

ON A CONJECTURE ON THE POWER SERIES COEFFICIENTS OF THE FUNCTION $U(n, x)$

ALAIN TOGBÉ AND SHICHUN YANG

Dedicated to Professor Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Soit

$$U(n, x) = 1 - \frac{1}{n} + \frac{1}{n} \left(\frac{1 + (1 - \frac{1}{n})x}{1 - \frac{1}{n}x} \right)^n$$

et $P_k(n)$ le polynôme défini par

$$\frac{1}{n-1} (U(n, x) - (1-x)^{-1}) = \sum_{k=2}^{\infty} \frac{P_k(n)x^k}{k! n^{k-1}}.$$

Nous montrons que si n est un entier relatif et $n \geq 2$, alors $k!$ divise $(n-1)P_k(n)$.

ABSTRACT. Let

$$U(n, x) = 1 - \frac{1}{n} + \frac{1}{n} \left(\frac{1 + (1 - \frac{1}{n})x}{1 - \frac{1}{n}x} \right)^n$$

and let the polynomial $P_k(n)$ be defined by

$$\frac{1}{n-1} (U(n, x) - (1-x)^{-1}) = \sum_{k=2}^{\infty} \frac{P_k(n)x^k}{k! n^{k-1}}.$$

We prove that if n is an integer and $n \geq 2$, then $k!$ divides $(n-1)P_k(n)$.

1. Introduction

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ be the sets of all real numbers, rational numbers, and positive integers respectively. The inequality between the arithmetic and geometric means is deduced from

$$1 + x \leq e^x.$$

One can refer to §4.2 of [1]. When $x < 1$, with a change of variable, the above inequality becomes

$$e^x \leq \frac{1}{1-x}.$$

Suppose that

$$U(n, x) = 1 - \frac{1}{n} + \frac{1}{n} \left(\frac{1 + (1 - \frac{1}{n})x}{1 - \frac{1}{n}x} \right)^n.$$

Then, for any real numbers n and x such that $n \geq 1$ and $-\frac{n}{n-1} < x < n$, Seon-Hong Kim [3] proved that

$$e^x \leq U(n, x).$$

The equality holds if and only if $x = 0$. Moreover, for $0 \leq x < 1$ and $1 \leq n \leq 2$, he showed that

$$(1.1) \quad e^x \leq U(n, x) \leq \frac{1}{1-x}.$$

If $x < 0$ and $0 \leq n \leq 1$, he obtained that

$$e^x \leq \frac{1}{1-x} \leq U(n, x).$$

He thought that inequality (1.1) can be strengthened to an inequality between the corresponding power series coefficients. When $|x| < 1$, let us write

$$(1.2) \quad \frac{1}{n-1} (U(n, x) - (1-x)^{-1}) = \sum_{k=2}^{\infty} \frac{P_k(n)x^k}{k! n^{k-1}}.$$

Here

$$\begin{aligned} P_2(n) &= n - 2, \\ P_3(n) &= n^3 - 2n^2 - 6, \\ P_4(n) &= n^5 - 5n^4 + 18n^3 - 48n^2 + 12n - 24. \end{aligned}$$

The unique real zeros of the polynomials P_2 , P_3 and P_4 are given by 2, 2.7776..., and 3.5934... respectively. Kim set many conjectures related to the polynomials P_k (see page 411 of [3]).

The main result of this note is the following.

Theorem 1. *If n is an integer, and $n \geq 2$, then*

$$k! \mid (n-1)P_k(n).$$

The case when $n = 2$ in Theorem 1 was conjectured by Kim [3]. After proving Theorem 1, we will raise a question.

2. Proof of Theorem 1

We will use the known fact that if $h \geq 2$ and $|t| < 1$, then

$$\frac{1}{(1-t)^h} = \sum_{k=0}^{\infty} \frac{(k+h-1)(k+h-2) \cdots (k+1)}{(h-1)!} t^k.$$

Therefore we get

$$\begin{aligned}
 f(n, x) &= \left(\frac{1 + (1 - \frac{1}{n})x}{1 - \frac{x}{n}} \right)^n \\
 &= \left(\frac{n}{1 - \frac{x}{n}} - (n-1) \right)^n \\
 &= \sum_{j=0}^n \left((-1)^j \binom{n}{j} \left(\frac{n}{1 - \frac{x}{n}} \right)^{n-j} (n-1)^j \right) \\
 &= \sum_{j=0}^{n-2} \left((-1)^j \binom{n}{j} n^{n-j} (n-1)^j \left(\sum_{k=0}^{\infty} \frac{(k+n-j-1)!}{(n-j-1)! k!} \left(\frac{x}{n} \right)^k \right) \right) \\
 &\quad + (-1)^{n-1} \binom{n}{n-1} n(n-1)^{n-1} \sum_{k=0}^{\infty} \left(\frac{x}{n} \right)^k + (-1)^n (n-1)^n,
 \end{aligned}$$

because when $|u| < 1$,

$$\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k.$$

Then from equation (1.2), if $k \geq 2$, we have

$$\begin{aligned}
 P_k(n) &= \frac{1}{n(n-1)} \left[\sum_{j=0}^{n-2} \left((-1)^j \binom{n}{j} n^{n-j} (n-1)^j \frac{(k+n-j-1)!}{(n-j-1)! k!} \cdot \frac{1}{n^k} \right) \right. \\
 &\quad \left. + (-1)^{n-1} n^2 (n-1)^{n-1} \cdot \frac{1}{n^k} - n \right] k! n^{k-1} \\
 &= \frac{1}{n-1} \left[\sum_{j=0}^{n-2} \left((-1)^j \binom{n}{j} n^{n-j-2} (n-1)^j \frac{(k+n-j-1)!}{(n-j-1)! k!} \right) \right. \\
 &\quad \left. + (-1)^{n-1} (n-1)^{n-1} - n^{k-1} \right] k!.
 \end{aligned}$$

We conclude that there is a positive integer A such that

$$P_k(n) = \frac{1}{n-1} A k!.$$

Thus $k! \mid (n-1)P_k(n)$.

3. Final remarks

We already wrote the values of $P_2(n)$, $P_3(n)$, $P_4(n)$. If we compute a further more, we get:

$$\begin{aligned}
 P_5(n) &= n^7 - 9n^6 + 46n^5 - 124n^4 + 120n^3 - 360n^2 + 120n - 120, \\
 P_6(n) &= n^9 - 14n^8 + 101n^7 - 424n^6 + 1200n^5 - 2940n^4 + 2280n^3 \\
 &\quad - 3120n^2 + 1080n - 720,
 \end{aligned}$$

$$\begin{aligned}
P_7(n) &= n^{11} - 20n^{10} + 197n^9 - 1168n^8 + 4656n^7 - 12858n^6 + 20580n^5 \\
&\quad - 41160n^4 + 32760n^3 - 30240n^2 + 10080n - 5040, \\
P_8(n) &= n^{13} - 27n^{12} + 351n^{11} - 2785n^{10} + 14960n^9 - 56972n^8 + 158760n^7 \\
&\quad - 367584n^6 + 465360n^5 - 616560n^4 + 453600n^3 - 322560n^2 \\
&\quad + 100800n - 40320, \\
&\dots
\end{aligned}$$

Question. As for P_2 , P_3 , and P_4 , one can check that P_5 , P_6 , P_7 , and P_8 are irreducible over $\mathbb{Q}$. Moreover, the polynomials P_2 to P_8 all have a unique real root whose integral part is equal to the index of the polynomial minus one. So we naturally ask the question: are these properties true for any P_k ? We believe the answer is affirmative but we do not have much evidence to conclude readily.

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REFERENCES

- [1] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, 1975.
- [2] L. G. Hua, *Introduction of Number Theory* (in Chinese), Beijing Science Press, 1979.
- [3] S.-H. Kim, *Densely algebraic bounds for the exponential function*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 237–241.

A. TOGBÉ, MATH. DEPT., PURDUE U. NORTH CENTRAL, 1401 S, U.S. 421, WESTVILLE IN 46391, USA
 atogbe@pnc.edu

S. YANG, DEPT. OF MATH., ABA TEACHER'S COLLEGE WENCHUAN, SICHUAN, 623000, P. R. CHINA
 ysc1020@sina.com