

ON THE SIZE OF THE INTERSECTION OF TWO LUCAS SEQUENCES OF DISTINCT TYPE

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Dedicated to Professor Paulo Ribenboim on his 80th birthday.

RÉSUMÉ. Soient a et b des entiers positifs avec a qui n'est pas un carré parfait et $b > 1$. On montre que si $\max\{a, b\} > 4.233 \cdot 10^{52}$, alors il y a au plus deux solutions entières positives (x, k) à l'équation diophantienne $x^2 - a \left(\frac{b^k - 1}{b - 1}\right)^2 = 1$. De plus, si a et b sont des entiers positifs quelconques, le nombre de solutions est au plus trois.

ABSTRACT. Let a and b be positive integers, with a not a perfect square and $b > 1$. It is proved that if $\max\{a, b\} > 4.233 \cdot 10^{52}$, then there are at most two positive integer solutions (x, k) to the Diophantine equation $x^2 - a \left(\frac{b^k - 1}{b - 1}\right)^2 = 1$. Moreover, for the remaining set of pairs of positive integers a, b , the number of solutions is at most three.

1. Introduction

The study of simultaneous Pell equations of the form

$$x^2 - ay^2 = 1 \quad \text{and} \quad z^2 - by^2 = 1,$$

along with generalizations thereof, has a long history and continues to receive considerable interest. It is well known that such systems of equations have finitely many solutions in positive integers (x, y, z) . What is quite impressive, however, is that upper bounds for the number of positive integer solutions have been proved for infinite families of generalized systems of simultaneous Pell equations. The bound obtained, for a fixed family of equations, is absolute and hence, does not depend on the coefficients of any particular equation in the family. For instance, for any fixed pair of nonsquare positive integers a and b , the above system of Pell equations is known to have at most two solutions in positive integers by a recent result of Bennett, Cipu, Mignotte and Okazaki [3].

An alternative viewpoint is that of intersecting two Lucas sequences. The set of positive integers y arising from solutions to the Pell equation $x^2 - ay^2 = 1$ is well known to be a Lucas sequence, and so the aforementioned result of Bennett *et al.* states that the intersection of two such Lucas sequences contains at most two positive integers.

In the present paper, we follow this line of thought in a slightly different direction. In particular, we consider the family of Lucas sequences of the form $\left\{\frac{b^k-1}{b-1}\right\}$, with $b > 1$ a fixed positive integer. We obtain an upper bound for the size of the intersection of such a Lucas sequence together with a Lucas sequence consisting of solutions y to the Pell equation $x^2 - ay^2 = 1$.

More precisely, let a and b denote positive integers, a not a perfect square and $b > 1$. We consider the number of positive integer solutions to

$$(1) \quad x^2 - a \left(\frac{b^k - 1}{b - 1} \right)^2 = 1.$$

This equation is evidently equivalent to the simultaneous Diophantine equations

$$(2) \quad x^2 - ay^2 = 1 \quad \text{and} \quad y = \frac{b^k - 1}{b - 1}.$$

Our main results are the following.

Theorem 1.1. *If $\max\{a, b\} > 4.233 \cdot 10^{52}$, then equation (1) has at most 2 positive integer solutions (x, k) .*

Theorem 1.2. *Equation (1) has at most 3 positive integer solutions (x, k) .*

The paper is organized as follows. In Section 2, we recall some properties related to Pell equations. Moreover, we particularly prove that, when $b = 2m$, then equation (1) has exactly one solution, except when $b = 2$ and $m = 2^u$, in which cases we have two solutions. See Propositions 2.3 and 2.4. In Sections 3 and 4, we use gap principles to ensure that the solutions of equation (1) don't cluster together. Theorem 1.1 is proved in Section 5 by means of a result of Matveev on linear forms in logarithms. We use the same method to show Theorem 1.2 in Section 6. Some remarks and a conjecture are made in the last section. We hope to completely solve the problem in another project.

2. Preliminary results

For a nonsquare positive integer a , we let (T_1, U_1) denote the smallest positive integer solution of the Pell equation

$$x^2 - ay^2 = 1,$$

and we define α to be

$$\alpha = T_1 + U_1\sqrt{a}.$$

For $j \geq 1$, the sequences $\{T_j\}$ and $\{U_j\}$ are defined by

$$\alpha^j = T_j + U_j\sqrt{a}.$$

For $k \geq 1$, we define the sequence $\{H_k\}$ by

$$H_k = \frac{b^k - 1}{b - 1}.$$

The main goal of this paper is to study the equation

$$(3) \quad U_j = H_k.$$

Note that (3) can alternatively be written in the form

$$(4) \quad y = \frac{\alpha^j - \alpha^{-j}}{2\sqrt{a}} = \frac{b^k - 1}{b - 1}.$$

We begin by stating some well known properties of the above sequences. See the related work done by Yuan in [10].

Lemma 2.1. (1) If $U_k \neq 1$, then $U_k \mid U_n$ if and only if $k \mid n$.

$$(2) \quad \gcd(U_i, U_j) = U_{\gcd(i,j)}.$$

$$(3) \quad T_k \mid T_n \text{ if and only if } n/k \text{ is an odd integer.}$$

$$(4) \quad \text{If } H_k \neq 1, \text{ then } H_k \mid H_n \text{ if and only if } k \mid n.$$

$$(5) \quad \gcd(H_k, H_l) = H_{\gcd(k,l)}.$$

$$(6) \quad U_{2mn \pm k} \equiv \pm U_k \pmod{U_n}.$$

$$(7) \quad H_{mn+k} \equiv H_k \pmod{H_n}.$$

Proof. The first five statements are well known properties of Lucas-Lehmer sequences. In order to confirm (6), notice that

$$U_{2mn \pm k} \mp U_k = 2T_{mn \pm k}U_{mn},$$

and hence from (1), one obtains $U_n \mid U_{mn}$, and thus

$$U_n \mid (U_{2mn \pm k} \mp U_k).$$

The congruence in (7) is a consequence of the relation $H_{mn+k} - H_k = b^k \cdot H_{mn}$. $\square$

We will henceforth assume that equation (3) has at least one positive integer solution, with the smallest one being given by

$$U_{j_1} = H_{k_1}.$$

Lemma 2.2. Let j, k be any positive solution of equation (3). Then j_1 divides j and k_1 divides k .

Proof. Assume that $j > j_1$ and $k > k_1$ are indices for which $U_j = H_k$. Let $g_1 = \gcd(j, j_1)$ and $g_2 = \gcd(k, k_1)$. Then the above lemma shows that

$$U_{g_1} = \gcd(U_{j_1}, U_j) = \gcd(H_{k_1}, H_k) = H_{g_2},$$

and so by the minimality of j_1 and k_1 , $g_1 = j_1$ and $k_1 = g_2$. From this, it follows that $j_1 \mid j$ and $k_1 \mid k$. $\square$

This lemma allows us to simplify the situation considerably by passing from an arbitrary nonsquare positive integer a , to one integer of the form $m^2 - 1$, for $m > 1$, as follows.

Let $\beta = \alpha^{j_1}$, $m = T_{j_1}$, $A = U_{j_1}^2 a = m^2 - 1$. For $j \geq 1$, put

$$\beta^j = t_j + u_j \sqrt{A}.$$

Then $u_j = U_{jj_1}/U_{j_1}$. Similarly, let $B = b^{k_1}$ and for $k \geq 1$, define

$$h_k = \frac{B^k - 1}{B - 1}.$$

Then $h_k = H_{kk_1}/H_{k_1}$. Thus, if j and k are integers for which $H_k = U_j$ then, on one hand, k_1 divides k and j_1 divides j . On the other hand, as $U_{j_1} = H_{k_1}$, we see that $U_j = H_k$ if and only if $U_j/U_{j_1} = H_k/H_{k_1}$. By the above definitions, this occurs if and only if $u_{j/j_1} = h_{k/k_1}$. The result of this analysis is that the number of solutions to $U_j = H_k$ is equal to the number of solutions to $u_j = h_k$. Hence, in order to get a bound for the number of solutions to (3), it is sufficient to consider the special case that a is of the form $m^2 - 1$.

Thus, we revert to the original notation, with the only change being that $a = m^2 - 1$. Note that this entails that $(j_1, k_1) = (1, 1)$. Also, (2) becomes

$$(5) \quad x^2 - (m^2 - 1)y^2 = 1 \quad \text{and} \quad y = \frac{b^k - 1}{b - 1}.$$

Let us deal first with the case $b = 2m$. The first few values of U_j are the following.

$$\begin{aligned} U_1 &= 1, & U_2 &= 2m, & U_3 &= 4m^2 - 1, & U_4 &= 8m^3 - 4m, \\ U_5 &= 16m^4 - 12m^2 + 1 & \text{and} & & U_6 &= 32m^5 - 32m^3 + 6m. \end{aligned}$$

Using the fact that $\{U_j\}$ is a linear recurrence, it follows that

$$U_j \equiv 0 \pmod{m} \quad \text{or} \quad U_j \equiv \pm 1 \pmod{m^2}.$$

The following result completely solves equation (5) for the case $b = 2m$, $m > 1$.

Proposition 2.3. *If $b = 2m$ with $m > 1$, then equation (5) has only the trivial solution $(x, y, k) = (m, 1, 1)$.*

Proof. Our goal is to prove that if $U_j = H_k$ and $b = 2m$, then $k < 2$, which in turn immediately implies that $k = 1$. We can exclude $k = 2$ because in this case, $H_k = b + 1 = 2m + 1$ and $2m + 1$ is not equal to U_j for any j , since $m > 1$. Hence, assume now that $k \geq 3$. Then we have

$$\begin{aligned} H_k &= (b^k - 1)/(b - 1) \\ &= b^{k-1} + \cdots + b^2 + b + 1 \\ &\equiv 2m + 1 \pmod{m^2}. \end{aligned}$$

As noted above,

$$U_j \equiv 0 \pmod{m} \quad \text{or} \quad U_j \equiv \pm 1 \pmod{m^2}.$$

The first of these three possibilities is obviously not possible. The second case where $U_j \equiv -1 \pmod{m^2}$ implies that $m = 2$ and $b = 4$. Similarly, the third case where $U_j \equiv 1 \pmod{m^2}$ implies that $m = 2$ and hence that $b = 4$. But one can see that

$$\begin{aligned} H_k &= (4^k - 1)/(4 - 1) \\ &= 4^{k-1} + \cdots + 4^2 + 4 + 1 \\ &\equiv 5 \pmod{16}, \end{aligned}$$

hence $j \equiv 1 \pmod{4}$. It is an easy exercise to show that if $j \equiv 1 \pmod{4}$, then $U_j \equiv 1 \pmod{16}$. This ends the proof of the proposition. $\square$

In the next result, we completely solve equation (5) for the special case $b = 2$.

Proposition 2.4. *Suppose $b = 2$. Then equation (5) has only the positive solution $(x, y, k) = (m, 1, 1)$, except only if $m = 2^u$ where u is a positive integer. In this case, equation (5) just has two solutions*

$$(x, y, k) = (2^u, 1, 1) \quad \text{and} \quad (2^{3u+2} - 3 \cdot 2^u, 2^{2u+2} - 1, 2u + 2).$$

Proof. Assume that (x, y, k) is a positive integer solution of equation (5). Then there exists a positive integer l such that $U_l = 2^k - 1$. Thus, l is odd since U_l is an even integer if l is even. So we consider

$$U_{2i+1} = 2^k - 1, \quad 0 \leq i \in \mathbb{Z}.$$

Since $U_1 = 1$, it follows that

$$\begin{aligned} U_{2i+1} &= mU_{2i} + T_{2i} \\ &= 2mT_iU_i + 2T_i^2 - 1 \end{aligned}$$

and so, because $U_{2i+1} = 2^k - 1$,

$$\begin{aligned} 2mT_iU_i + 2T_i^2 &= T_i(2mU_i + 2T_i) \\ &= 2^k. \end{aligned}$$

In particular, as T_i divides 2^k , $T_i = 2^v$ for some nonnegative integer v . This forces $i = 1$, and hence that $T_1 = 2^u$ with $u > 0$ and $U_{2i+1} = U_3$.

Therefore, the only solution (l, k) to $U_l = 2^k - 1$ is $l = k = 1$, unless $m = T_1 = 2^u$ for some integer $u > 0$. In this case, $U_3 = 2^{2u+2} - 1$ and $x = T_3 = 2^{3u+2} - 3 \cdot 2^u$ is the only other solution. Thus, this finishes the proof of the case $b = 2$. $\square$

We henceforth assume that equation (2) has at least three positive integer solutions (x_i, y_i, k_i) , with $1 = k_1 < k_2 < k_3$. Assume further that if another solution (x, y, k) exists, then $k_3 < k$. Thus, we obtain the following result.

Lemma 2.5. *There are positive integers p and $q > 1$ with the property that*

$$j_3 = pj_2 + \varepsilon, \quad k_3 = qk_2 + \varepsilon \quad \text{and} \quad \varepsilon \in \{0, 1\}.$$

Furthermore, if $\varepsilon = 1$, then $2 \mid p$.

Proof. If $j_2 \mid j_3$, then $k_2 \mid k_3$, and evidently the statements in the lemma hold. Otherwise, let $j_3 = 2p_1j_2 \pm r$ with $0 < r < j_2$ and $k_3 = qk_2 + s$ with $0 < s < k_2$ for some integers p, s, q, r . Using Lemma 2.2, we see that

$$\pm U_r \equiv H_s \pmod{U_{j_2}}.$$

Since $r < j_2$, it follows that $U_r < (1/2)U_{j_2}$, and similarly $H_s < (1/2)H_{k_2}$. Therefore, the terms in the above congruence are smaller than half of the modulus, forcing the equality $U_r = H_s$, and also that the sign in the congruence is positive. Therefore,

$$(r, s) = (j_1, k_1) = (1, 1), \quad j_3 = 2p_1j_2 + 1 \quad \text{and} \quad k_3 = qk_2 + 1.$$

To complete the proof, we only need to show that $q > 1$. Suppose that $q = 1$. Then we have

$$\begin{aligned} U_{j_3} - 1 &= H_{k_3} - 1 \\ &= H_{j_2+1} - 1 \\ &= bH_{k_2} \\ &= bU_{j_2}. \end{aligned}$$

But as

$$T_{j_2} > U_{j_2} = H_{k_2} \geq H_2 \geq b + 1 > b,$$

we see that

$$\begin{aligned} U_{j_3} - 1 &\geq U_{2j_2+1} - 1 \\ &= T_{j_2}U_{j_2} \\ &> bU_{j_2} \\ &= U_{j_3} - 1, \end{aligned}$$

which is a contradiction. Thus, the proof is done. $\square$

3. Gap principles

In this section, we apply some gap principles to the solutions of equation (1). Let

$$\Lambda_i = j_i \log \alpha - k_i \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right),$$

for a positive integer i . We will use the inequalities

$$\frac{t}{1+t} < \log(1+t) < t$$

and

$$t - \frac{t^2}{2} < \log(1+t) < t,$$

for $t > -1$. From (4), we get

$$\frac{b-1}{2\sqrt{a}} \cdot \frac{\alpha^{j_i}}{b^{k_i}} - 1 = \frac{1}{b^{k_i}} \left(\frac{b-1}{2\sqrt{a}} \alpha^{-j_i} - 1 \right).$$

Put $t_i = e^{\Lambda_i} - 1$, that is, $\Lambda_i = \log(1+t_i)$. For $i = 2, 3$, we have

$$b-1 < b+1 = H_2 \leq U_{j_i} = \frac{\alpha^{j_i} - \alpha^{-j_i}}{2\sqrt{a}} < \frac{\alpha^{j_i}}{2\sqrt{a}},$$

which yields

$$-1 < \frac{b-1}{2\sqrt{a}} \alpha^{-j_i} - 1 < \frac{1}{4a} - 1.$$

Thus, we obtain that

$$-\frac{1}{b^{k_i}} < t_i < \left(-1 + \frac{1}{4a} \right) \frac{1}{b^{k_i}}.$$

From the inequality $\log(1 + t_i) < t_i$, we have

$$\Lambda_i < \left(-1 + \frac{1}{4a}\right) \frac{1}{b^{k_i}}.$$

Notice that $-1 < t_i < 0$. From the inequalities

$$\begin{aligned} \log(1 + t_i) &> \frac{t_i}{1 + t_i} \\ &= 1 - \frac{1}{1 + t_i} \\ &> 1 - \frac{1}{1 - \frac{1}{b^{k_i}}} \\ &= -\frac{1}{b^{k_i} - 1}, \end{aligned}$$

we get

$$(6) \quad \left(1 - \frac{1}{4a}\right) \frac{1}{b^{k_i}} < -\Lambda_i < \frac{1}{b^{k_i} - 1}, \quad i = 2, 3.$$

The next result gives us bounds for U_j and H_k .

Lemma 3.1. *For $j, k > 1$, we have*

$$\alpha^{j-1} < U_j < (2m)^{j-1}$$

and

$$b^{k-1} < H_k < (b+1)^{k-1}.$$

Proof. We proceed by induction. First, notice that

$$U_2 = 2\sqrt{a+1} > \alpha = \sqrt{a+1} + \sqrt{a}.$$

Suppose now that

$$U_j = \frac{\alpha^j - \bar{\alpha}^j}{2\sqrt{a}} > \alpha^{j-1}.$$

Then we have

$$\begin{aligned} U_{j+1} &= \frac{\alpha^{j+1} - \bar{\alpha}^{j+1}}{2\sqrt{a}} \\ &= \frac{\alpha \cdot \alpha^j - \alpha \cdot \bar{\alpha}^j - \bar{\alpha} \cdot \bar{\alpha}^j + \alpha \cdot \bar{\alpha}^j}{2\sqrt{a}} \\ &= \alpha \cdot \frac{\alpha^j - \bar{\alpha}^j}{2\sqrt{a}} + \frac{\bar{\alpha}^j(\alpha - \bar{\alpha})}{2\sqrt{a}} \\ &> \alpha^j + \frac{\bar{\alpha}^j(\alpha - \bar{\alpha})}{2\sqrt{a}} \\ &> \alpha^j. \end{aligned}$$

For the upper bound for U_j , we obtain the result by induction from the fact that

$$U_j = \begin{cases} j & \text{if } j = 0, 1, \\ 2mU_{j-1} - U_{j-2} < 2mU_{j-1} & \text{if } j > 1. \end{cases}$$

If we rewrite

$$H_k = b^{k-1} + b^{k-2} + \cdots + 1,$$

we get

$$b^{k-1} < H_k < (b+1)^{k-1}.$$

So the lemma is proved. $\square$

We will apply gap principles for separate cases by comparing the size of b and $2m$. That is what we will do in the next lemmas.

Lemma 3.2. *For $i = 2, 3$, we have*

$$(7) \quad \begin{cases} j_i \leq k_i, p < q & \text{if } b \leq 2m - 1, \\ j_i > k_i, p > q & \text{if } b \geq 2m + 1. \end{cases}$$

Moreover, $j_i = k_i$ if and only if $j_2 = k_2 = 2$ and $b = 2m - 1$.

Proof. First, let us determine some relations between j_i and k_i . If $b \leq 2m - 1$, then by Lemma 3.1,

$$(2m - 1)^{j_i-1} < \alpha^{j_i-1} < y_i < b^{k_i} \leq (2m - 1)^{k_i}.$$

Therefore, $j_i - 1 < k_i$ implies $j_i \leq k_i$. If $b \geq 2m + 1$, then by Lemma 3.1,

$$(2m + 1)^{k_i-1} \leq b^{k_i-1} < y_i < (2m)^{j_i-1} < (2m + 1)^{j_i-1},$$

and thus $j_i > k_i$.

Assume that $j_i = k_i = n \geq 2$ for some $2 \leq i \leq 3$, in the case $b \leq 2m - 1$. This implies $U_n = H_n$. If $n = 2$, then $U_2 = b + 1$ and $H_2 = 2m$, hence $b = 2m - 1$. But for $n \geq 3$, $U_n > H_n$, which we prove by induction. This is evidently true when $n = 3$, since $U_3 = 4m^2 - 1$ and

$$\begin{aligned} H_3 &= b^2 + b + 1 \\ &\leq (2m - 1)^2 + (2m - 1) + 1 \\ &= 4m^2 - 2m + 1. \end{aligned}$$

Notice that the sequences $\{U_n\}$ and $\{H_n\}$ satisfy $U_{n+1} = 2mU_n - U_{n-1}$, $n > 1$ and $H_{n+1} = bH_n + 1$, respectively. Thus we have

$$U_{n+1} > 2mU_n - U_n \geq bU_n \geq b(H_n + 1) > H_{n+1}.$$

Next, we consider the following linear forms in logarithms

$$\begin{aligned} \Lambda_2 &= j_2 \log \alpha - k_2 \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right), \\ \Lambda_3 &= j_3 \log \alpha - k_3 \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right), \end{aligned}$$

and the equations

$$j_3 = pj_2 + \varepsilon \quad \text{and} \quad k_3 = qk_2 + \varepsilon, \quad \varepsilon \in \{0, 1\}.$$

Then we obtain

$$(8) \quad (p - q)j_2 \log \alpha = \Lambda_3 - q\Lambda_2 - \varepsilon \log \frac{\alpha}{b} - (q - 1) \log \left(\frac{2\sqrt{a}}{b-1} \right).$$

(i) *Case* $b \leq 2m - 1$. Since $\Lambda_3 < 0$ and $b \leq 2m - 1 < \alpha$, we obtain, from equation (8), that

$$(p - q)j_2 \log \alpha < -q\Lambda_2 - (q - 1) \log \left(\frac{2\sqrt{a}}{b - 1} \right).$$

Notice that

$$\frac{2\sqrt{a}}{b - 1} = \frac{\sqrt{4m^2 - 4}}{b - 1} \geq \frac{\sqrt{b^2 + 2b - 3}}{b - 1} = \sqrt{\frac{b + 3}{b - 1}}.$$

Thus we have

$$\begin{aligned} \log \left(\frac{2\sqrt{a}}{b - 1} \right) &\geq \frac{1}{2} \log \left(1 + \frac{4}{b - 1} \right) \\ &> \frac{2}{b - 1} - \frac{4}{(b - 1)^2} \\ &= \frac{2b - 6}{(b - 1)^2}. \end{aligned}$$

From the above relations and equation (6), we get

$$\begin{aligned} (p - q)j_2 \log \alpha &< \frac{q}{b^{k_2} - 1} - (q - 1) \frac{2b - 6}{(b - 1)^2} \\ &\leq \frac{q}{b^2 - 1} - (q - 1) \frac{2b - 6}{(b - 1)^2}. \end{aligned}$$

The right hand side of the inequality is negative except when $k_2 = 2$ and $b = 2, 3$. We know that

$$b + 1 = H_{k_2} = U_{j_2} \geq U_2 = 2m.$$

Since $b \leq 2m - 1$, we obtain $b = 2m - 1$. Therefore, in the cases $k_2 = 2$ and $b = 2, 3$, this implies that $b = 3$ and $m = 2$. Again, we have

$$\begin{aligned} (p - q)j_2 \log \alpha &< \frac{2}{b^{k_2} - 1} - \frac{1}{2} \log \left(1 + \frac{4}{b - 1} \right) \\ &= \frac{2}{3^2 - 1} - \frac{1}{2} \log 3 \\ &< 0. \end{aligned}$$

Then we get $p < q$ for $b \neq 3$.

(ii) *Case* $b \geq 2m + 1$. Since

$$b \geq 2m + 1 > \alpha \quad \text{and} \quad b - 1 \geq 2m > 2\sqrt{a},$$

using equation (8), we get

$$(p - q)j_2 \log \alpha > \Lambda_3 - q\Lambda_2.$$

From equation (6), we have

$$\begin{aligned}
 \Lambda_3 - q\Lambda_2 &> \Lambda_3 - \Lambda_2 \\
 &> -\frac{1}{b^{k_3} - 1} + \frac{1}{b^{k_2}} \left(1 - \frac{1}{4a}\right) \\
 &> -\frac{2}{b^{2k_2}} + \frac{1}{2b^{k_2}} \\
 &> 0.
 \end{aligned}$$

Thus we obtain $p > q$.

This completes the proof of the lemma. $\square$

Lemma 3.3. *If $\varepsilon = 0$, then we have*

$$(9) \quad \begin{cases} k_3 > k_2 \alpha^{(j_2-1)/2} & \text{if } b \leq 2m - 1, \\ j_3 > j_2 b^{(k_2-1)/2} & \text{if } b \geq 2m + 1. \end{cases}$$

Moreover, $k_3 > 2m$ for $j_2 = 2$ in the first case.

Proof. If $\varepsilon = 0$, then $j_3 = pj_2$ and $k_3 = qk_2$. As

$$\begin{aligned}
 y_3 &= U_{pj_2} \\
 &= \frac{(x_1 + y_1\sqrt{a})^{pj_2} - (x_1 - y_1\sqrt{a})^{pj_2}}{2\sqrt{a}} \\
 &= \frac{(x_2 + y_2\sqrt{a})^p - (x_2 - y_2\sqrt{a})^p}{2\sqrt{a}} \\
 &= \sum_{t=0}^{[(p-1)/2]} \binom{p}{2t+1} x_2^{p-2t-1} y_2^{2t+1} a^t \\
 &\equiv p x_2^{p-1} y_2 \pmod{ay_2^3}
 \end{aligned}$$

and $x_2^2 \equiv 1 \pmod{ay_2^2}$, we have that

$$(10) \quad \left(\frac{y_3}{y_2}\right)^2 \equiv p^2 x_2^{2p-2} \equiv p^2 \pmod{ay_2^2}.$$

Moreover,

$$(11) \quad \frac{y_3}{y_2} = \frac{b^{qk_2} - 1}{b^{k_2} - 1} = \sum_{t=0}^{q-1} \binom{q}{t} b^{kt} \equiv \sum_{t=0}^{q-1} 1 = q \pmod{b^{k_2} - 1},$$

from which we deduce that

$$p^2 \equiv q^2 \pmod{y_2}.$$

By Lemma 3.2, $p \neq q$ and therefore, $\max\{p, q\} \geq \sqrt{y_2}$. If $b \leq 2m - 1$, then from Lemma 3.2, we have $p < q$. Hence, we get

$$k_3 = qk_2 \geq k_2 \sqrt{y_2} \geq k_2 \alpha^{(j_2-1)/2}.$$

If $b \geq 2m - 1$, then from Lemma 3.2, we have $p > q$. Then, we obtain

$$j_3 = pj_2 \geq j_2 \sqrt{y_2} \geq j_2 b^{(k_2-1)/2}.$$

If $j_2 = 2$, then $px_2^{p-1} \equiv q \pmod{y_2}$. Since we have $x_2 = T_2 = 2m^2 - 1$ and $y_2 = U_2 = 2m$, we get

$$p(-1)^{p-1} \equiv q \pmod{2m}.$$

Then $p + q \geq 2m$ and as $p < q$, we get $q > m$. This implies that $k_3 = qk_2 > k_2m$. Therefore, the result is obtained. $\square$

Lemma 3.4. We have

$$(12) \quad \begin{cases} mp > bq & \text{if } b < m/2, \\ mp < bq & \text{if } b \geq 2m + 1. \end{cases}$$

Proof. Let us consider that

$$\Lambda_3 = (pj_2 + \varepsilon) \log \alpha - (qk_2 + \varepsilon) \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right).$$

(i) *Case $b < m/2$.* This implies $b - 1 < 2\sqrt{a}$. We then obtain that

$$\begin{aligned} qk_2 \log b &= pj_2 \log \alpha + \varepsilon \log \frac{\alpha}{b} - \log \left(\frac{2\sqrt{a}}{b-1} \right) - \Lambda_3 \\ &< pj_2 \log \alpha + \varepsilon \log \frac{\alpha}{b} - \Lambda_3 \\ &< pj_2 \log \alpha + \varepsilon \log \frac{\alpha}{b} + \frac{1}{b^{k_3} - 1}. \end{aligned}$$

From Lemma 3.1 and equation (4), it follows that $\alpha^{j_2-1} < y_2 < b^{k_2}$ whence

$$(j_2 - 1) \log \alpha < k_2 \log b,$$

and we obtain the inequality

$$q(j_2 - 1) \log \alpha < pj_2 \log \alpha + \varepsilon \log \frac{\alpha}{b} + \frac{1}{b^{k_3} - 1}.$$

This inequality implies that

$$\begin{aligned} q(j_2 - 1) &< pj_2 + \left(\varepsilon \log \frac{\alpha}{b} + \frac{1}{b^{k_3} - 1} \right) / \log \alpha \\ &< pj_2 + 1, \end{aligned}$$

and so $q(j_2 - 1) \leq pj_2$. Since $j_2 > 1$, we have $q \leq 2p$ and therefore, $mp > 2bp \geq bq$.

(ii) *Case $b \geq 2m + 1$.* This implies $b - 1 > 2\sqrt{a}$. We use a similar method. From the definition of Λ_3 ,

$$\begin{aligned} pj_2 \log \alpha &< qk_2 \log b + \varepsilon \log \frac{b}{\alpha} + \log \left(\frac{2\sqrt{a}}{b-1} \right) + \Lambda_3 \\ &< qk_2 \log b + \varepsilon \log \frac{b}{\alpha}. \end{aligned}$$

From the relation $b^{k_2-1} < y_2 < \alpha^{j_2}$, we have

$$(k_2 - 1) \log b < j_2 \log \alpha.$$

Thus, we get

$$p(k_2 - 1) \log b < qk_2 \log b + \varepsilon \log \frac{b}{\alpha},$$

from which it follows that

$$p(k_2 - 1) < qk_2 + \varepsilon \left(1 - \frac{\log \alpha}{\log b}\right) < qk_2 + 1.$$

From this inequality, we conclude that $p(k_2 - 1) \leq qk_2$. Since $k_2 > 1$, we get $p \leq 2q$ and therefore, $bq \geq (2m + 1)q > 2mq > mp$. The proof is now completed. $\square$

Lemma 3.5. *If $\varepsilon = 1$, then we have*

$$(13) \quad \begin{cases} j_3 > j_2 \alpha^{(j_2-1)/2}/m & \text{and} & k_3 > k_2 \alpha^{(j_2-1)/2}/m & \text{if } b < m/2, \\ j_3 > j_2 b^{(k_2-3)/2} & \text{and} & k_3 > k_2 b^{(k_2-3)/2} & \text{if } b \geq 2m + 1. \end{cases}$$

Proof. If $\varepsilon = 1$, then $j_3 = pj_2 + 1$, $k_3 = qk_2 + 1$, and $2 \mid p$. Put $p = 2p_1$. We have

$$\begin{aligned} T_{j_3} + U_{j_3} \sqrt{a} &= (x_1 + y_1 \sqrt{a})^{2p_1 j_2 + 1} \\ &= (x_1 + y_1 \sqrt{a}) (x_2 + y_2 \sqrt{a})^{2p_1} \\ &= (x_1 + y_1 \sqrt{a}) (x_2^2 + ay_2^2 + 2x_2 y_2 \sqrt{a})^{p_1} \\ &\equiv (x_1 + y_1 \sqrt{a}) (1 + 2x_2 y_2 \sqrt{a})^{p_1} \\ &\equiv (x_1 + y_1 \sqrt{a}) (1 + 2p_1 x_2 y_2 \sqrt{a}) \pmod{ay_2^2}. \end{aligned}$$

Notice that $(x_1, y_1) = (m, 1)$. We get

$$\begin{aligned} y_3 &= U_{j_3} \\ &\equiv 2p_1 m x_2 y_2 + 1 \\ &= pm x_2 y_2 + 1 \pmod{ay_2^2}. \end{aligned}$$

Thus,

$$((y_3 - 1)/y_2)^2 \equiv p^2 m^2 \pmod{ay_2}.$$

Also, from the relation

$$y_3 - 1 = \frac{b^{qk_2+1} - 1}{b - 1} - 1 = b \frac{b^{qk_2} - 1}{b - 1},$$

it follows that

$$((y_3 - y_1)/y_2)^2 \equiv q^2 b^2 \pmod{y_2}.$$

We deduce that

$$m^2 p^2 \equiv b^2 q^2 \pmod{y_2}.$$

If $b < m/2$ or $b \geq 2m + 1$, then Lemma 3.4 yields $mp \neq bq$ and $\max\{mp, bq\} > \sqrt{y_2}$. If $b < m/2$, then Lemma 3.2 and Lemma 3.4 give that $p < q$ and $mp > bq$. Using this, we obtain the inequalities

$$j_3 > pj_2 > j_2 \sqrt{y_2}/m > j_2 \alpha^{(j_2-1)/2}/m$$

and

$$k_3 > qk_2 > pk_2 > k_2 \sqrt{y_2}/m > k_2 \alpha^{(j_2-1)/2}/m.$$

If $b \geq 2m + 1$, then by Lemma 3.2 and Lemma 3.4, it follows that $p > q$ and $mp < bq$. Therefore,

$$k_3 > qk_2 > k_2 \sqrt{y_2}/b > k_2 b^{(k_2-3)/2}$$

and

$$j_3 > pj_2 > qj_2 > j_2\sqrt{y_2}/b > j_2b^{(k_2-3)/2}.$$

This ends the proof. $\square$

Now, we study the remaining case, b and m satisfying the inequality

$$m/2 \leq b \leq 2m - 1.$$

We will assume that $b \geq 3$. Recall the property (6), for $i = 2, 3$, that is,

$$\frac{1}{b^{k_i}} \left(1 - \frac{1}{4a}\right) < -\Lambda_i < \frac{1}{b^{k_i} - 1}.$$

Since $b \leq 2m - 1$, we have $b < 2\sqrt{a} < \alpha$. From

$$\Lambda_1 = \log \alpha - \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right),$$

it follows that

$$\begin{aligned} -\Lambda_1 &= \log \left(\frac{2\sqrt{a}}{\alpha} \right) + \log \left(\frac{b}{b-1} \right) \\ &< \log \left(\frac{b}{b-1} \right) \\ &< \frac{1}{b-1}. \end{aligned}$$

Moreover,

$$\alpha - 2\sqrt{a} = m - \sqrt{m^2 - 1} = \frac{1}{\alpha}.$$

Therefore,

$$\begin{aligned} -\Lambda_1 &= \log \left(\frac{2\sqrt{a}}{\alpha} \right) + \log \left(\frac{b}{b-1} \right) \\ &= -\log \left(1 + \frac{\alpha - 2\sqrt{a}}{2\sqrt{a}} \right) + \log \left(1 + \frac{1}{b-1} \right) \\ &= -\log \left(1 + \frac{1}{2\sqrt{a}\alpha} \right) + \log \left(1 + \frac{1}{b-1} \right) \\ &> -\frac{1}{2\sqrt{a}\alpha} + \frac{1}{b-1} - \frac{1}{2(b-1)^2} \\ &> -\frac{1}{b^2} + \frac{1}{b-1} - \frac{1}{2(b-1)^2}. \end{aligned}$$

If $b \geq 3$, then

$$-\Lambda_1 > -\frac{1}{b^2} + \frac{1}{b-1} - \frac{1}{2(b-1)^2} > \frac{1}{b^2-1} \geq \frac{1}{b^{k_2}-1} > -\Lambda_2.$$

Therefore, for $3 < b \leq 2m - 1$, we have

$$(14) \quad 0 < -\Lambda_3 < -\Lambda_2 < -\Lambda_1 < \frac{1}{b-1}.$$

It is easy to see that the above inequality is also true if (j_i, k_i) , $i = 1, 2, 3$, is any of the three solutions of equation (4) with $j_1 < j_2 < j_3$.

The idea of the following result comes from [3].

Lemma 3.6. *Let (j_i, k_i) , $i = 1, 2, 3$, be any three positive solutions of equation (4). Suppose that the positive integers m and b satisfy $b^{k_1} > 5$ and $b \leq 2m - 1$. Then we have*

$$j_3 - j_2 \geq R(b^{k_1} - 1) \log b,$$

where

$$R = \begin{vmatrix} k_3 - k_2 & k_3 - k_1 \\ j_3 - j_2 & j_3 - j_1 \end{vmatrix} \in \mathbb{N}.$$

Proof. From our suppositions, we have 3 points

$$(t_i, u_i) = (j_i \log \alpha, k_i \log b), \quad i = 1, 2, 3,$$

on the curve

$$(15) \quad e^u - 1 = \frac{b-1}{\sqrt{a}} \sinh(t), \quad t \geq \log \alpha.$$

Therefore, we write

$$u = \log \left(1 + \frac{b-1}{\sqrt{a}} \sinh(t) \right).$$

We note that since $t \geq \log \alpha$, we have $u \geq \log b$. We implicitly differentiate (15) to find that

$$(16) \quad \frac{du}{dt} e^u = \frac{b-1}{\sqrt{a}} \cosh(t).$$

From (15), we deduce that $\frac{du}{dt} > 0$. Since $a = m^2 - 1$ and $b \leq 2m - 1$, we have

$$\frac{(b-1)^2}{a} \leq \frac{(2m-2)^2}{(m^2-1)} < 4.$$

Notice that

$$e^u \geq e^{u_1} = b^{k_1} \geq b > 5.$$

By (15) and (16),

$$\begin{aligned} \left(\frac{du}{dt} \right)^2 &= \frac{1}{e^{2u}} \cdot \frac{(b-1)^2}{a} \cosh(t)^2 \\ &= \frac{1}{e^{2u}} \cdot \frac{(b-1)^2}{a} (\sinh(t)^2 + 1) \\ &= \left(1 - \frac{1}{e^u} \right)^2 + \frac{1}{e^{2u}} \cdot \frac{(b-1)^2}{a} \\ &= 1 + \frac{1}{e^{2u}} \left(1 + \frac{(b-1)^2}{a} - 2e^u \right) \\ &< 1, \end{aligned}$$

which implies that

$$(17) \quad 0 < \frac{du}{dt} < 1.$$

Similarly, the implicit differentiation of (16) yields

$$(18) \quad \frac{d^2u}{dt^2}e^u + \left(\frac{du}{dt}\right)^2 e^u = \frac{b-1}{\sqrt{a}} \sinh(t) = e^u - 1.$$

Therefore,

$$\begin{aligned} \frac{d^2u}{dt^2} &= 1 - \frac{1}{e^u} - \left(\frac{du}{dt}\right)^2 \\ &= -\frac{1}{e^u} - \frac{1}{e^{2u}} \left(1 + \frac{(b-1)^2}{a} - 2e^u\right) \\ &= \frac{1}{e^{2u}} \left(e^u - 1 - \frac{(b-1)^2}{a}\right) \\ &> \frac{1}{e^{2u}}(b-5) \\ (19) \quad &> 0. \end{aligned}$$

The Mean Value Theorem, together with (17) and (19), implies that

$$\begin{aligned} 0 &< \frac{u_3 - u_2}{t_3 - t_2} - \frac{u_3 - u_1}{t_3 - t_1} \\ &< 1 - \frac{u_3 - u_1}{t_3 - t_1} \\ &= \frac{\Lambda_3 - \Lambda_1}{t_3 - t_1} \\ &< \frac{-\Lambda_1}{t_3 - t_1} \\ &< \frac{1}{(b^{k_1} - 1)(t_3 - t_1)}, \end{aligned}$$

and it follows that

$$0 < \frac{k_3 - k_2}{j_3 - j_2} - \frac{k_3 - k_1}{j_3 - j_1} < \frac{1}{(j_3 - j_1)(b^{k_1} - 1) \log b}.$$

Since

$$R = \begin{vmatrix} k_3 - k_2 & k_3 - k_1 \\ j_3 - j_2 & j_3 - j_1 \end{vmatrix},$$

we have that R is a positive integer and thus, $j_3 - j_2 > R(b^{k_1} - 1) \log b$. Therefore, we get the result. $\square$

4. Small j_2 or k_2

In this section, we consider the small values of j_2 or k_2 .

Lemma 4.1. *Suppose $\varepsilon = 1$ and $m \geq 5$. If $k_2 = 2$, then*

$$(20) \quad \begin{cases} k_3 \geq b + 1 & \text{if } j_2 = 2, \\ j_3 > k_3 \geq 0.6b & \text{if } j_2 = 3, \\ j_3 > 7.6b^{2/3} & \text{if } j_2 = 4, \\ j_3 > 4.65b^{1/2} & \text{if } j_2 = 5, \\ j_3 > 192b^{2/5} & \text{if } j_2 = 6, \\ j_3 \geq 1.97j_2b^{\frac{j_2-3}{2j_2-2}} & \text{if } j_2 \geq 7. \end{cases}$$

Proof. We have $y_2 = b + 1 = U_{j_2}$.

(1) If $j_2 = 2$, then $b = 2m - 1$. Since $U_n > H_n$, we have that if $n > 2$, then $j_3 < k_3$ for

$$j_2 = k_2 = 2 \pmod{2m - 2}$$

as well as

$$U_j \equiv j \pmod{2m - 2} \quad \text{and} \quad H_k \equiv k \pmod{2m - 2}.$$

Therefore,

$$j_3 \equiv k_3 \pmod{2m - 2},$$

and hence

$$k_3 > j_3 + 2m - 2 > 2m = b + 1.$$

(2) Suppose $j_2 = 3$. Then

$$y_2 = U_3 = 4m^2 - 1 = b + 1,$$

so $b = 4m^2 - 2$. Reducing modulo $4m^2 - 3$, we get a sequence having period 12 and

$$U_j \equiv 0, \pm 1, \pm 2, \pm 2m \pmod{4m^2 - 3}.$$

Combining $H_{k_3} \equiv k_3 \pmod{b - 1}$, we have

$$k_3 + 2m \geq 4m^3 - 3,$$

that is,

$$k_3 \geq 4m^2 - 2m - 3.$$

When $m \geq 2$, we get

$$4m^2 - 2m - 3 > 0.6(4m^2 - 1).$$

Thus,

$$j_3 > k_3 > 0.6(b + 1) > 0.6b.$$

(3) Suppose $j_2 = 4$. Then

$$y_2 = U_4 = 8m^3 - 4m = b + 1 \quad \text{and} \quad x_2 = T_4 = 8m^4 - 8m^2 + 1.$$

Since $\varepsilon = 1$, in the proof of Lemma 3.5,

$$y_3 \equiv pmx_2y_2 + 1 \pmod{ay_2^2},$$

and we obtain

$$(21) \quad \frac{y_3 - 1}{y_2} \equiv pmx_2 \equiv p(8m^5 - 8m^3 + m) \equiv -mp \pmod{U_4}.$$

Moreover,

$$\frac{y_3 - 1}{y_2} = b \frac{b^{qk_2} - 1}{b^{k_2} - 1} = b \frac{b^{2q} - 1}{b^2 - 1} \equiv bq \equiv -q \pmod{b + 1}.$$

Thus,

$$mp \equiv q \pmod{y_2}.$$

Since $b = U_4 - 1 > 2m + 1$, Lemma 3.2 shows that $p > q$. Therefore, $p > y_2/m$.

When $m \geq 2$, we get

$$y_2 = b + 1 = 8m^3 - 4m \geq 7m^3,$$

and so

$$j_3 > pj_2 > j_2y_2/(y_2/7)^{1/3} \geq 1.9j_2(b + 1)^{2/3} > 7.6b^{2/3}.$$

(4) Suppose $j_2 = 5$. Then

$$y_2 = U_5 = 16m^4 - 12m^2 + 1 = b + 1$$

and

$$x_2 = T_5 = 16m^5 - 20m^3 + 5m.$$

In this case, we have that

$$\begin{aligned} -q &\equiv pmx_2 \\ &\equiv 2p_1m(16m^5 - 20m^3 + 5m) \\ &\equiv -p_1(4m^2 - 1) \pmod{y_2} \end{aligned}$$

and hence

$$(22) \quad p_1(4m^2 - 1) \equiv q \pmod{y_2}.$$

Since $2p_1 = p > q$, $p > 2y_2/(4m^2 - 1)$ and $m \geq 2$,

$$(4m^2 - 1)^2 < 1.08(16m^4 - 12m^2 + 1) = 1.08y_2.$$

Therefore,

$$j_3 > pj_2 > 2j_2y_2/(4m^2 - 1) > 0.93j_2y_2^{1/2} > 4.65b^{1/2}.$$

(5) Suppose $j_2 = 6$. Then

$$y_2 = U_6 = 32m^5 - 32m^3 + 6m$$

and

$$x_2 = T_6 = 32m^6 - 48m^4 + 18m^2 - 1.$$

Using a similar method as above, we deduce the congruence

$$p(4m^3 - 2m) \equiv q \pmod{y_2}.$$

When $m \geq 2$, we get $4m - 2 < y_2^{3/5}/32$. Therefore,

$$j_3 > pj_2 > 2j_2y_2/(4m^3 - 2m) > 32j_2y_2^{2/5} > 192b^{2/5}.$$

(6) Suppose $j_2 \geq 7$. Since $\varepsilon = 1$, by the same method used in the proof of Lemma 3.5, we get

$$(23) \quad m^2p^2 \equiv b^2q^2 \pmod{y_2}.$$

As $y_2 = b + 1$, we have

$$m^2p^2 \equiv q^2 \pmod{y_2},$$

and we remark that $b + 1 = U_{j_2} \geq U_6 > 2m + 1$. From Lemma 3.2, $p > q$, and thus $m^2p^2 > y_2$. The inequalities

$$b + 1 = U_{j_2} > \alpha^{j_2-1} > (1.97m)^{j_2-1}$$

imply

$$m < \frac{1}{1.97}(b + 1)^{\frac{1}{j_2-1}}.$$

Therefore,

$$p > \frac{\sqrt{y_2}}{m} > 1.97(b + 1)^{\frac{1}{2} - \frac{1}{j_2-1}}$$

and

$$j_3 = pj_2 + \varepsilon \geq pj_2 > 1.97j_2b^{\frac{j_2-3}{2j_2-2}}.$$

This completes the proof. $\square$

Lemma 4.2. Suppose $\varepsilon = 1$ and $m \geq 5$. If $k_2 = 3$, then $j_2 \geq 7$ is odd, and we have

$$(24) \quad j_3 > 0.49(j_2 - 1)(2.12b^{\frac{1}{2}} - 1).$$

Proof. We have

$$y_2 = H_3 = b^2 + b + 1 = U_{j_2}.$$

Since y_2 is odd, from the expression of U_j , j_2 is also odd. If $j_2 = 3$, then from

$$y_2 = 4m^2 - 1 = b^2 + b + 1,$$

it follows that $7 = 16m^2 - (2b + 1)^2$. This implies that $m = b = 1$, which is not possible. If $j_2 = 5$, then from

$$y_2 = 16m^4 - 12m^2 + 1 = b^2 + b + 1,$$

it follows that $8 = (8m^2 - 3)^2 - (2b + 1)^2$. So we obtain $m = 0$ and $b = 0$, which is impossible. Thus $j_2 \geq 7$. Notice that if $y_2 = b^2 + b + 1$, then by the same method as in the proof of Lemma 3.5,

$$(25) \quad m^2p^2 \equiv b^2q^2 \equiv -(b + 1)q^2 \pmod{y_2}.$$

Now, from (25), we deduce that $m^2p^2 + (b + 1)q^2 \geq y_2$.

If $(b + 1)q^2 > m^2p^2$, then $(b + 1)q^2 > y_2/2$. Thus

$$q > \sqrt{\frac{b^2 + b + 1}{2(b + 1)}} > \frac{b^{\frac{1}{2}}}{\sqrt{2}}.$$

Therefore, we have

$$k_3 \geq qk_2 = 3q \geq 2.12b^{\frac{1}{2}}.$$

From Lemma 3.1, we have $(2m)^{j_3-1} > b^{k_3-1}$. Hence,

$$j_3 > (k_3 - 1) \left(\frac{\log b}{\log(2m)} \right) - 1,$$

and from the fact that

$$(b+1)^2 > b^2 + b + 1 = U_{j_2} > \alpha^{j_2-1} > (1.97m)^{j_2-1},$$

we get the inequality $b > (1.97m)^{\frac{j_2-1}{2}} - 1$. Therefore,

$$\begin{aligned} j_3 &> (k_3 - 1) \frac{\log b}{\log(2m)} \\ &\geq (2.12b^{\frac{1}{2}} - 1) \left(\frac{\log \left((1.97m)^{\frac{j_2-1}{2}} - 1 \right)}{\log(2m)} \right). \end{aligned}$$

If $m \geq 2$ and $d \geq 3$, then

$$(1.97m)^d - 1 > (1.95m)^d \quad \text{and} \quad \log(1.95m) > 0.98 \log(2m),$$

and therefore we obtain

$$j_3 > 0.49(j_2 - 1)(2.12b^{\frac{1}{2}} - 1).$$

If $m^2 p^2 \geq (b+1)q^2$, then $m^2 p^2 \geq y_2/2$. Since

$$y_2 = U_{j_2} > \alpha^{j_2-1} > (1.97m)^{j_2-1},$$

we get

$$m < \frac{1}{1.97} y_2^{\frac{1}{j_2-1}}.$$

Thus

$$\begin{aligned} j_3 &= pj_2 + \varepsilon \\ &\geq pj_2 \\ &> \frac{1.97}{\sqrt{2}} j_2 y_2^{\frac{1}{2} - \frac{1}{j_2-1}} \\ &\geq 1.39 j_2 (b^2 + b + 1)^{\frac{1}{2} - \frac{1}{j_2-1}} \\ &> 0.49(j_2 - 1)(2.12b^{\frac{1}{2}} - 1). \end{aligned}$$

Therefore, the result is obtained. □

Lemma 4.3. *If $\varepsilon = 1$, $m \geq 5$ and $j_2 = 2$, then $2 \mid k_2$ and*

$$(26) \quad k_3 \geq 2k_2m.$$

Proof. If $j_2 = 2$, then

$$y_2 = U_2 = 2m = H_{k_2}$$

is even. Since

$$y_2 = 1 + b + \dots + b^{k_2-1},$$

we have that k_2 must be even. Since $\varepsilon = 1$, by Lemma 2.5, we have

$$j_3 = pj_2 + 1, \quad k_3 = qk_2 + 1 \quad \text{and} \quad 2 \mid p.$$

Using the method in the proof of Lemma 3.5, it follows that

$$y_3 = U_{j_3} \equiv 2p_1mx_2y_2 + 1 \pmod{ay_2^2}.$$

Thus,

$$(y_3 - 1)/y_2 \equiv 2p_1mx_2 \pmod{ay_2^2}.$$

Moreover,

$$\begin{aligned} (y_3 - 1)/y_2 &= b(b^{qk_2} - 1)/(b^{k_2} - 1) \\ &\equiv bq \pmod{(b^{k_2} - 1)}. \end{aligned}$$

We see therefore that

$$(27) \quad bq \equiv 2p_1mx_2 \equiv 0 \pmod{y_2}.$$

It follows that $y_2 \mid bq$, and since

$$\gcd(b, y_2) = (b, 1 + b + b^2 + \cdots + b^{k_2-1}) = 1,$$

we see that $y_2 \mid q$. This implies that $q \geq y_2$. Therefore,

$$k_3 \geq qk_2 \geq k_2y_2 \geq 2k_2m.$$

This ends the proof of the lemma. $\square$

Lemma 4.4. *If $\varepsilon = 1$, $m \geq 5$, $j_2 = 3$ and $b < m/2$, then $k_2 \geq 5$ and*

$$(28) \quad k_3 > 0.76k_2(4m^2 - 1)^{1 - \frac{1}{k_2-1}}.$$

Proof. In this case, we have that $y_2 = U_3 = 4m^2 - 1$. By Lemma 3.4, $j_3 \leq k_3$. If $k_3 = j_3 = 3$, then $4m^2 - 1 = b^2 + b + 1$, which implies that $(4m)^2 - (2b + 1)^2 = 7$, which is impossible. If $k_3 = 4$, then we obtain a solution to $(2m)^2 = b^3 + b^2 + b + 2$. Using MAGMA [4] to find all integer points on this curve, we obtain that the only ones are

$$(2m, b) = (-1, \pm 1), (2, \pm 4).$$

This means that $m = 1$ and $b = 4$. Therefore, $k_3 \geq 5$. Since $\varepsilon = 1$ and $j_3 = 2p_1j_2 + 1$, we obtain

$$y_3 \equiv 2p_1mx_2y_2 + 1 \pmod{y_2^2}.$$

Notice that $(x_2, y_2) = (4m^3 - 3m, 4m^2 - 1)$. We get

$$\begin{aligned} (y_3 - 1)/y_2 &\equiv 2p_1m(4m^3 - 3m) \\ &\equiv 2p_1m(-2m) \\ &\equiv -4m^2p_1 \\ &\equiv -p_1 \\ &= -p/2 \pmod{4m^2 - 1}. \end{aligned}$$

Moreover, we have

$$(y_3 - 1)/y_2 \equiv bq \pmod{y_2}.$$

Then one gets

$$p/2 + bq \geq y_2.$$

As $b < m/2$, from Lemma 3.2, we have $p < q$. Thus, we obtain

$$(b + 1/2)q > y_2 = 4m^2 - 1.$$

Therefore, we have $q > (4m^2 - 1)/(b + 1/2)$. Since

$$4m^2 - 1 = y_2 = H_{k_2} > b^{k_2-1},$$

we get $4m^2 \geq b^{k_2-1}$. Then we have

$$b + 1/2 < (4m^2)^{1/(k_2-1)} + 1/2 < 1.3(4m^2 - 1)^{1/(k_2-1)},$$

with $m \geq 3$. So

$$q > 0.76(4m^2 - 1)^{1-1/(k_2-1)}$$

implies

$$k_3 > 0.76k_2(4m^2 - 1)^{1-1/(k_2-1)}.$$

This completes the proof of the Lemma. $\square$

Lemma 4.5. *If $\varepsilon = 1$, $j_2 = 4$ and $b < m/2$, then $2 \mid k_2$, $k_2 \geq 6$ and*

$$(29) \quad k_3 \geq k_2 \alpha^3 / (2m).$$

Proof. In this case, we have

$$y_2 = U_4 = 8m^3 - 4m = H_{k_2} \geq H_5 > b^5.$$

We use equation (21) to obtain

$$\frac{y_3 - 1}{y_2} \equiv -mp \pmod{y_2}.$$

Combining this congruence and

$$\frac{y_3 - 1}{y_2} \equiv bq \pmod{y_2},$$

it follows that

$$mp + bq \equiv 0 \pmod{y_2}.$$

Using Lemma 3.4, we get $mp > bq$ and so

$$2mp > mp + bq \geq y_2.$$

This implies that $p > y_2/(2m)$ and, by Lemma 3.2, we see that $q > p$. Therefore,

$$k_3 > qk_2 > pk_2 > k_2 y_2 / (2m) > k_2 \alpha^3 / (2m).$$

So the lemma is proved. $\square$

5. Proof of Theorem 1.1

Now, let us recall the following result on linear forms in logarithms which is due to Matveev [7, Corollary 2.3].

Lemma 5.1. *Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers, different from 0 or 1, and let $\log \alpha_1, \dots, \log \alpha_n$ represent determinations of their logarithms. Let D be the degree over $\mathbb{Q}$ of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$, and let $b_1, \dots, b_n$ be rational integers. Define*

$$B = \max\{|b_1|, \dots, |b_n|\}$$

and, for $1 \leq i \leq n$,

$$A_i = \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\},$$

where $h(\alpha)$ denotes the absolute logarithmic Weil height of α . Assume that the number $\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_n$ does not vanish. Then we have

$$|\Lambda| \geq \exp\{-C(n, \varkappa) D^2 A_1 \cdots A_n \log(eD) \log(eB)\},$$

where $\varkappa = 1$ if $\mathbb{K} \subset \mathbb{R}$ and $\varkappa = 2$ otherwise, and

$$C(n, \varkappa) = \min \left\{ \frac{1}{\varkappa} \left(\frac{1}{2} en \right)^\varkappa 30^{n+3} n^{3.5}, 2^{6n+20} \right\}.$$

Now, we apply this lemma with $n = 3$ for

$$(30) \quad \Lambda_3 = j_3 \log \alpha - k_3 \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right).$$

Here,

$$D = 2, \quad b_1 = j_3, \quad b_2 = -k_3, \quad b_3 = -1, \quad \alpha_1 = \alpha, \quad \alpha_2 = b \quad \text{and} \quad \alpha_3 = \frac{2\sqrt{a}}{b-1}.$$

Notice that $\alpha = m + \sqrt{m^2 - 1}$ is a root of $X^2 - 2mX + 1 = 0$. We have

$$h(\alpha_1) = \frac{1}{2} \log \alpha, \quad h(\alpha_2) = \log b \quad \text{and} \quad h(\alpha_3) = \frac{1}{2} \log \max\{4a, (b-1)^2\}.$$

Therefore, we can take

$$A_1 = \log \alpha, \quad A_2 = 2 \log b, \quad A_3 = \log \max\{4a, (b-1)^2\} \quad \text{and} \quad B = \max\{j_3, k_3\}.$$

Using Lemma 5.1, we get the inequality

$$(31) \quad \log |\Lambda_3| > -1.8828 \cdot 10^{12} \cdot \log \alpha \cdot \log b \cdot \log \max\{4a, (b-1)^2\} \\ \cdot \log \max\{ej_3, ek_3\}.$$

From (6), we have

$$|\Lambda_3| = -\Lambda_3 < \frac{1}{b^{k_3} - 1} < \frac{2}{b^{k_3}}.$$

Thus,

$$(32) \quad k_3 < 3.766 \cdot 10^{12} \cdot \log \alpha \cdot \log \max\{4a, (b-1)^2\} \cdot \log \max\{ej_3, ek_3\}.$$

So $y_3 = H_{k_3} < b^{k_3}$ and Lemma 3.1 imply $y_3 > \alpha^{j_3-1}$. This yields

$$(j_3 - 1) \log \alpha < k_3 \log b.$$

Thus, we also have

$$(33) \quad j_3 - 1 < 3.766 \cdot 10^{12} \cdot \log b \cdot \log \max\{4a, (b-1)^2\} \cdot \log \max\{ej_3, ek_3\}.$$

When $b \leq 2m - 1$, we have

$$4a = 4m^2 - 4 > (b-1)^2$$

and from Lemma 3.2, we get $j_3 \leq k_3$. When $b \geq 2m + 1$, by a similar argument we have $4a < (b-1)^2$ and $j_3 > k_3$. Therefore, from (32) and (33) respectively, we obtain

$$(34) \quad \begin{cases} \frac{k_3}{\log(ek_3)} < 7.532 \cdot 10^{12} (\log(2m))^2 & \text{if } b \leq 2m - 1, \\ \frac{j_3 - 1}{\log(ej_3)} < 7.532 \cdot 10^{12} (\log b)^2 & \text{if } b \geq 2m + 1. \end{cases}$$

Suppose that equation (2) has at least three positive integer solutions (x_i, y_i, k_i) , with $1 = k_1 < k_2 < k_3$. By the details in Section 2, we only need to consider $a = m^2 - 1$, namely equations (5). Let $\alpha = m + \sqrt{m^2 - 1}$ be such that $(m, 1)$ is the smallest solution of the Pell equation $x^2 - (m^2 - 1)y^2 = 1$ and $j_i, i = 1, 2, 3$, defined by (3). We will combine the upper and lower bounds for j_i or k_i , in three separate cases.

(i) *Case $b < m/2$.* Since $b \neq 2m - 1$, from Lemma 3.2, we get $j_i < k_i, i = 2, 3$. When $\varepsilon = 0$, by Lemma 3.3, we obtain

$$k_3 > \min\{2m, 3\alpha\} = 2m.$$

Suppose now $\varepsilon = 1$. If $j_2 \geq 5$, then $k_2 \geq 6$ and, by Lemma 3.5, we have

$$k_3 \geq k_2 \alpha^{(j_2-1)/2} / m \geq 6\alpha^2 / m.$$

If $2 \leq j_2 \leq 4$, from Lemmas 4.3, 4.4 and 4.5, we have

$$k_3 \geq \min\left\{6m, 3.8(4m^2 - 1)^{3/4}, 6\alpha^3 / (2m)\right\} = 6m.$$

Therefore, we get

$$(35) \quad k_3 \geq \min\{2m, 6\alpha^2 / m, 6m\} = 2m.$$

By (34) and (35), we obtain

$$(36) \quad m < 3.766 \cdot 10^{12} (\log(2m))^2 (1 + \log(2m)).$$

This implies $m < 2.621 \cdot 10^{17}$.

(ii) *Case $b \geq 2m + 1$.* From Lemma 3.2, we get $j_i > k_i$ for $i = 2, 3$. When $\varepsilon = 0$, from Lemma 3.3, we have

$$j_3 \geq j_2 b^{(k_2-1)/2} \geq 3b^{1/2}.$$

Suppose now $\varepsilon = 1$. If $k_2 \geq 4$, then $j_2 \geq 5$. By Lemma 3.5, we have

$$j_3 \geq j_2 b^{(k_2-3)/2} \geq 5b^{1/2}.$$

If $2 \leq k_2 \leq 3$ with $b > 11000$, then from Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned} j_3 &\geq \min\left\{0.6b, 7.6b^{2/3}, 4.65b^{1/2}, 192b^{2/5}, 13.79b^{1/3}, 2.94\left(2.12b^{1/2} - 1\right)\right\} \\ &\geq 13.79b^{1/3}. \end{aligned}$$

Therefore, if $b > 11000$, then we have

$$(37) \quad j_3 \geq \min \left\{ 3b^{1/2}, 5b^{1/2}, 13.79b^{1/3} \right\} \geq 13.79b^{1/3}.$$

By (34) and (37), we obtain

$$(38) \quad b^{\frac{1}{3}} < 5.392 \cdot 10^{11} (\log b)^2 \left(1 + \log \left(13.79b^{\frac{1}{3}} \right) \right).$$

This implies $b < 4.233 \cdot 10^{52}$.

(iii) *Case $m/2 \leq b \leq 2m - 1$.* In this case, we know that $k_3 > j_3$. Assume that $b > 5$. By Lemma 3.6, we get

$$j_3 - 1 > j_3 - j_2 > (b - 1) \log b.$$

From (34), we have

$$\begin{aligned} \frac{(b-1) \log b}{1 + \log((b-1) \log b)} &< 7.532 \cdot 10^{12} \cdot (\log(2m))^2 \\ &\leq 7.532 \cdot 10^{12} \cdot (\log(4b))^2. \end{aligned}$$

This implies that $b < 1.253 \cdot 10^{16}$. Thus, one can see that $m \leq 2b < 2.506 \cdot 10^{16}$.

Observe that the special case $b = 2m$ is already proved by Proposition 2.3. Altogether, if

$$\begin{aligned} \max\{a, b\} &= \max\{m^2 - 1, b\} \\ &> \max\{6.870 \cdot 10^{34}, 6.432 \cdot 10^{32}, 1.253 \cdot 10^{16}, 4.233 \cdot 10^{52}\} \\ &= 4.233 \cdot 10^{52}, \end{aligned}$$

then equation (2) has at most two positive integer solutions (x, y, k) . This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.2

Suppose equation (2) has at least four positive integer solutions (x_i, y_i, k_i) , with $1 = k_1 < k_2 < k_3 < k_4$. We use a method similar to that in the above sections. We have the following lemma.

Lemma 6.1. *There are positive integers p' and $q' > 1$ with the property that*

$$j_4 = p'j_3 + \varepsilon', \quad k_4 = q'k_3 + \varepsilon', \quad \varepsilon' \in \{0, 1\}$$

or

$$j_4 = p'j_3 + j_2, \quad k_4 = q'k_3 + k_2.$$

Furthermore, if $\varepsilon \neq 0$, we have $2 \mid p$.

Proof. See the proof of Lemma 2.5. □

Lemma 6.2. *If $j_4 = p'j_3 + j_2$ and $k_4 = q'k_3 + k_2$, then*

$$(39) \quad \begin{cases} p' < q', \quad x_2 p' > b^{k_2} q' & \text{if } b < m/2, \\ p' > q', \quad x_2 p' < b^{k_2} q' & \text{if } b \geq 2m + 1. \end{cases}$$

Proof. See the proof of Lemma 3.2 and Lemma 3.4. $\square$

Using the above lemmas, we get the following.

Lemma 6.3. *If $j_4 = p'j_3 + j_2$ and $k_4 = q'k_3 + k_2$, then*

$$(40) \quad \begin{cases} k_4 > k_3 \alpha^{(j_3-1)/2} / x_2 & \text{if } b < m/2, \\ j_4 > j_3 b^{(k_3-1)/2} / b^{k_2} & \text{if } b \geq 2m+1. \end{cases}$$

Proof. The proof is similar to that of Lemma 3.5. $\square$

Now, assume that if another solution (x, y, k) exists, then $k_4 < k$. Let $a = m^2 - 1$ and $\alpha = m + \sqrt{m^2 - 1}$ be the smallest solution of the Pell equation $x^2 - (m^2 - 1)y^2 = 1$ and $j_i, i = 1, 2, 3, 4$, defined by (3).

Applying Matveev's result (Lemma 5.1) to

$$(41) \quad \Lambda_4 = j_4 \log \alpha - k_4 \log b - \log \left(\frac{2\sqrt{a}}{b-1} \right),$$

we obtain the following upper bounds of j_4 and k_4 which are similar to (34).

$$(42) \quad \begin{cases} \frac{k_4}{\log(ek_4)} < 7.532 \cdot 10^{12} (\log(2m))^2 & \text{if } b \leq 2m-1, \\ \frac{j_4-1}{\log(ej_4)} < 7.532 \cdot 10^{12} (\log b)^2 & \text{if } b \geq 2m+1. \end{cases}$$

(i) *Case $b < m/2$.* From Lemma 3.2, if $b \neq 2m-1$, then we have $j_i < k_i$, $i = 2, 3$. From Lemma 6.3, we have

$$k_4 > k_3 \alpha^{(j_3-1)/2} / x_2.$$

Since

$$x_2 + y_2 \sqrt{a} = T_{j_2} + U_{j_2} \sqrt{a} = \alpha^{j_2},$$

we have

$$x_2 = T_{j_2} = (\alpha^{j_2} + \alpha^{-j_2})/2.$$

From $m \geq 2b+1 \geq 5$, we get $\alpha^{j_2} \geq (5 + \sqrt{24})^2 > 9.8$. Moreover, $\alpha^{-j_2} < 0.103$ implies $x_2 < 0.506\alpha^{j_2}$. Thus, we obtain

$$(43) \quad k_4 > 1.976k_3 \alpha^{(j_3-1)/2-j_2}.$$

When $\varepsilon = 0$, by Lemma 3.3, we get

$$k_3 > 2m, \quad j_2 = 2$$

and

$$k_3 > k_2 \alpha^{(j_2-1)/2} \geq (j_2+1) \alpha^{(j_2-1)/2}, \quad j_2 \geq 3.$$

Since $b \geq 3$, we have

$$\begin{aligned} j_3 - 1 &> (k_3 - 1) \frac{\log b}{\log(2m)} \\ &\geq 1.098(k_3 - 1) \frac{1}{\log(2m)}. \end{aligned}$$

Thus, we get

$$j_3 - 1 > \begin{cases} 1.098(2m - 1)/\log(2m) & \text{if } j_2 = 2, \\ 1.098((j_2 + 1)\alpha^{(j_2-1)/2} - 1)/\log(2m) & \text{if } j_2 \geq 3. \end{cases}$$

Combining this and (43) yields

$$(44) \quad k_4 > \begin{cases} 3.952m\alpha^{(1.098m-0.549)/\log(2m)-2} & \text{if } j_2 = 2, \\ 7.904\alpha^{(2.196\alpha-0.549)/\log(2m)-2} & \text{if } j_2 \geq 3. \end{cases}$$

From this and (42), we deduce

$$\frac{m\alpha^{(1.098m-0.549)/\log(2m)-2}}{1 + \log(3.952m\alpha^{0.3465(2m-1)/\log(2m)-2})} < 1.906 \cdot 10^{12} (\log(2m))^2,$$

for $j_2 = 2$, and

$$\frac{\alpha^{(2.196\alpha-0.549)/\log(2m)-2}}{1 + \log(7.904\alpha^{(1.4595\alpha-0.1535)/\log(2m)-2})} < 9.529 \cdot 10^{11} (\log(2m))^2,$$

for $j_2 \geq 3$. This implies $m < 38$ and $m < 10$, respectively.

Now, we consider the case $\varepsilon = 1$. If $j_2 \geq 5$, then we have $1.979m < \alpha$ since $m \geq 5$. By Lemma 3.5, we get

$$(45) \quad k_3 > k_2\alpha^{(j_2-1)/2}/m > 1.979k_2\alpha^{(j_2-3)/2}$$

and

$$(46) \quad j_3 > j_2\alpha^{(j_2-1)/2}/m > 1.979j_2\alpha^{(j_2-3)/2}.$$

Thus, inequalities (43), (45) and (46) imply

$$\begin{aligned} k_4 &> 3.91k_2\alpha^{\frac{1}{2}(j_3-j_2-4)} \\ &> 3.91k_2\alpha^{\frac{1}{2}(j_2(1.979\alpha^{\frac{1}{2}(j_2-3)}-1)-4)} \\ &\geq 23.46\alpha^{4.947\alpha-4.5}. \end{aligned}$$

This and (42) give

$$\frac{\alpha^{4.947\alpha-4.5}}{1 + \log(23.46\alpha^{4.947\alpha-4.5})} < 3.211 \cdot 10^{11} (\log(2m))^2.$$

Thus, we get $m < 3$. But $m \geq 5$, so this is impossible.

If $j_2 = 4$, then using Lemma 4.5 yields

$$k_3 > k_2\alpha^3/(2m) > 0.989k_2\alpha^2.$$

As $mp < bq$, we obtain $j_3 > 1.96j_2\alpha^2$. This bound is larger than those obtained by inequalities (45) and (46). Moreover, $j_2 = 4 < 5$. So we also get a contraction.

If $2 \leq j_2 \leq 3$, then in the proof of Theorem 1.1 we have seen that $k_3 \geq 6m$. As

$$j_3 - 1 > (k_3 - 1) \log b / \log(2m),$$

we have

$$j_3 - 1 > 1.098(6m - 1)/\log(2m).$$

So from (43), we obtain

$$(47) \quad \begin{aligned} k_4 &> 11.856m\alpha^{\frac{1}{2}(1.098(6m-1)/\log(2m)-2j_2)} \\ &\geq 11.856m\alpha^{(3.294m-0.549)/\log(2m)-3}. \end{aligned}$$

Combining this and (42) gives

$$\frac{m\alpha^{(3.294m-0.549)/\log(2m)-3}}{1 + \log(11.856m\alpha^{(3.294m-0.549)/\log(2m)-3})} < 6.353 \cdot 10^{11} (\log(2m))^2.$$

This implies $m < 13$.

(ii) *Case $b \geq 2m + 1$.* From Lemma 3.2, we get $j_i > k_i$, $i = 2, 3$. Using Lemma 6.3, we have

$$(48) \quad j_4 > j_3 b^{(k_3-1)/2-k_2}.$$

When $\varepsilon = 0$, Lemma 3.3 implies

$$j_3 > j_2 b^{(k_2-1)/2}.$$

Since

$$k_3 - 1 > (j_3 - 1) \frac{\log \alpha}{\log(b+1)}$$

and

$$\alpha = m + \sqrt{m^2 - 1} \geq 2 + \sqrt{3} > 3.732,$$

we have

$$\begin{aligned} k_3 - 1 &> 1.316(j_3 - 1)/\log(b+1) \\ &> 1.316(j_2 b^{(k_2-1)/2} - 1)/\log(b+1). \end{aligned}$$

Using this and (48), we obtain

$$\begin{aligned} j_4 &> 3b^{0.658(3b^{\frac{1}{2}(k_2-1)}-1)/\log(b+1)-\frac{1}{2}(k_2+1)} \\ &\geq 3b^{(1.974b^{\frac{1}{2}}-0.658)/\log(b+1)-1.5}. \end{aligned}$$

The above inequality and (42) give

$$\frac{b^{(1.974b^{\frac{1}{2}}-0.658)/\log(b+1)-1.5}}{1 + \log\left(3b^{(1.974b^{\frac{1}{2}}-0.658)/\log(b+1)-1.5}\right)} < 2.511 \cdot 10^{12} (\log b)^2.$$

So we obtain $b < 546$.

Suppose $\varepsilon = 1$. If $k_2 \geq 4$, then we have $j_2 \geq 5$. Using Lemma 3.5, we get

$$j_3 \geq j_2 b^{(k_2-3)/2} \quad \text{and} \quad k_3 \geq k_2 b^{(k_2-3)/2}.$$

Combining this and inequality (48), we have

$$j_4 > j_2 b^{\frac{1}{2}k_2(b^{\frac{1}{2}(k_2-3)}-1)-2} \geq 5b^{2b^{\frac{1}{2}}-4}.$$

Then from (42), we deduce

$$\frac{5b^{2b^{\frac{1}{2}}-4} - 1}{1 + \log(5b^{2b^{\frac{1}{2}}-4})} < 7.532 \cdot 10^{12} (\log b)^2.$$

This implies $b < 44$.

If $2 \leq k_2 \leq 3$, we assume $b \geq 600$. From Lemma 4.1, comparing the minimum value of j_3 , we have

$$j_3 \geq \min \{4.65b^{1/2}, 13.97b^{1/3}\}$$

with $k_2 = 2$. From Lemma 4.2, we have $j_3 > 6.233b^{1/2} - 2.94$ with $k_2 = 3$. Since $4.65b^{1/2}$ and $6.233b^{1/2} - 2.94$ are both larger than $3b^{1/2}$, this has already been considered in case $\varepsilon = 0$. Then we only need to consider $j_3 > 13.97b^{1/3}$. For any given j_3 , we have

$$k_3 - 1 > 1.136(j_3 - 1)/\log(b + 1).$$

From (48) we get

$$j_4 > 13.97b^{(9.192b^{\frac{1}{3}} - 0.658)/\log(b+1) - \frac{5}{3}}.$$

This and (42) give us

$$\frac{b^{(9.192b^{\frac{1}{3}} - 0.658)/\log(b+1) - \frac{5}{3}}}{1 + \log\left(13.97b^{(9.192b^{\frac{1}{3}} - 0.658)/\log(b+1) - \frac{5}{3}}\right)} < 5.392 \cdot 10^{11} (\log b)^2.$$

This implies $b < 96$.

(iii) *Case $m/2 \leq b \leq 2m - 1$.* If $b > 2$, we have $b^{k_2} > 5$ and $b \leq 2m - 1$. Since (j_i, k_i) , $i = 2, 3, 4$, are three positive solutions of (4), by Lemmas 3.2 and 3.6, we have

$$(49) \quad k_4 > j_4 > j_4 - j_3 > (b^{k_2} - 1) \log b.$$

Therefore, we use (42) and (48) to obtain

$$(50) \quad \frac{(b^{k_2} - 1) \log b}{1 + \log((b^{k_2} - 1) \log b)} < 7.532 \cdot 10^{12} (\log(2m))^2 < 7.532 \cdot 10^{12} (\log(4b))^2.$$

Suppose that $m \geq 600$ and $b \geq 600$. Then (50) implies $k_2 \leq 6$.

Moreover, from Lemma 3.2, we have $j_2 \leq k_2 - 1$ or $j_2 = k_2 = 2$. If $j_2 \leq k_2 - 1$, using Lemma 3.1, we get

$$(m/2)^{k_2-1} \leq b^{k_2-1} < H_{k_2} = y_2 = U_{j_2} < (2m)^{j_2-1} \leq (2m)^{k_2-2}.$$

We deduce the inequality

$$m < \frac{2(2m)^{k_2-2}}{(m/2)^{k_2-2}} = 2 \cdot 4^{k_2-2} \leq 2 \cdot 4^4 = 512.$$

This contradicts the fact that $m \geq 600$.

If $j_2 = k_2 = 2$, then $b = 2m - 1$. We have

$$\begin{aligned} \frac{(b^2 - 1) \log b}{1 + \log((b^2 - 1) \log b)} &< 7.532 \cdot 10^{12} (\log(2m))^2 \\ &= 7.532 \cdot 10^{12} (\log(b + 1))^2. \end{aligned}$$

Then we obtain that $b < 74011748$.

Now, in order to cover the values not obtained, we will use a Diophantine approximation algorithm, the so-called Baker-Davenport reduction method. Combining this and the lower bounds given in Section 5, we can check the existence of a fourth solution of equation (5) for a given m and b .

Lemma 6.4. *Assume that M is a positive integer. Let P/Q be a convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequation

$$0 < s\kappa - t + \mu < AB^{-s}$$

in integers s and t with

$$\frac{\log(AQ/\eta)}{\log B} \leq s \leq M.$$

Applying the above lemma, we get the following.

Lemma 6.5. *Suppose $2 < b, m < 600$ or $2 < b = 2m - 1 < 74011748$. Then equation (5) has no fourth positive integer solution (x_4, y_4, k_4) .*

Here are a few remarks about the computations. The program was developed in PARI/GP running with 100 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find one that satisfies the condition. We applied the algorithm in three cases.

(i) $2 < b < 600$, $1 < m < 600$ and $b < 2m - 1$. Since $m < 600$, by (42), we get $k_4 < 3.79 \cdot 10^{14}$. From the definition of Λ_4 given by (41), we take

$$s = k_4, \quad t = j_4, \quad \kappa = \frac{\log b}{\log \alpha}, \quad \mu = \frac{\log(2\sqrt{a}/(b-1))}{\log \alpha}$$

and

$$A = \frac{2}{\log \alpha}, \quad B = b, \quad M = 3.79 \cdot 10^{14}.$$

The use of the second convergent was needed in 5466 cases (2.04%). The third convergent was used in 199 cases (0.075%), the fourth in 34 cases (0.013%), the fifth in 7 cases and the sixth when $(m, b) = (398, 530), (599, 467)$. In all cases, we obtained $k_4 \leq 39$. If $b \leq m/2$, from (44), (47) and the fact $m \geq 7$, we have $k_4 > 176$. This leads to a contradiction. If $m/2 < b < 2m - 1$, from (49), we get $k_4 > (b^{k_2} - 1) \log b$. We obtain $3 \leq b \leq 5$. This implies $3 \leq m \leq 9$. Again we take $M = 40$ and we ran again the program. We obtained $k_4 \leq 9$, but the inequality $(b^{k_2} - 1) \log b < 9$ is impossible when $b \geq 3$. The computations took about 10 minutes.

(ii) $5 \leq 2m + 1 \leq b < 600$. As $b < 600$ and from (42), we obtain $j_4 < 3.09 \cdot 10^{14}$. Thus using the definition of Λ_4 given by (41), we take

$$s = j_4, \quad t = -k_4, \quad \kappa = -\frac{\log \alpha}{\log b}, \quad \mu = \frac{\log(2\sqrt{a}/(b-1))}{\log b}$$

and

$$A = \frac{2}{\log b}, \quad B = b, \quad M = 3.09 \cdot 10^{14}.$$

The use of the second convergent was needed in 1811 cases (2.04%), the third in 62 cases (0.07%) and the fourth in 7 cases (0.008%). In all cases, we obtained $j_4 \leq 22$. Then we ran again the program with $M = 23$ and we obtained a contradiction. It took about 10 minutes to do all the computations.

(iii) $2 < b = 2m - 1 < 74011748$. Again, by (42), we get $k_4 < 2.67 \cdot 10^{15}$. We take s, t, κ, μ, A, B as in the first case and $M = 2.67 \cdot 10^{15}$.

The use of the second convergent was needed in 526836 cases (1.42%). The third convergent was used in 17859 cases (0.05%), the fourth in 3720 cases (0.01%), the fifth, sixth, seventh and eighth respectively in 655, 150, 27 and 7 cases. The tenth was used when $(m, b) = (3974950, 7949899)$. In all cases, we obtained $k_4 \leq 35$. This and $k_4 > (b^2 - 1) \log b$ imply $(m, b) = (2, 3)$. We ran again the program with $M = 36$, and we got $k_4 \leq 7$. This also leads to a contradiction. In about 20 hours, we did all the computations.

7. Final remark

We have seen that there exist infinitely many pairs (m, b) such that equation (5) has at least two solutions (x, y, k) . Namely, when

$$(m, b) = ((1 + b + \dots + b^{r-1})/2, b) \quad \text{with } r > 1,$$

then equation (5) has the two solutions

$$(x, y, k) = (m, 1, 1), (2m^2 - 1, 2m, r),$$

and when

$$(m, b) = (m, U_s - 1) \quad \text{with } s > 1,$$

equation (5) has the two solutions

$$(x, y, k) = (m, 1, 1), (T_s, U_s, 2).$$

Conjecture 7.1. *Equation (5) has at most 1 solution (m, b, j, k) with $\min(j, k) > 0$ except when $m = (1 + b + \dots + b^{k-1})/2$ and $k = 2$ or $b = U_j - 1$ and $j = 2$.*

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