

ON AN ANALOGUE OF DABOUSSI'S THEOREM ON SOME SEMIGROUPS OF INTEGERS GENERATED BY SOME SUBSETS OF THE PRIMES

IMRE KÁTAI

Dedicated to Professor Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Soit $1, \beta_1, \dots, \beta_t$ des nombres réels rationnellement indépendants, et soit $I_1, I_2, \dots, I_t \subseteq [0, 1)$ des ensembles tels que chacun d'eux est une réunion finie d'intervalles. Soit $\mathcal{P}^*$ l'ensemble des nombres premiers p pour lesquels $\{\beta_j p\} \in I_j$ pour $j = 1, \dots, t$. Soit $\mathcal{B}^*$ le semi-groupe engendré par $\mathcal{P}^*$. Nous prouvons que si α est un nombre irrationnel satisfaisant à d'autres conditions, alors les valeurs de $F(n) + \alpha n$, lorsque n parcourt $\mathcal{B}^*$, sont uniformément distribuées pour toute fonction arithmétique additive F .

ABSTRACT. Let $1, \beta_1, \dots, \beta_t$ be rationally independent real numbers, and let $I_1, I_2, \dots, I_t \subseteq [0, 1)$ be sets such that each of them is a finite union of intervals. Let $\mathcal{P}^*$ be the set of prime numbers p for which $\{\beta_j p\} \in I_j$ for $j = 1, \dots, t$. Let $\mathcal{B}^*$ be the semigroup generated by $\mathcal{P}^*$. We prove that if α is an irrational number satisfying some further conditions, then the values $F(n) + \alpha n$ for n running through $\mathcal{B}^*$ are uniformly distributed for all additive arithmetical functions F .

1. Introduction

We shall use the following standard notations: $\mathcal{P}$ is the set of prime numbers, p with or without an index denotes a prime number, $\{x\}$ is the fractional part of x , and $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$.

Let $e(x) = e^{2\pi i x}$, let $\mathcal{M}$ be the set of complex valued multiplicative functions, and let $\mathcal{M}_1 = \{f \in \mathcal{M} : |f(n)| \leq 1, n \in \mathbb{N}\}$. Let $\mathcal{A}$ be the set of additive arithmetical functions. Let $\lambda(E)$ be the Lebesgue measure of the set $E (\subseteq \mathbb{R})$.

If $\mathcal{P}^*$ is a subset of the primes, then $\mathcal{B}^* = \mathcal{B}(\mathcal{P}^*)$ denotes the semigroup generated by the elements of $\mathcal{P}^*$. Let

$$\pi_{\mathcal{P}^*}(x) = \#\{p \leq x \mid p \in \mathcal{P}^*\} \quad \text{and} \quad N_{\mathcal{B}^*}(x) = \#\{n \leq x \mid n \in \mathcal{B}^*\}.$$

From a theorem of E. Wirsing [7] we know that if

$$(1.1) \quad \pi_{\mathcal{P}^*}(x) = (1 + o_x(1))\kappa x / (\log x),$$

with some fixed $\kappa \in (0, 1]$, then

$$(1.2) \quad N_{\mathcal{B}^*}(x) = (1 + o_x(1))cx(\log x)^{1-\kappa},$$

where $c = c(\kappa)$ is a positive constant.

Daboussi proved (see [1, 2]) that for all irrational α ,

$$\sup_{f \in \mathcal{M}_1} \frac{1}{x} \left| \sum_{n \leq x} f(n)e(n\alpha) \right| = \varrho_x(\alpha) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Hence one can read that $F(n) + \alpha n$ is uniformly distributed mod 1 for every $F \in \mathcal{A}$, and an upper bound for the discrepancy can be given with a sequence which does not depend on F and which tends to zero as $x \rightarrow \infty$.

We are interested in the following question: For what $\mathcal{P}^*$ does the relation

$$\sup_{f \in \mathcal{M}_1} \frac{1}{N_{\mathcal{B}^*}(x)} \left| \sum_{\substack{n \leq x \\ n \in \mathcal{B}^*}} f(n)e(n\alpha) \right| = \tilde{\varrho}_x(\alpha) \rightarrow 0 \quad \text{as } x \rightarrow \infty$$

hold for every irrational α with the exception of countably many values?

We are far from being able to give a complete answer for this question. Here we shall construct such sets $\mathcal{P}^*$ for which a somewhat weaker assertion is true.

2. Construction of $\mathcal{P}^*$

Let $I_1, \dots, I_t \subseteq [0, 1)$ be sets, each of them being a finite union of intervals. Let $\beta_1, \dots, \beta_t$ be irrational numbers, and let $1, \beta_1, \dots, \beta_t$ be linearly independent over $\mathbb{Q}$. Let

$$\mathcal{P}^* = \{p \mid \{\beta_j p\} \in I_j, j = 1, \dots, t\}.$$

Let $f_j(x)$ be the characteristic function of I_j in $[0, 1)$, i.e.,

$$f_j(x) = \begin{cases} 1 & \text{if } x \in I_j, \\ 0 & \text{if } x \in [0, 1) \setminus I_j. \end{cases}$$

Let us extend this function by

$$f_j(x + k) = f_j(x), \quad k \in \mathbb{Z}.$$

Let the Fourier series of $f_j(x)$ be $\sum_{n \in \mathbb{Z}} a_j^{(n)} e(nx)$. It is clear that

$$a_0^{(j)} = \int_0^1 f_j(x) dx = \lambda(I_j),$$

and that $|a_j^{(n)}| \leq 1$ for every j and n . (Even $a_j^{(n)} = \mathcal{O}(1/n)$ is true, but the constant may depend on I_j .)

Let $\kappa = \lambda(I_1) \cdots \lambda(I_t) (= a_0^{(1)} a_0^{(2)} \cdots a_0^{(t)})$. Let

$$s(p) = f_1(\beta_1 p) \cdots f_t(\beta_t p).$$

It is clear that

$$s(p) = \begin{cases} 1 & \text{if } p \in \mathcal{P}^*, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Delta > 0$ be a small number, and let

$$f_j^{(\Delta)}(x) = \frac{1}{(2\Delta)^2} \int_{-\Delta}^{\Delta} \int_{-\Delta}^{\Delta} f_j(x + u_1 + u_2) du_1 du_2.$$

Since

$$\kappa(n) = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} e(nu) du = \frac{e(n\Delta) - e(-n\Delta)}{4\pi i n \Delta},$$

the Fourier coefficients $b_n^{(j)}$ of $f_j^{(\Delta)}(x)$ are

$$b_n^{(j)} = a_n^{(j)} \kappa^2(n).$$

But $|a_n^{(j)}| \leq 1$, and therefore

$$|b_n^{(j)}| \leq \kappa^2(n) \leq \min^2 \left(1, \frac{2}{|n|\Delta} \right).$$

It is clear that

$$\begin{cases} f_j^{(\Delta)}(x) = 1 & \text{if } [x - 2\Delta, x + 2\Delta] \subset I_j, \\ f_j^{(\Delta)}(x) = 0 & \text{if } [x - 2\Delta, x + 2\Delta] \cap I_j = \emptyset, \text{ and} \\ 0 \leq f_j^{(\Delta)}(x) \leq 1 & \text{holds for every } x. \end{cases}$$

Let

$$s_{\Delta}(p) = f_1^{(\Delta)}(\beta_1 p) \cdots f_t^{(\Delta)}(\beta_t p).$$

If $s_{\Delta}(p) \neq s(p)$, then $f_j^{(\Delta)}(\beta_j p) \neq f_j(\beta_j p)$ holds for at least one j ; consequently, the distance of $\beta_j p \bmod 1$ to the closest endpoint of the interval in I_j is less than 2Δ .

Suppose that $u_1, \dots, u_R$ are the endpoints of the intervals in I_j and let

$$H_j = \bigcup_{i=1}^R [u_i - 2\Delta, u_i + 2\Delta].$$

Then $\lambda(H_j) \leq 4\Delta R$. If $f_j^{(\Delta)}(\beta_j p) \neq f_j(\beta_j p)$ then $\beta_j p \bmod 1 \in H_j$.

From a well-known theorem of I. M. Vinogradov $\beta_j p \bmod 1$ is uniformly distributed; consequently the number of the primes $p \leq x$ satisfying $\{\beta_j p\} \in H_j$ is less than $(5\Delta R)\pi(x)$ if x is large enough. See [5] or [6]. We have only finitely many β 's, so we conclude that

$$(2.1) \quad \#\{p \leq x \mid s(p) \neq s_{\Delta}(p)\} \leq c_1 \Delta \pi(x)$$

when $x > x_0$. Here c_1 is a constant which may depend on $I_1, \dots, I_t$ and $\beta_1, \dots, \beta_t$.

Let us choose a large K such that, for $j = 1, \dots, T$, we have

$$(2.2) \quad \left| f_j^{(\Delta)}(u) - \sum_{|n| \leq K} b_n^{(j)}(nu) \right| \leq \Delta.$$

Let

$$s_{\Delta,K}(p) = \sum_{n_1=-K}^K \cdots \sum_{n_t=-K}^K b_{n_1} \cdots b_{n_t} e((n_1\beta_1 + \cdots + n_t\beta_t)p).$$

From (2.2) we obtain that

$$(2.3) \quad |s_{\Delta}(p) - s_{\Delta,K}(p)| \leq t \Delta, \quad p \in \mathcal{P}.$$

3. Formulation of the theorems

Condition A. The number α satisfies this condition if there exists a function $g(x)$ tending to infinity such that

$$\min_{\substack{1 \leq V \leq g(x)(\log x)^{1-\kappa} \\ V \in \mathbb{N}}} x(\log x)^{-31} \|V\alpha\| > 1$$

holds for every large x .

Let

$$M(x) = \sum_{\substack{m \leq x \\ m \in \mathcal{B}^*}} \frac{1}{m} \quad \text{and} \quad M(x, v, l) = \sum_{\substack{m \leq x \\ m \equiv l \pmod{v} \\ m \in \mathcal{B}^*}} \frac{1}{m}.$$

Condition B. We say that this condition holds for $\mathcal{B}^*$ if there exists a function $g(x)$ tending to infinity such that

$$\sum_{Y_x \leq V \leq g(x)(\log x)^{1-\kappa}} \max_{l \pmod{V}} \frac{M(x, v, l)}{M(x)} \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

when $Y_x \rightarrow \infty$ arbitrarily slowly.

Theorem 1. Assume that $I_1, \dots, I_t, \beta_1, \dots, \beta_t$ and $\mathcal{P}^*$ are chosen as in §2. Let α be such a number for which $m\alpha + n_1\beta_1 + \cdots + n_t\beta_t$ is irrational for every choice of $m, n_1, \dots, n_t \in \mathbb{Z}$, except the case $m = n_1 = \cdots = n_t = 0$. Assume furthermore that either Condition A or Condition B holds. Then

$$\sup_{f \in \mathcal{M}_1} \frac{1}{N_{\mathcal{B}^*}(x)} \left| \sum_{\substack{n \leq x \\ n \in \mathcal{B}^*}} f(n) e(n\alpha) \right| = \varrho_x(\alpha) \rightarrow 0$$

as $x \rightarrow \infty$.

As a special case we formulate the following result.

Theorem 2. Under the conditions stated in Theorem 1, we have

$$\frac{1}{N_{\mathcal{B}^*}(x)} \left| \sum_{\substack{n \leq x \\ n \in \mathcal{B}^*}} e(n\alpha) \right| \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

Theorem 1 can be deduced from Theorem 2 by repeating the argument applied in [4]. What we need is a weak variant of the Turán-Kubilius inequality, namely for $\mathcal{R}$ a finite set of primes $p_1 < \dots < p_T$, with $p_j \in \mathcal{P}^*$, and

$$A_{\mathcal{R}} = \sum_{l=1}^T \frac{1}{p_l}, \quad \omega_{\mathcal{R}}(n) = \sum_{\substack{p|n \\ p \in \mathcal{R}}} 1,$$

we have

$$(3.1) \quad \limsup_{x \rightarrow \infty} \frac{1}{N_{\mathcal{B}^*}(x)} \sum_{\substack{n \leq x \\ n \in \mathcal{B}^*}} (\omega_{\mathcal{R}}(n) - A_{\mathcal{R}})^2 \leq c_1 A_{\mathcal{R}},$$

where c_1 is an absolute constant. Inequality (3.1) can be deduced by observing that for a fixed $u \in (0, \infty)$,

$$\lim_{x \rightarrow \infty} \frac{N_{\mathcal{B}^*}(xu)}{N_{\mathcal{B}^*}(x)} = u,$$

which is a consequence of (1.2).

If the condition stated in Theorem 1 holds for α , then it also holds for $j\alpha$ with $j = 1, 2, 3, \dots$. By using the Hermann Weyl's criterion, from Theorem 1 we obtain the next theorem.

Theorem 3. *Assume that the condition A or B holds. For an arbitrary additive arithmetical function F let*

$$D_x(F, B) = \sup_{[u, v) \subset [0, 1)} \left| \frac{1}{N_{\mathcal{B}^*}(x)} \# \{n \leq x, n \in \mathcal{B}^*, \{F(n) + \alpha n\} \in [u, v)\} - (v - u) \right|.$$

Then

$$\sup_{F \in \mathcal{A}} D_x(F, \mathcal{B}^*) \rightarrow 0 \quad \text{as } x \rightarrow \infty.$$

4. Auxiliary lemmas for trigonometric sums with a prime variable

Lemma 4. *Let y be a large number and assume that*

$$R \leq q \leq \frac{y}{R}, \quad 1 \leq R \leq y^{1/4}, \quad (a, q) = 1 \quad \text{and} \quad \left| \delta - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$

Then

$$\left| \sum_{p \leq y} e(\delta p) \right| \ll \frac{y}{\sqrt{R}} (\log y)^{16}.$$

This lemma is essentially due to I. M. Vinogradov. An explicit form and proof can be found in the book of Montgomery [5] (Corollary 16.3, page 142).

Lemma 5. *Let*

$$\tau = \frac{(\log x)^B}{x}, \quad 1 \leq q \leq (\log x)^{B/2}, \quad (a, q) = 1 \quad \text{and} \quad \left| \delta - \frac{a}{q} \right| \leq \frac{1}{q\tau}.$$

Then

$$\sum_{p \leq x} e(\delta p) \ll \frac{lix}{\varphi(q)}.$$

This can be proved by partial integration from Lemma 3.1 in Vaughan [6].

5. Proof of Theorem 2

Let $\varepsilon > 0$ be an arbitrary positive number. It is enough to prove that

$$\frac{1}{N_{\mathcal{B}^*}(x)} \left| \sum_{\substack{n \leq x \\ n \in \mathcal{B}^*}} e(n\alpha) \right| < \varepsilon \quad \text{if } x > x_0(\varepsilon).$$

We shall choose positive constants $\varepsilon_1, \varepsilon_2, \dots$ such that $\varepsilon_1 + \varepsilon_2 + \dots \leq \varepsilon$.

Let $P(n)$ be the largest prime factor of n . Let $\delta_1 > 0$ be so small that

$$(5.1) \quad \#\{n \leq x \mid n \in \mathcal{B}^*, P(n) \leq x^{\delta_1}\} < \varepsilon_1 N_{\mathcal{B}^*}(x)$$

holds for every $x > x_1$. Then we have

$$\sum_{\substack{\sqrt{x} \leq n \leq x \\ n \in \mathcal{B}^* \\ P(n) < x^{\delta_1}}} 1 \leq \frac{2}{\log x} \sum_{\substack{n \in \mathcal{B}^* \\ P(n) < x^{\delta_1}}} \log n \leq \frac{2}{\log x} \sum_{p < x^{\delta_1}} (\log p) N_{\mathcal{B}^*}\left(\frac{x}{p}\right),$$

and from (1.1), $N_{\mathcal{B}^*}\left(\frac{x}{p}\right) \leq \frac{c}{p} N_{\mathcal{B}^*}(x)$ with an absolute constant c . We obtain that the left-hand side of (5.1) is less than

$$\sqrt{x} + \frac{2N_{\mathcal{B}^*}(x)}{\log x} \cdot \sum_{p < x^{\delta_1}} \frac{\log p}{p} < 3\delta_1 N_{\mathcal{B}^*}(x),$$

if x is large enough. Thus (5.1) holds if $\delta_1 = \varepsilon_1/3$.

Let $\mathcal{N}_1 = \{n \leq x \mid n \in \mathcal{B}^*, P(n) > x^{\delta_1}\}$. Let us write each n as mp , where $p = P(n)$. Then

$$\sum_{n \in \mathcal{N}_1} e(\alpha n) = \sum_{\substack{m \leq x^{1-\delta_1} \\ m \in \mathcal{B}^*}} B_m,$$

where

$$B_m = B_m^{(1)} - B_m^{(2)},$$

with

$$B_m^{(1)} = \sum_{\substack{p < \frac{x}{m} \\ p \in \mathcal{P}^*}} e(\alpha mp) \quad \text{and} \quad B_m^{(2)} = \sum_{p \leq P(m)} e(\alpha mp).$$

Summing over all integers m , and $p \in \mathcal{P}$, we proved in [3] that

$$\sum_m B_m^{(2)} \ll \frac{1}{\varepsilon^2} \frac{x}{(\log x)^2} + \frac{1}{\varepsilon} \frac{x}{\log x} \ll o_x(1) N_{\mathcal{B}^*}(x).$$

It remains to estimate $\sum B_m^{(1)}$.

We have

$$\begin{aligned} B_m^{(1)} &= \sum_{p < \frac{x}{m}} e(\alpha m p) s(p) \\ &= \sum_{n_1=-K}^K \dots \sum_{n_t=-K}^K \sum_{\substack{p \leq \frac{x}{m} \\ p \in \mathcal{P}}} e((\alpha m + \beta_1 n_1 + \dots + \beta_t n_t)p) \\ &\quad + \mathcal{O}\left(\Delta\pi\left(\frac{x}{m}\right)\right). \end{aligned}$$

Here we used (2.1) and (2.3).

Let

$$(5.2) \quad E(m \mid n_1, \dots, n_t) = \sum_{\substack{p \leq \frac{x}{m} \\ p \in \mathcal{P}}} e((\alpha m + \beta_1 n_1 + \dots + \beta_t n_t)p).$$

Let $\gamma_m = \alpha m + \eta$, $\eta = n_1 \beta_1 + \dots + n_t \beta_t$. To estimate (5.2) we shall use Lemmas 4 and 5. Let $\tau = \frac{x}{(\log x)^{30}}$. Let $\frac{a_m}{q_m}$ be such a rational number for which

$$(5.3) \quad \left| \gamma_m - \frac{a_m}{q_m} \right| < \frac{1}{q_m \tau}, \quad \text{with } q_m \leq \tau,$$

holds. From Lemma 4 we obtain that

$$E(m \mid n_1, \dots, n_t) \ll \frac{\frac{x}{m}}{\log^2 \frac{x}{m}} \quad \text{if } q_m \geq (\log x)^4.$$

By using Lemma 5, we obtain that

$$E(m \mid n_1, \dots, n_t) \ll \frac{x}{q_m m \log \frac{x}{m}}.$$

Summing over all m with $m > Y_x$, $Y_x \rightarrow \infty$ we obtain that

$$\sum_{q_m > Y_x} E(m \mid n_1, \dots, n_t) = o_x(1) N_{\mathcal{B}^*}(x).$$

Now we estimate $\sum_{\substack{m \in \mathcal{B}^* \\ q_m = Q}} |E(m \mid n_1, \dots, n_t)|$.

Let $m_1 < \dots < m_R$ be those numbers in $\mathcal{B}^*$ for which (5.3) holds with $q_m = Q$. Let

$$\nu = \min_{j=1, \dots, R-1} (m_{j+1} - m_j).$$

Then

$$\sum_{\substack{m \in \mathcal{B}^* \\ q_m = Q}} E(m \mid n_1, \dots, n_t) \ll \frac{x}{\delta_1 \nu},$$

which is $o_x(1) N_{\mathcal{B}^*}(x)$ if $\frac{\nu}{(\log x)^{1-\kappa}} \rightarrow \infty$ as $x \rightarrow \infty$. Let us consider the case where $\nu \leq Y_x (\log x)^{1-\kappa}$, $Y_x \rightarrow \infty$ slowly. Assume $\nu = m_{j+1} - m_j$. Then, from (5.3), applied for $m = m_j, m_{j+1}$ we have

$$\left| \gamma_{m_{j+1}} - \gamma_{m_j} - \frac{a_{m_{j+1}} - a_{m_j}}{Q} \right| < \frac{2}{Q\tau},$$

and so

$$\left| \nu\alpha - \frac{B}{Q} \right| < \frac{2}{Q\tau},$$

that is to say,

$$\min_{Q\nu \leq Y_x Q(\log x)^{1-\kappa}} \|Q\nu\alpha\| \leq \frac{2}{q\tau}.$$

This cannot hold if Condition A is satisfied.

Assume the fulfillment of Condition B. Let $\nu_j = m_j - m_1$. From (5.3) we obtain

$$|\gamma_{m_j} - \gamma_{m_1}| = \left| \nu_j\alpha - \frac{B_j}{Q} \right| < \frac{2}{Q\tau},$$

whence

$$\left| \alpha - \frac{B_j}{Q\nu_j} \right| < \frac{2}{Q\tau\nu_j}.$$

Since $\nu_j < x^{1-\delta_1}$, $\left| \frac{B_i}{Q\nu_i} - \frac{B_j}{Q\nu_j} \right|$ is either 0 or at least $\frac{1}{Q\nu_i\nu_j}$. But at the same time, it is at most $\frac{2}{Q\tau} \left(\frac{1}{\nu_i} + \frac{1}{\nu_j} \right)$, and this is impossible.

Let $\frac{B_i}{Q\nu_i} = \frac{U}{V}$ and $(U, V) = 1$. Then $B_i = \frac{UQ\nu_i}{V}$ is an integer for every i , and so $m_i \equiv m_1 \pmod{\frac{V}{QV}}$ for $i = 1, 2, \dots$

Since α is irrational, V cannot be bounded as $x \rightarrow \infty$. Then there is a function $Y_x \rightarrow \infty$ for which $\left| \alpha - \frac{U}{V} \right| < \frac{2}{Q\tau}$ does not hold if $V \leq Y_x$.

Assume that $V > g(x)(\log x)^{1-\kappa}$. Then

$$\sum_{\substack{m \leq x^{1-\delta_1} \\ m \equiv m_1 \pmod{V}}} \frac{x}{m \log \frac{x}{m}} \ll \frac{1}{\delta_1} \frac{x(\log x)}{(\log x)V} \ll \frac{1}{\delta_1 g(x)} N_{\mathcal{B}^*}(x).$$

Finally, it remains to consider the case when $Y_x \leq V \leq g(x)(\log x)^{1-\kappa}$. Hence we assume that Condition B is satisfied. Then

$$\begin{aligned} \sum_{Y_x \leq V \leq g(x)(\log x)^{1-\kappa}} \sum_{\substack{m \leq x^{1-\delta_1} \\ m \equiv m_1 \pmod{V} \\ m \in \mathcal{B}^*}} \frac{x}{m \log \frac{x}{m}} \\ \ll \frac{1}{\delta_1} \frac{x}{\log x} \sum_{Y_x \leq V \leq g(x)(\log x)} \max_{l \pmod{V}} M(x, v, l) \\ \ll o_x(1) N_{\mathcal{B}^*}(x). \end{aligned}$$

Collecting inequalities we obtain our Theorem 2.

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I. KÁTAI, COMPUTER ALGEBRA DEPT., EÖTVÖS LORÁND U., 1117 BUDAPEST, PÁZMÁNY PÉTER
SÉTÁNY I/C, HUNGARY
katali@compalg.inf.elte.hu