topology τ of a locally convex solid vector lattice is said to be *Lebesgue* if each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E converges to 0 for the topology τ .

Let E' be the topological dual of E . For each $f \in E'$, let $P_f(x) = |f|(|x|)$ for each $x \in E$. The *absolute weak topology* $|\sigma|(E, E')$ is the locally convex solid topology on E generated by the family of lattice seminorms $\{P_f \mid f \in E'\}$. Similarly, $|\sigma|(E', E)$ is the locally convex solid topology on E' generated by the family of lattice seminorms $\{P_x \mid x \in E\}$ where $P_x(f) = |f|(|x|)$ for each $f \in E'$. For more information about locally convex solid topologies, we refer the reader to the book of Aliprantis and Burkinshaw [1].

An *operator* between two Banach lattices is a bounded linear mapping. An operator $T: E \rightarrow F$ is *positive* if $T(x) \geq 0$ in F whenever $x \geq 0$ in E . The operator T is *regular* if $T = T_1 - T_2$ for two positive operators T_1 and T_2 from E into F . It is well known that every positive linear mapping on a Banach lattice is continuous. A regular operator between two Banach lattices E and F is said to be *AM-compact* if it carries each order bounded subset of E onto a relatively compact subset of F .

3. On duality of order absolutely weakly compact operators

Let us recall that if an operator $T: E \rightarrow F$ between two Banach lattices is positive, then its adjoint $T': F' \rightarrow E'$ is also positive.

As for AM-compact operators, the class of order absolutely weakly compact operators has the following property.

Proposition 3.1. *Let E , F and G be Banach lattices. If*

$$E \xrightarrow{T} F \xrightarrow{S} G$$

are regular operators and T or S is order absolutely weakly compact, then $S \circ T$ is order absolutely weakly compact.

Proof. Assume that T is order absolutely weakly compact and let $x \in E^+$. Then $T[-x, x]$ is $|\sigma|(F, F')$ -precompact. Since S is regular, it is $|\sigma|(F, F')$ - $|\sigma|(G, G')$ -continuous (See Exercise 11, p. 180 of [3]). So $S(T[-x, x])$ is $|\sigma|(G, G')$ -precompact. And hence $S \circ T$ is order absolutely weakly compact.

Assume now that S is order absolutely weakly compact and let $x \in E^+$. As T is regular, $T[-x, x]$ is order bounded. So $S(T[-x, x])$ is $|\sigma|(G, G')$ -precompact. Hence $S \circ T$ is order absolutely weakly compact. $\square$

We need the duality Theorem of Grothendieck ([13, Thm 3, p. 51]).

Lemma 3.2. *Let $T: E \rightarrow F$ be an operator between two Banach lattices. Then T maps order intervals of E onto $|\sigma|(F, F')$ -precompact subsets of F if and only if T' maps order intervals of F' onto $|\sigma|(E', E)$ -precompact subsets of E' .*

Contrary to AM-compact operators (see [8] and [12]), the class of order absolutely weakly compact operators satisfies the domination property. In fact, we have the following lemma; see [3, Theorem 5.11].

Lemma 3.3. *Suppose that S and T are two operators from a Banach lattice E into another Banach lattice F such that $0 \leq S \leq T$ and T is order absolutely weakly compact. Then S is order absolutely weakly compact.*

As a consequence, we obtain the following result.

Corollary 3.4. *Let T be a regular operator from a Banach lattice E into another Banach lattice F . If the module of T exists and is order absolutely weakly compact, then T is order absolutely weakly compact.*

Proof. Since $-|T| \leq T \leq |T|$, we have $0 \leq T + |T| \leq 2|T|$. As $|T|$ is order absolutely weakly compact, it follows from Lemma 3.3 that $T + |T|$ is order absolutely weakly compact, and hence T is order absolutely weakly compact. $\square$

Recall that a norm $\|\cdot\|$ of a Banach lattice E is *order continuous* if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E , the sequence (x_α) converges to 0 for the norm $\|\cdot\|$. Recall also that a nonzero element x of a vector lattice E is *discrete* if the order ideal generated by x equals the subspace generated by x . The vector lattice E is *discrete* if it admits a complete disjoint system of discrete elements.

Proposition 3.5. *Let E and F be two Banach lattices. If E' or F' is discrete, then every regular operator T from E into F is order absolutely weakly compact.*

Proof. Assume that E' is discrete and let $x \in E^+$. By Theorem 4.1 of Duhoux [11], the order interval $[-x, x]$ is $|\sigma|(E, E')$ -precompact. So $T[-x, x]$ is $|\sigma|(F, F')$ -precompact. This proves that T is order absolutely weakly compact.

Assume now that F' is discrete and let $x \in E^+$. As T is regular, $T[-x, x]$ is order bounded. So by Theorem 4.1 of Duhoux [11], $T[-x, x]$ is $|\sigma|(F, F')$ -precompact. Hence T is order absolutely weakly compact. $\square$

In the following, we give a characterization of Banach lattices for which each positive operator is order absolutely weakly compact.

Proposition 3.6. *Let E be a Banach lattice. The following assertions are equivalent :*

- (1) E' is discrete.
- (2) For each Banach lattice F , every regular operator T from E into F is order absolutely weakly compact.
- (3) Every regular operator T from E into E is order absolutely weakly compact.
- (4) The identity operator $\text{Id}_E: E \rightarrow E$ is order absolutely weakly compact.

Proof. Assertion (1) implies assertion (2) by Proposition 3.5. Because, obviously, (2) implies (3), which implies (4), it remains to show that (4) implies (1). Since the identity operator $\text{Id}_E: E \rightarrow E$ is order absolutely weakly compact, every order interval of E is $|\sigma|(E, E')$ -precompact. So by Theorem 4.1 of Duhoux [11], E' is discrete. $\square$

As a consequence of Proposition 3.6, we derive the following corollary.

Corollary 3.7. *Let E be a Banach lattice with an order continuous norm. Then E is discrete if and only if E' is discrete.*

Proof. Assume that E is discrete. Since the norm of E is order continuous, it follows from Corollary 21.13 of [1] that each order interval of E is norm compact. Hence the identity operator $\text{Id}_E: E \rightarrow E$ is AM-compact and then order absolutely weakly compact. Now, Proposition 3.6 shows that the Banach lattice E' is discrete.

Conversely, if E' is discrete, it follows from Proposition 3.6 that the identity operator $\text{Id}_E: E \rightarrow E$ is order absolutely weakly compact. As the norm of E is order continuous, the operator Id_E is AM-compact. Finally, Corollary 21.13 of [1] implies that E is discrete. $\square$

Recall that a Banach lattice E is said to be an *AM-space* if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have

$$\|x + y\| = \max\{\|x\|, \|y\|\}.$$

The Banach lattice E is an *AL-space* if its topological dual E' is an AM-space.

Remarks 3.8. (1) There exist operators that are not order absolutely weakly compact. In fact, since the Banach lattice $(l^\infty)'$ is not discrete (see [6, Cor. 3.5]), it

follows from Proposition 3.6 that the identity operator Id_{l^∞} , of the Banach lattice l^∞ , is not order absolutely weakly compact.

(2) An order absolutely weakly compact is not necessary AM-compact. In fact, since the topological dual c' is discrete, it follows from Proposition 3.6 that the identity operator Id_c , of the Banach lattice c of all convergent sequences, is order absolutely weakly compact but it is not AM-compact.

Now, we study the problem of duality for the class of order absolutely weakly compact. For the first time, we prove that an operator is always order absolutely weakly compact if its adjoint is order absolutely weakly compact.

Theorem 3.9. *Let T be a regular operator from a Banach lattice E into another Banach lattice F . If the adjoint $T': F' \longrightarrow E'$ is order absolutely weakly compact, then T is order absolutely weakly compact.*

Proof. In fact, if the adjoint operator T' is order absolutely weakly compact, then the subset $T'([-y', y'])$ is precompact for the topology $|\sigma|(E', E'')$. Since $|\sigma|(E', E)$ is weaker than $|\sigma|(E', E'')$, it follows that $T'([-y', y'])$ is precompact for $|\sigma|(E', E)$. Now, an application of Lemma 3.2 gives that for each $x \in E^+$, the set $T([-x, x])$ is precompact for $|\sigma|(F, F')$. Hence T is order absolutely weakly compact. $\square$

However, as mentioned at the beginning of this paper, the converse is false in general. In fact, the identity operator Id_{l^1} , of the Banach lattice l^1 , is order absolutely weakly compact but its adjoint, the identity operator Id_{l^∞} of l^∞ , is not order absolutely weakly compact.

The following result gives some sufficient conditions under which the adjoint is order absolutely weakly compact whenever the operator is order absolutely weakly compact.

Theorem 3.10. *Let E be a Banach lattice. The following assertions are equivalent :*

- (1) *Every positive order absolutely weakly compact operator from E into E admits an adjoint which is also order absolutely weakly compact.*
- (2) *The norm of E' is order continuous.*

Proof. (2) implies (1) : If T is order absolutely weakly compact, then the set $T([-x, x])$ is precompact for $|\sigma|(E, E')$ for each $x \in E^+$. Now, Lemma 3.2 implies that for each $y' \in (E')^+$, the subset $T'([0, y'])$ is precompact for $|\sigma|(E', E)$ which is always a Lebesgue topology on E' . By our hypothesis, the topology defined by the norm of E' is Lebesgue ; hence it follows from Theorem 1.1 of [4] that the order bounded set $T'([0, y'])$ is also precompact for the norm of E' . This proves that T' is AM-compact and hence T' is order absolutely weakly compact.

(1) implies (2) : Assume that the norm of E' is not order continuous. Then it follows from the proof of Theorem 1 of Wickstead [14], that E contains a sublattice, which is isomorphic to l^1 , and there exists a positive projection P from E into l^1 . We consider the operator

$$T = i \circ P: E \longrightarrow l^1 \longrightarrow E$$

where i is the inclusion operator of l^1 in E . Note that $P \circ T \circ i = \text{Id}_{l^1}$. It is clear that T is AM-compact and hence T is order absolutely weakly compact. But its adjoint

$$T': E' \longrightarrow l^\infty \longrightarrow E'$$

is not order absolutely weakly compact. Otherwise, the operator

$$i' \circ T' \circ P': l^\infty \longrightarrow l^\infty$$

would be order absolutely weakly compact by Proposition 3.1. Hence $\text{Id}_{l^\infty} = i' \circ T' \circ P'$ is order absolutely weakly compact. This gives a contradiction. $\square$

More generally, if $E \neq F$, we obtain the following.

Theorem 3.11. *Let E and F be Banach lattices. If each positive order absolutely weakly compact operator from E into F admits an adjoint which is also order absolutely weakly compact, then one of the followings assertions is valid :*

- (1) *The norm of E' is order continuous.*
- (2) *The norm of F' is order continuous.*

Proof. Assume that the norms of E' and F' are not order continuous. Then E (resp. F) contains a sublattice isomorphic to l^1 and there exists a positive projection

$$P_1: E \longrightarrow l^1 \quad (\text{resp. } P_2: F \longrightarrow l^1).$$

We denote by

$$i_1: l^1 \longrightarrow E \quad (\text{resp. } i_2: l^1 \longrightarrow F)$$

the canonical injection of l^1 into E (resp. l^1 into F). We consider the composed operator

$$T = i_2 \circ P_1: E \longrightarrow l^1 \longrightarrow F.$$

It is an order absolutely weakly compact operator because $T = i_2 \circ \text{Id}_{l^1} \circ P_1$ and the identity operator Id_{l^1} is order absolutely weakly compact.

However, the operator $T' = P_1' \circ i_2'$ is not order absolutely weakly compact. Otherwise, the composed operator

$$\text{Id}_{l^\infty} = i_1' \circ T' \circ P_2': l^\infty \longrightarrow l^\infty,$$

would be order absolutely weakly compact. But this is impossible. $\square$

Remarks 3.12. (1) The first necessary condition of Theorem 3.11 is sufficient. In fact, if $T: E \longrightarrow F$ is order absolutely weakly compact, then $T([-x, x])$ is precompact for $|\sigma|(F, F')$ for each $x \in E^+$. Now, Lemma 3.2 implies that for each $y' \in (F')^+$, the subset $T'([0, y'])$ is precompact for $|\sigma|(E', E)$ which is always a Lebesgue topology on E' . By our hypothesis, the topology defined by the norm of E' is Lebesgue; hence it follows from Theorem 1.1 of [4] that the order bounded set $T'([0, y'])$ is also precompact for the norm of E' . This proves that T' is AM-compact and hence T' is order absolutely weakly compact.

(2) If F' is discrete and its norm is order continuous, then the adjoint of every operator from E into F is order absolutely weakly compact. In fact, by Corollary 21.13 of [1], each order interval of F' is norm compact. Then every operator

$$T': F' \longrightarrow E'$$

is AM-compact and hence order absolutely weakly compact.

REFERENCES

- [1] C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Pure and Applied Mathematics, Vol. 76, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, xii+198 pp.
- [2] C. D. Aliprantis and O. Burkinshaw, *Dunford–Pettis operators on Banach lattices*, Trans. Amer. Math. Soc. **274** (1982), no. 1, 227–238.
- [3] C. D. Aliprantis and O. Burkinshaw, *Positive operators*, Reprint of the 1985 original, Springer, Dordrecht, 2006, xx+376 pp.
- [4] C. D. Aliprantis, O. Burkinshaw and M. Duhoux, *Compactness properties of abstract kernel operators*, Pacific J. Math. **100** (1982), no. 1, 1–22.
- [5] B. Aqzzouz and J. Hmichane, *The duality problem for the class of order weakly compact operators*, Glasg. Math. J. **51** (2009), no. 1, 101–108.
- [6] B. Aqzzouz et R. Nourira, *Sur les opérateurs précompacts positifs*, C. R. Math. Acad. Sci. Paris **337** (2003), no. 8, 527–530.
- [7] B. Aqzzouz, R. Nourira, and L. Zraoula, *Compactness of AM-compact operators on Banach lattices*, Ann. Sci. Math. Québec **31** (2007), no. 1, 1–11.
- [8] B. Aqzzouz, R. Nourira and L. Zraoula, *Compactness properties of operators dominated by AM-compact operators*, Proc. Amer. Math. Soc. **135** (2007), no. 4, 1151–1157 (electronic).
- [9] B. Aqzzouz, R. Nourira, and L. Zraoula, *The duality problem for the class of AM-compact operators on Banach lattices*, Canad. Math. Bull. **51** (2008), no. 1, 15–20.
- [10] P. G. Dodds, *σ -weakly compact mappings of Riesz spaces*, Trans. Amer. Math. Soc. **214** (1975), 389–402.
- [11] M. Duhoux, *Order precompactness in topological Riesz spaces*, J. London Math. Soc. (2) **23** (1981), no. 3, 509–522.
- [12] D. H. Fremlin, *Riesz spaces with the order-continuity property. I.*, Math. Proc. Cambridge Philos. Soc. **81** (1977), no. 1, 31–42.
- [13] A. P. Robertson and W. Robertson, *Topological vector spaces*, Second edition, Cambridge Tracts in Mathematics and Mathematical Physics, No. 53, Cambridge University Press, London-New York, 1973, viii+172 pp.
- [14] A. W. Wickstead, *Converses for the Dodds–Fremlin and Kalton–Saab theorems*, Math. Proc. Cambridge Philos. Soc. **120** (1996), no. 1, 175–179.
- [15] A. C. Zaanen, *Riesz spaces. II.*, North-Holland Mathematical Library, **30** North-Holland Publishing Co., Amsterdam, 1983, xi+720 pp.

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