

KOREVAAR CLASSES OF ENTIRE FUNCTIONS AS UNIFORM LIMITS OF POLYNOMIALS WITH ZEROS IN HALF PLANES AND VERTICAL STRIPS

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RÉSUMÉ. Une classe de fonctions entières qui peuvent être approximées uniformément sur un compact par des polynômes dont les racines sont contenues dans un demi-plan est appelée une classe de Korevaar. Une caractérisation des éléments de ces classes découle d’une étude de J. Korevaar sur les fonctions entières approximées par des polynômes dont les racines forment un ensemble ayant une structure de demi-plan. Les résultats de Korevaar donnent des conditions nécessaires et suffisantes sur l’appartenance des intégrales de Laplace à ces classes. Comme application de ces résultats, on obtient des conditions sur les coefficients des fonctions entières pour qu’elles appartiennent à ces classes. Nous donnons aussi des conditions sur les intégrales de Laplace pour avoir des racines imaginaires.

ABSTRACT. Classes of entire functions that are uniformly approximated on compacta by polynomials having zeros contained in a half plane are referred to as Korevaar classes. Characterization of members in this class follows from general investigations by J. Korevaar on entire functions that are approximated by polynomials with their zeros restricted to sets having half plane structure. Korevaar’s results yield necessary and sufficient conditions for membership of Laplace integrals in Korevaar classes. Applications include conditions on the coefficients of entire functions for belonging in Korevaar classes and for Laplace integrals to have imaginary zeros.

1. Korevaar classes for structured sets

The Hadamard product characterization of the entire functions that are uniform limits on compacta (that is, compact subsets) of polynomials with only non negative zeros was discovered by Laguerre in 1882. A related problem of characterizing entire functions when the approximating polynomials have only real zeros was solved by Pólya in 1915. Subsequent investigations for entire functions approximated by polynomials with zeros all located in sectors with aperture at most π were conducted by Pólya, Lindwart and Obrechhoff. The most general investigations in this area are attributed to J. Korevaar in 1949 in [8]. These results include the characterization of entire functions that are uniformly approximated by polynomials with all zeros restricted to sets having a half plane or strip structure. In particular, only results pertaining to entire functions

that are uniformly approximated by polynomials with zeros restricted to sets having these types of structures are considered below. In the twentieth century, researches on the Riemann Hypothesis have employed the class of entire functions that are uniform limits on compacta of polynomials with only real zeros. In modern literature, this class of entire functions is referred to as the Laguerre-Pólya class. An historic survey of these classic results can be found in [9].

We have the following notations for the left and right half planes:

$$\begin{cases} \Lambda(-\infty, \sigma) = \{z \mid \Re(z) < \sigma\}, \\ \Lambda(\sigma', \infty) = \{z \mid \Re(z) > \sigma'\}, \\ \Lambda(-\infty, \sigma] = \{z \mid \Re(z) \leq \sigma\}, \\ \Lambda[\sigma', \infty) = \{z \mid \Re(z) \geq \sigma'\}, \end{cases}$$

as well as the vertical strips

$$\Lambda(\sigma', \sigma) = \Lambda(-\infty, \sigma) \cap \Lambda(\sigma', \infty)$$

and

$$\Lambda[\sigma', \sigma] = \Lambda(-\infty, \sigma] \cap \Lambda[\sigma', \infty).$$

An unbounded set L is said to have *left half plane structure* (LHPS) if we have the following:

- (i) $L \subset \Lambda(-\infty, \sigma)$,
- (ii) $\delta(L) = \lim_{r \rightarrow \infty} \sup_{|z| \geq r} \{\Re(z) \mid z \in L\} > -\infty$.

Then, $\delta = \delta(L)$ is called the *asymptotic abscissa* of L . An unbounded set R is said to have *right half plane structure* (RHPS) if we have the following:

- (i) $R \subset \Lambda(\sigma', \infty)$,
- (ii) $\delta'(R) = \lim_{r \rightarrow \infty} \inf_{|z| \geq r} \{\Re(z) \mid z \in R\} < \infty$.

Then, $\delta' = \delta'(R)$ is called the *asymptotic abscissa* of R . If S has both LHPS and RHPS, then S is said to have *vertical strip structure* (VSS) with *asymptotic abscissae* $\delta' \leq \delta$ given above. These definitions are consistent with the seminal work of J. Korevaar in [8]. The cases $\delta = -\infty$ and $\delta' = \infty$ are included in this original work but not needed here, and thus excluded below.

If a set S that has either the half plane or the vertical strip structure, then the *Korevaar class* corresponding to S , denoted $\mathcal{K}(S)$, is defined as the set of entire functions that are uniform limits on compacta of polynomials whose zeros are all in S . Entire functions in Korevaar classes are members of the class of entire functions having *order* at most 2. If Φ is a member of a Korevaar class, then it necessarily has the product form

$$(1) \quad \Phi(z) = e^{\lambda z^2} \phi(z),$$

where λ is complex, and ϕ is an entire function having genus at most 1 (see [8]). This means that ϕ has Hadamard product given by

$$(2) \quad \phi(z) = C e^{\xi z} z^\eta \prod_j \left(1 - \frac{z}{z_j}\right) e^{z/z_j},$$

where C and ξ are complex, η is a non negative integer, and $\sum_j |z_j|^{-2} < \infty$ (see [2]). In particular, ϕ has order at most 2 and minimal type so that the two factors of (1) have distinct growth rates (see [2] or [11]). Taking the logarithmic derivative of (2) yields the formulas

$$(3) \quad \eta = \left[z \left(\frac{\phi'}{\phi} \right) (z) \right]_{z \rightarrow 0} \quad \text{and} \quad \xi = \left[\left(\frac{\phi'}{\phi} \right) (z) - \frac{\eta}{z} \right]_{z \rightarrow 0},$$

which are applied below. Korevaar classes are obviously closed under ordinary multiplication of functions. A standard argument gives that Korevaar classes corresponding to closed sets are closed under uniform convergence on compacta. Indeed, one has $\overline{\mathcal{K}(S)} = \mathcal{K}(\overline{S})$. The Gauss-Lucas Theorem (see [6] or [10]) yields the *differentiation property*:

$$(4) \quad \Phi \in \mathcal{K}(S) \implies \Phi' \in \mathcal{K}(\text{co}(S)),$$

where $\text{co}(S)$ is the closed convex hull of S . Moreover, a routine argument gives the *translation property*:

$$(5) \quad \Phi \in \mathcal{K}(S) \iff \Phi_\sigma \in \mathcal{K}(S - \sigma),$$

where $\Phi_\sigma(z) = \Phi(z + \sigma)$. Necessary and sufficient conditions for membership in Korevaar classes of structured sets are given in the next theorem (see [8]).

Theorem 1.1. *Let Φ be an entire function given by (1) with λ non negative, and let ϕ be given by (2) with $\eta = 0$ if S does not contain 0. Moreover, let z_j be a non zero element of S for all j . Suppose $\sum_j \Re(1/z_j)$ is convergent, and let $\kappa = \kappa(\phi)$ denote the value*

$$(6) \quad \kappa = \Re(\xi) + \sum_j \Re(1/z_j).$$

Then, we have the following:

- (i) *If S has LHPS and asymptotic abscissa δ with $\kappa \geq -2\lambda\delta$, then $\Phi \in \mathcal{K}(S)$.*
- (ii) *If S has RHPS and asymptotic abscissa δ' with $\kappa \leq -2\lambda\delta'$, then $\Phi \in \mathcal{K}(S)$.*
- (iii) *If S has VSS and asymptotic ascissae $\delta' \leq \delta$ with $-2\lambda\delta \leq \kappa \leq -2\lambda\delta'$, then $\Phi \in \mathcal{K}(S)$.*

Conversely, if $\Phi \in \mathcal{K}(S)$, then all the corresponding conditions stated above for Φ hold.

In view of Theorem 1.1, if the sets L and R have LHPS and RHPS, respectively, then $S = L \cap R$ has VSS and asymptotic abscissae $\delta' \leq \delta$ with $\mathcal{K}(S) = \mathcal{K}(L) \cap \mathcal{K}(R)$. The last theorem yields the following corollary regarding families of sets having LHPS.

Corollary 1.2. *For n a positive integer, let L_n have LHPS and asymptotic abscissa of δ_n with $L = \liminf_{n \rightarrow \infty} L_n$ unbounded. Then, L has LHPS with asymptotic abscissa*

$$\delta = \limsup_{n \rightarrow \infty} \delta_n,$$

and

$$\mathcal{K}(L) = \bigcap_{n=1}^{\infty} \mathcal{K}(S_n),$$

where $S_n = \bigcup_{m=n}^{\infty} L_m$ for n a positive integer.

Proof. Since $L = \bigcap_{n=1}^{\infty} S_n$, one gets that L has LHPS. Putting

$$s_n = \lim_{r \rightarrow \infty} \sup_{|z| \geq r} \{\Re(z) \mid z \in S_n\}$$

for $n \geq 1$, we have $s_i \geq s_{i+1}$ for all $i \geq 1$, and $\delta = \inf \{s_n \mid n \geq 1\}$. It follows that $s_n = \sup \{\delta_m \mid m \geq n\}$ for all positive integer n , so that

$$\delta = \lim_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} \delta_n.$$

Since, for all n , $L \subseteq S_n$, one gets $\mathcal{K}(L) \subseteq \mathcal{K}(S_n)$ and thus,

$$\mathcal{K}(L) \subseteq \bigcap_{n=1}^{\infty} \mathcal{K}(S_n).$$

If $\Phi \in \bigcap_{n=1}^{\infty} \mathcal{K}(S_n)$, then $z_j \in S_n$ for all n and for all j , so that $z_j \in L$ for all j . If κ is given by (6) and $\epsilon > 0$, then $\delta - \epsilon > s_n$ for n sufficiently large, so that

$$\kappa \geq -2\lambda s_n > -2\lambda(\delta - \epsilon),$$

for n sufficiently large. Letting $\epsilon \rightarrow 0$ gives $\kappa \geq -2\lambda\delta$ and thus, $\Phi \in \mathcal{K}(L)$. $\square$

The argument supporting the next corollary on families of sets with RHPS is similar to the argument above.

Corollary 1.3. *For n a positive integer, let R_n have RHPS and asymptotic abscissa δ'_n with $R = \liminf_{n \rightarrow \infty} R_n$ unbounded. Then, R has RHPS with asymptotic abscissa*

$$\delta' = \liminf_{n \rightarrow \infty} \delta'_n,$$

and

$$\mathcal{K}(L) = \bigcap_{n=1}^{\infty} \mathcal{K}(S_n),$$

where $S_n = \bigcup_{m=n}^{\infty} R_m$ for $n \geq 1$.

Special cases of the last corollaries pertaining to nested structured sets are given in the next two corollaries.

Corollary 1.4. *Let $L_1 \supseteq L_2 \supseteq \dots$ be a nested sequence of sets having LHPS with $L = \bigcap_{n=1}^{\infty} L_n$ unbounded. Let δ_n be the asymptotic abscissa of L_n for $n \geq 1$. Then, L has LHPS with asymptotic abscissa*

$$\delta = \lim_{n \rightarrow \infty} \delta_n,$$

and with

$$\mathcal{K}(L) = \bigcap_{n=1}^{\infty} \mathcal{K}(L_n).$$

Corollary 1.5. *Let $R_1 \supseteq R_2 \supseteq \dots$ be a nested sequence of sets having RHPS with $R = \bigcap_{n=1}^{\infty} R_n$ unbounded. Let δ'_n be the asymptotic abscissa of R_n for $n \geq 1$. Then, R has RHPS with asymptotic abscissa*

$$\delta' = \lim_{n \rightarrow \infty} \delta'_n,$$

and with

$$\mathcal{K}(R) = \bigcap_{n=1}^{\infty} \mathcal{K}(R_n).$$

Now, denote the Korevaar class corresponding to the open half planes $\Lambda(-\infty, \sigma)$, $\Lambda(\sigma', \infty)$ and the vertical strip $\Lambda(\sigma', \sigma) = \Lambda(-\infty, \sigma) \cap \Lambda(\sigma', \infty)$ by $\mathcal{K}(-\infty, \sigma)$, $\mathcal{K}(\sigma', \infty)$, and $\mathcal{K}(\sigma', \sigma)$, respectively. Similar notation applies to the Korevaar class corresponding to closed half planes $\Lambda(-\infty, \sigma]$, $\Lambda[\sigma', \infty)$ and the vertical strip $\Lambda[\sigma', \sigma]$. The next corollary follows directly from Theorem 1.1.

Corollary 1.6. *Let Φ be given by (1) with $\lambda \geq 0$, and let ϕ be given by (2) where $\sigma' < \Re(z_j) < \sigma$ for all j , and $\eta = 0$ if $\sigma \leq 0$ or $\sigma' \geq 0$. Suppose $\kappa = \kappa(\phi)$ is given by (6). Then one has:*

- (i) $\kappa \geq -2\lambda\sigma \iff \Phi \in \mathcal{K}(-\infty, \sigma)$,
- (ii) $\kappa \leq -2\lambda\sigma' \iff \Phi \in \mathcal{K}(\sigma', \infty)$,
- (iii) $-2\lambda\sigma \leq \kappa \leq -2\lambda\sigma' \iff \Phi \in \mathcal{K}(\sigma', \sigma)$.

In particular, the relation $\mathcal{K}(\sigma', \sigma) = \mathcal{K}(-\infty, \sigma) \cap \mathcal{K}(\sigma', \infty)$ holds.

By Corollary 1.4, the Korevaar class of $\Lambda(-\infty, \sigma]$ is given by the relation

$$\mathcal{K}(-\infty, \sigma] = \bigcap_{n=1}^{\infty} \mathcal{K}(-\infty, \sigma + 1/n).$$

Thus, $\Phi \in \mathcal{K}(-\infty, \sigma]$ is characterized by $\eta = 0$ if $\sigma < 0$, and $\Re(z_j) \leq \sigma$ for all j with $\kappa \geq -2\lambda\sigma$, where κ is given by (6). Similarly for membership in Korevaar classes corresponding to $\Lambda[\sigma', \infty)$ and $\Lambda[\sigma', \sigma]$. These observations are summarized in the following corollary.

Corollary 1.7. *Let Φ be given by (1) with $\lambda \geq 0$, and ϕ given by (2), where $\sigma' \leq \Re(z_j) \leq \sigma$ for all j , and $\eta = 0$ if $\sigma < 0$ or $\sigma' > 0$. Suppose $\kappa = \kappa(\phi)$ is given by (6). Then one has:*

- (i) $\kappa \geq -2\lambda\sigma \iff \Phi \in \mathcal{K}(-\infty, \sigma]$,
- (ii) $\kappa \leq -2\lambda\sigma' \iff \Phi \in \mathcal{K}[\sigma', \infty)$,
- (iii) $-2\lambda\sigma \leq \kappa \leq -2\lambda\sigma' \iff \Phi \in \mathcal{K}[\sigma', \sigma]$.

In particular, the relation $\mathcal{K}[\sigma', \sigma] = \mathcal{K}(-\infty, \sigma] \cap \mathcal{K}[\sigma', \infty)$ holds.

2. Algebraic characterization of the Korevaar classes

For an entire function given by

$$(7) \quad \Phi(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k,$$

the n -th *Jensen polynomial* associated with $\Phi(z)$ is defined by

$$(8) \quad g_n(z) = \sum_{k=0}^n \binom{n}{k} a_k z^k,$$

for $n \geq 1$. The polynomials $\tilde{g}_n(z) = g_n(z/n)$ have the approximation property

$$(9) \quad \tilde{g}_n \rightarrow \Phi \quad \text{as } n \rightarrow \infty,$$

uniformly on compacta (see [3]).

A polynomial in the class $\mathcal{K}(-\infty, 0)$ is called a *Hurwitz polynomial*. The next theorem includes a condition for Jensen polynomials to be Hurwitz polynomials.

Theorem 2.1. *Let Φ be given by (7) and let g_n , for $n \geq 1$, be given by (8). Suppose L has LHPS and asymptotic abscissa $\delta \leq 0$. Finally, let $\Phi \in \mathcal{K}(L)$ be given by (1) where $\lambda \geq 0$, ϕ given by (2), and let $\kappa = \kappa(\phi)$ be given by (6). Then, there exists $\epsilon \geq 0$ such that the zeros of g_n are all in $\Lambda(-\infty, -\epsilon]$, for $n \geq 1$. Moreover, if $\kappa > -2\lambda\delta$ and $\Phi(0) \neq 0$, then $\epsilon > 0$.*

Proof. Put $r = \kappa + 2\lambda\delta \geq 0$, and fix $\Re(w) \geq -r$. With $\Psi(z) = e^{wz}\Phi(z)$, it follows that $\Psi \in \mathcal{K}(L)$. Differentiation yields the relation

$$\Psi^{(n)}(z) = C e^{wz} \sum_{k=0}^n \binom{n}{k} \Phi^{(k)}(z) w^{n-k},$$

for $n \geq 0$. By (4), $\Psi^{(n)} \in \mathcal{K}(\text{co}(L))$ and hence, $\Psi^{(n)}(z) \neq 0$ if $\Re(z) > 0$, for $n \geq 0$. Letting $z \rightarrow 0$, Hurwitz theorem now gives that $g_n(w^{-1})$ is non zero whenever $\Re(w) > -r$, for $n \geq 0$.

If $r = 0$, then the zeros of g_n are in $\Lambda(-\infty, 0]$, for $n \geq 0$. If $r > 0$, then the zeros of g_n are in the disc

$$|z + 1/2r| \leq 1/2r,$$

for $n \geq 0$. In the last case, $\Phi(0) \neq 0$ implies $g_n(0) \neq 0$, for $n \geq 0$. $\square$

Corollary 2.2. *Let Φ be an entire function given by (1) with $\lambda \geq 0$, and $\Phi(0) \neq 0$ such that ϕ is given by (2) and $\kappa = \kappa(\phi)$ is given by (6). Suppose $\Phi \in \mathcal{K}(-\infty, 0]$ and $\kappa > 0$. Then, the Jensen polynomials associated with $\Phi(z)$ are all Hurwitz polynomials. Conversely, if $\epsilon > 0$ and the Jensen polynomials associated with $\Psi(z) = e^{-\epsilon z}\Phi(z)$ are all Hurwitz polynomials, then $\Phi \in \mathcal{K}(-\infty, 0]$ and $\kappa > 0$.*

Proof. By Theorem 2.1, if $\Phi \in \mathcal{K}(-\infty, 0]$ with $\kappa > 0$, then there exists $\epsilon \geq 0$ such that the zeros of the Jensen polynomials g_n are all in $\Lambda(-\infty, -\epsilon]$, for $n \geq 1$. Thus, the Jensen polynomials associated with $\Phi(z)$ are all Hurwitz polynomials. If $\epsilon > 0$ and the Jensen polynomials associated with $\Psi(z) = e^{-\epsilon z}\Phi(z)$ are all Hurwitz polynomials, then $\Psi \in \mathcal{K}(-\infty, 0]$, by (9). Thus, $\Phi(z) = e^{\epsilon z}\Psi(z)$ is in $\mathcal{K}(-\infty, 0]$ with $\kappa - \epsilon \geq 0$, by Corollary 1.7. $\square$

The Jensen polynomials provide an algebraic characterization for Korevaar classes corresponding to closed half planes, given in the next two corollaries. This terminology is adopted from the original terminology applied to the Laguerre-Polya classes (see [4]).

Corollary 2.3. *Let Φ_σ be an entire function and let g_n , for $n \geq 1$, be the Jensen polynomials associated with Φ_σ . A necessary and sufficient condition for Φ to belong to $\mathcal{K}(-\infty, \sigma]$ is $g_n \in \mathcal{K}(-\infty, 0]$ for $n \geq 1$.*

Proof. From (5), $\Phi \in \mathcal{K}(-\infty, \sigma]$ if and only if $\Phi_\sigma \in \mathcal{K}(-\infty, 0]$. It suffices to show the result for $\Phi_\sigma \in \mathcal{K}(-\infty, 0]$. To this end, the necessity follows from Theorem 2.1, and the sufficiency follows from Property (9). $\square$

The corresponding result follows directly from the *reflection property*:

$$(10) \quad \Phi \in \mathcal{K}(S) \iff \Phi^- \in \mathcal{K}(-S),$$

where $\Phi^-(z) = \Phi(-z)$.

Corollary 2.4. *Let Φ_σ be an entire function and let g_n , for $n \geq 1$, be the Jensen polynomials associated with Φ_σ . A necessary and sufficient condition for Φ to belong to $\mathcal{K}[\sigma, \infty)$ is $g_n \in \mathcal{K}[0, \infty)$ for $n \geq 1$.*

3. Membership conditions for exponential polynomials

If the polynomial

$$p(z) = \sum_{m=0}^n \mu_m z^m$$

has complex coefficients, then the zeros of p are included in a closed annulus having spectral radii $\rho'(p) \leq \rho(p)$, which is the tightest closed annulus containing the zeros of p . For exponential polynomials $\varphi(z) = p(e^z)$, the zeros of φ are included in the closed strip having spectral abscissae $\sigma'(\varphi) \leq \sigma(\varphi)$, given by

$$\sigma'(\varphi) = \ln[\rho'(p)] \quad \text{and} \quad \sigma(\varphi) = \ln[\rho(p)],$$

where $\Lambda[\sigma'(\varphi), \sigma(\varphi)]$ is the tightest closed strip containing all of the zeros of φ . Moreover, $\sigma' \leq \sigma'(\varphi)$ and $\sigma \geq \sigma(\varphi)$ are called *left inclusion abscissae* (LIA) and *right inclusion abscissae* (RIA) of φ , respectively. For $m = 1, 2, \dots, n$, if $\mu_0 \mu_n \neq 0$ and $\gamma_m > 0$ satisfies

$$\sum_{m=1}^n \gamma_m^{-1} \leq 1,$$

then a classic estimate (see [13]) locates all the zeros of p in a closed disc with radius

$$\rho = \max \left\{ \sqrt[m]{\gamma_m \left| \frac{\mu_{n-m}}{\mu_n} \right|} \mid m = 1, 2, \dots, n \right\}.$$

The next two lemmas are needed in the next section. The first lemma adapts this result for estimating the inclusion abscissae of exponential polynomials.

Lemma 3.1. Let $\varphi(z) = \sum_{m=0}^n \mu_m e^{mz}$, where μ_m is complex for $m = 0, 1, \dots, n$ and $\mu_0 \mu_n \neq 0$. Then, the inequations (resp. equations)

$$(11) \quad \sum_{m=0}^{n-1} |\mu_m| e^{m\tau} \underset{(\quad)}{\leq} |\mu_n| e^{n\tau} \quad \text{and} \quad \sum_{m=1}^n |\mu_m| e^{m\tau} \underset{(\quad)}{\leq} |\mu_0|$$

have solutions (resp. unique solutions) σ and σ' , which are RIA and LIA of φ , respectively.

Proof. The function

$$L(\tau) = \sum_{m=0}^{n-1} |\mu_m| e^{-(n-m)\tau}$$

is continuous and decreasing with $L(-\infty) = \infty$ and $L(\infty) = 0$. Thus, there exists a solution (resp. unique solutions) σ to the first inequation (resp. equation) above since it is equivalent to

$$L(\tau) \leq |\mu_n| \quad (\text{resp. } L(\tau) = |\mu_n|).$$

By Lemma 3.2 below, the zeros of φ are located in the half plane $\Lambda(-\infty, \sigma]$, where

$$\sigma = \max \left\{ \frac{1}{m} \ln \left[\gamma_m \left| \frac{\mu_{n-m}}{\mu_n} \right| \right] \mid m = 1, 2, \dots, n \right\},$$

and $\sum_{m=1}^n \gamma_n^{-1} \leq 1$. The sequence

$$\gamma_m = |\mu_n / \mu_{n-m}| e^{m\sigma}, \quad m = 1, 2, \dots, n,$$

satisfies this inequation since σ solves the inequation (resp. equation) above. It follows that σ is a RIA of ϕ . If $\tilde{\mu}_m = \mu_{n-m}$ for all m and $\tilde{\varphi}(z) = \sum_{m=0}^n \tilde{\mu}_m e^{mz}$, then one has that $\varphi(z) = e^{nz} \tilde{\varphi}(-z)$. The equivalence

$$\sum_{m=1}^n |\mu_m| e^{m\tau} \underset{(\quad)}{\leq} |\mu_0| \quad \Longleftrightarrow \quad \sum_{m=0}^{n-1} |\tilde{\mu}_m| e^{m\tilde{\tau}} \underset{(\quad)}{\leq} |\tilde{\mu}_n| e^{n\tilde{\tau}},$$

where $\tilde{\tau} = -\tau$, gives that $\tilde{\sigma} = -\sigma'$ is a RIA of $\tilde{\varphi}$. Thus, σ' is a LIA of φ . $\square$

The second lemma on inclusion abscissae of exponential polynomials follows from theakeya-Eneström Theorem (see [6]).

Lemma 3.2. For $m = 0, 1, \dots, n$, let μ_m be positive and put $\sigma' = \ln(\rho')$ and $\sigma = \ln(\rho)$, where

$$\rho' = \min \left\{ \frac{\mu_k}{\mu_{k+1}} \mid k = 0, 1, \dots, n-1 \right\}$$

and

$$\rho = \max \left\{ \frac{\mu_k}{\mu_{k+1}} \mid k = 0, 1, \dots, n-1 \right\}.$$

Then, σ' and σ are LIA and RIA of $\varphi(z) = \sum_{m=0}^n \mu_m e^{mz}$, respectively.

The following lemma is needed below for applying Corollary 1.7 to exponential polynomials.

Lemma 3.3. Let $\varphi(z) = e^z - e^c$, where c is complex valued. Then, φ has the product form given by (2) with zeros given by $z_k = c + 2\pi ik$, for $k \in \mathbb{Z}$, where

$$\Re(\xi) + \sum_{k=-\infty}^{\infty} \Re(1/z_k) = 1/2$$

and $\Re(1/z_0) = 0$, in case $c = 0$. In particular, $\phi \in \mathcal{K}(-\infty, \sigma]$ with $\sigma = \Re(c)$.

Proof. Clearly, φ has order equal to 1 and infinite product given by (2) with zeros given by the sequence above. By (3), if $c \neq 0$, then $\eta = 0$ and

$$\xi = 1/(1 - e^c) = (1 - \coth(c/2))/2.$$

The logarithmic derivative of $\sinh(c/2)$ yields the relation

$$\frac{\coth(c/2)}{2} = \frac{1}{c} + \sum_{k=1}^{\infty} \left[\frac{1}{c + 2\pi ik} + \frac{1}{c - 2\pi ik} \right].$$

If $c = 0$, then $\eta = 1$ and $\xi = 1/2$. The result follows from these relations. $\square$

Corollary 3.4. Let Φ be given by (1) with $\lambda \geq 0$ and $\phi(z) = e^{\zeta z} \varphi(z)$, where

$$\varphi(z) = \sum_{m=0}^n \mu_m e^{mz}$$

with ζ and μ_m complex, and $\mu_0 \mu_n \neq 0$ for $m = 0, 1, \dots, n$. Then, ϕ is given by (2) with $\kappa = \Re(\zeta) + n/2$ given by (6), and where

- (i) $\Re(\zeta) \geq -n/2 - 2\lambda\sigma \iff \Phi \in \mathcal{K}(-\infty, \sigma]$,
- (ii) $\Re(\zeta) \leq -n/2 - 2\lambda\sigma' \iff \Phi \in \mathcal{K}[\sigma', \infty)$,
- (iii) $-n/2 - 2\lambda\sigma \leq \Re(\zeta) \leq -n/2 - 2\lambda\sigma' \iff \Phi \in \mathcal{K}[\sigma', \sigma]$,

whenever σ and σ' are RIA and LIA of φ .

Proof. Assume without loss of generality that σ and σ' are RIA and LIA of φ . Let ϕ be given by (2). Then, the zeros of $\varphi(z) = \sum_{m=0}^n \mu_m e^{mz}$ are all located in the strip $\Lambda[\sigma', \sigma]$. Factoring φ gives

$$\phi(z) = \mu_n e^{\zeta z} \prod_{m=1}^n \varphi_m(z)$$

with $\varphi_m(z) = e^z - e^{c_m}$, where c_m is complex, for $m = 1, 2, \dots, n$. By Lemma 3.3, φ_m has the product form

$$\varphi_m(z) = C_m e^{\xi_m z} z^{\eta_m} \prod_{k=-\infty}^{\infty} \left(1 - \frac{z}{z_{mk}} \right) e^{z/z_{mk}},$$

where C_m and ξ_m are complex, $\eta_m = 0$ or 1 , and $z_{mk} = c_m + 2\pi ik$, for $k \in \mathbb{Z}$. Moreover,

$$\Re(\xi_m) + \sum_k \Re(1/z_{mk}) = 1/2,$$

for $m = 1, 2, \dots, n$. Combining the relations

$$\begin{cases} \Re(\xi) = \Re(\zeta) + \sum_m \Re(\xi_m), \\ \sum_j \Re(1/z_j) = \sum_m \sum_k \Re(1/z_{mk}) \end{cases}$$

yields the sum

$$\Re(\xi) + \sum_j \Re(1/z_j) = \Re(\zeta) + n/2.$$

The result in all cases follows from this last relation and Corollary 1.7. $\square$

4. Membership conditions for Laplace Steiltjes integrals

For a complex valued function γ on $[0, \ell]$, define the variation

$$V_\gamma = \sup_{\mathcal{P}} \sum_{m=1}^n |\gamma(\tau_m) - \gamma(\tau_{m-1})|,$$

where the supremum is taken over all partitions $\mathcal{P}$ of the form

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = \ell$$

with

$$V_\gamma = \int_0^\ell d|\gamma|(t).$$

The subclass of all complex valued functions on $[0, \ell]$ having bounded variation, that is, $V_\gamma < \infty$, is denoted by $BV[0, \ell]$. If $\gamma \in BV[0, \ell]$, then the *Laplace integral* of $d\gamma$ is defined by the Stieltjes integral

$$\mathcal{L}\{d\gamma\}(z) = \int_0^\ell e^{zt} d\gamma(t).$$

The Laplace integral is clearly entire, and a calculation gives

$$|\mathcal{L}\{d\gamma\}(z)| \leq V_\gamma e^{\ell|z|},$$

so that $\mathcal{L}\{d\gamma\}$ has order at most one and finite type. A function $\gamma \in BV[0, \ell]$ is called *admissible* on $[0, \ell]$ if the boundary variations $\delta_\gamma(0) = |\gamma(0) - \gamma(0+)|$ and $\delta_\gamma(\ell) = |\gamma(\ell) - \gamma(\ell-)|$ are both non zero. Sedletsii [15] and Cartwright [1] have shown independently that a sufficient condition for the location of all of the zeros of the Laplace integral $\mathcal{L}\{d\gamma\}$ to be located in a right and left half planes is given by $\gamma(0+) \neq \gamma(0)$ and $\gamma(\ell-) \neq \gamma(\ell)$, respectively. Thus, if $\gamma \in BV[0, \ell]$ is admissible and if $\varphi = \mathcal{L}\{d\gamma\}$, then there exists a tightest closed strip $\Lambda[\sigma'(\varphi), \sigma(\varphi)]$ that includes all of the zeros of φ . Moreover, $\sigma' \leq \sigma'(\varphi)$ and $\sigma \geq \sigma(\varphi)$ are called *left inclusion abscissae* (LIA) and *right inclusion abscissae* (RIA) of φ , respectively. The next lemma illustrates an example of inclusion abscissae needed below.

Lemma 4.1. *Let $\gamma \in \text{BV}[0, \ell]$ be admissible, and let $\varphi = \mathcal{L}\{d\gamma\}$. Then, the inequations (resp. equations)*

$$(12) \quad \int_0^{\ell-} e^{t\tau} d|\gamma|(t) \underset{(\quad)}{\leq} \delta_\gamma(\ell) e^{\ell\tau} \quad \text{and} \quad \int_{0+}^{\ell} e^{t\tau} d|\gamma|(t) \underset{(\quad)}{\leq} \delta_\gamma(0)$$

have solutions (resp. unique solutions) denoted by σ and σ' , which are RIA and LIA of φ , respectively.

Proof. The function

$$L(\tau) = \int_0^{\ell-} e^{-\tau(\ell-t)} d|\gamma|(t)$$

is continuous and decreasing with $L(-\infty) = \infty$ and $L(\infty) = 0$. Thus, there exists a solution (resp. unique solutions) σ to the first inequation (resp. equation) above since it is equivalent to

$$L(\tau) \leq \delta_\gamma(\ell) \quad (L(\tau) = \delta_\gamma(\ell)).$$

If $\Re(z) = \tau > \sigma$, then the inequality

$$|\varphi(z)| \geq \delta_\gamma(\ell) e^{\ell\tau} - \int_0^{\ell-} e^{t\tau} d|\gamma|(t) = e^{\ell\tau} (\delta_\gamma(\ell) - L(\tau)) > 0$$

shows that σ is a RIA of φ . If $\tilde{\gamma}(t) = \gamma(\ell - t)$ for $0 \leq t \leq \ell$ and if $\tilde{\varphi} = \mathcal{L}\{d\tilde{\gamma}\}$, then $\varphi(z) = e^{\ell z} \tilde{\varphi}(-z)$. The equivalence

$$\int_{0+}^{\ell} e^{t\tau} d|\gamma|(t) \underset{(\quad)}{\leq} \delta_\gamma(0) \iff \int_0^{\ell-} e^{t\tilde{\tau}} d|\tilde{\gamma}|(t) \underset{(\quad)}{\leq} \delta_{\tilde{\gamma}}(\ell) e^{\ell\tilde{\tau}},$$

where $\tilde{\tau} = -\tau$, gives that $\tilde{\sigma} = -\sigma'$ is a RIA of $\tilde{\varphi}$. Thus, σ' is a LIA of φ . $\square$

If $\mathcal{P}_n$ is a uniform partition of $[0, \ell]$, then $\tau_m = m\iota_n$ for $m = 0, 1, \dots, n$, where $\iota_n = \ell/n$ when $n \geq 2$. By definition of the Steiltjes integral, one gets that, for $n \geq 2$, the sequence of sums

$$\mathcal{S}_n\{d\gamma\}(z) = \sum_{m=0}^{n-1} \mu_m e^{\tau_m z}$$

with $\mu_m = \gamma(\tau_{m+1}) - \gamma(\tau_m)$ and $m = 0, 1, \dots, n-1$ converges uniformly on compacta to $\mathcal{L}\{d\gamma\}$. Definitions for the LIA and RIA of the sums $\mathcal{S}_n\{d\gamma\}$, for $n \geq 2$, are identical to those above.

Lemma 4.2. *Let $\gamma \in \text{BV}[0, \ell]$ be admissible and let $\phi_n(z) = e^{-\ell z/2} \varphi_n(z)$, where $\varphi_n(z) = \mathcal{S}_n\{d\gamma\}(z)$, for $n \geq 2$. Then, the equations*

$$(13) \quad \sum_{m=0}^{n-2} |\mu_m| e^{\tau_m \tau} = \delta_n(\ell) e^{\ell\tau} \quad \text{and} \quad \sum_{m=1}^{n-1} |\mu_m| e^{\tau_m \tau} = \delta_n(0),$$

where $\delta_n(\ell) = |\gamma(\tau_n) - \gamma(\tau_{n-1})|$ and $\delta_n(0) = |\gamma(\tau_1) - \gamma(\tau_0)|$, have unique solutions σ_n and σ'_n that are LIA and RIA of φ_n , respectively. Moreover, $\phi_n \in \mathcal{K}[\sigma'_n, \sigma_n]$, for $n \geq 2$.

Proof. Let $\tilde{\phi}_n(z) = e^{-nz/2}\tilde{\varphi}_n(z)$, where

$$\tilde{\varphi}_n(z) = \varphi_n(z/\iota_n) = \sum_{m=0}^{n-1} \mu_n e^{mz}.$$

By Corollary 3.4, $\tilde{\sigma}_n$ and $\tilde{\sigma}'_n$ are LIA and RIA of $\tilde{\varphi}_n$, respectively, since they solve equations (11), for $n \geq 2$. Moreover, Corollary 3.4 gives $\tilde{\phi}_n \in \mathcal{K}[\tilde{\sigma}'_n, \tilde{\sigma}_n]$, for $n \geq 2$. Since, for $n \geq 2$, $\varphi_n(z) = \tilde{\varphi}_n(\iota_n z)$, one has that $\sigma_n = \tilde{\sigma}_n/\iota_n$ and $\sigma'_n = \tilde{\sigma}'_n/\iota_n$ are LIA and RIA of φ_n , and are the solutions to equations (13), respectively. It follows that $\phi_n \in \mathcal{K}[\sigma'_n, \sigma_n]$, for $n \geq 2$. $\square$

Lemma 4.3. *Let $\gamma \in \text{BV}[0, \ell]$ be admissible and let σ and σ' be solutions to equations (12), respectively. If σ_n and σ'_n are solutions to equations (13) then, for $n \geq 2$, $\sigma_n \rightarrow \sigma$ and $\sigma'_n \rightarrow \sigma'$ as $n \rightarrow \infty$.*

Proof. As in Lemma 4.1, σ is the solution to the equation $L(\tau) = \delta_\gamma(\ell)$, where

$$L(\tau) = \int_0^{\ell-\tau} e^{-\tau(\ell-t)} d|\gamma|(t)$$

is continuous and decreasing. Likewise, we have that σ_n is the solution to the equation $L_n(\tau) = \delta_n(\ell)$, where

$$L_n(\tau) = \sum_{m=0}^{n-1} |\mu_m| e^{-(\ell-\tau_m)\tau}$$

is continuous and decreasing, for $n \geq 2$. A routine argument gives $L_n(\tau) \rightarrow L(\tau)$ as $n \rightarrow \infty$ uniformly on compact intervals. It follows that $L_n^{-1}(\delta) \rightarrow L^{-1}(\delta)$ as $n \rightarrow \infty$ uniformly on compact intervals. Put

$$\sigma_n = L_n^{-1}(\delta_n(\ell)) \quad \text{and} \quad \hat{\sigma}_n = L_n^{-1}(\delta_\gamma(\ell)),$$

for $n \geq 2$. Since $\sigma = L^{-1}(\delta_\gamma(\ell))$, we get $|\hat{\sigma}_n - \sigma| \rightarrow 0$ as $n \rightarrow \infty$. Now, since $\delta_n(\ell) \rightarrow \delta_\gamma(\ell)$ as $n \rightarrow \infty$, we have $|\sigma_n - \hat{\sigma}_n| \rightarrow 0$ as $n \rightarrow \infty$, by uniform convergence. This gives $\sigma_n \rightarrow \sigma$ as $n \rightarrow \infty$. A similar argument gives $\sigma'_n \rightarrow \sigma'$ as $n \rightarrow \infty$. $\square$

Lemma 4.4. *Let $\gamma \in \text{BV}[0, \ell]$ be admissible and let $\hat{\phi}(z) = e^{-\ell z/2}\varphi(z)$, where $\varphi = \mathcal{L}\{d\gamma\}$. Suppose σ and σ' are solutions to equations (12), respectively. Then $\hat{\phi} \in \mathcal{K}[\sigma', \sigma]$.*

Proof. Put $\phi_n(z) = e^{-\ell z/2}\varphi_n(z)$, where $\varphi_n = \mathcal{S}_n\{d\gamma\}$, for $n \geq 2$. By Lemma 4.2, $\phi_n \in \mathcal{K}[\sigma'_n, \sigma_n]$, for $n \geq 2$. Denote

$$s_n = \sup_{m \geq n} \sigma_m \quad \text{and} \quad s'_n = \inf_{m \geq n} \sigma'_m,$$

for $n \geq 1$. Then, $s_n \rightarrow \sigma$ and $s'_n \rightarrow \sigma'$ as $n \rightarrow \infty$. Moreover, if $n \geq m$, then $[s'_n, s_n] \subseteq [s'_m, s_m]$ and thus, $\mathcal{K}[s'_n, s_n] \subseteq \mathcal{K}[s'_m, s_m]$. Since $\mathcal{K}[\sigma'_n, \sigma_n] \subseteq \mathcal{K}[s'_n, s_n]$ for $n \geq 2$, it follows that $\phi_n \in \mathcal{K}[s'_m, s_m]$ for $n \geq m$. Since $\phi_n \rightarrow \hat{\phi}$ as $n \rightarrow \infty$ uniformly on compacta, then $\hat{\phi} \in \mathcal{K}[s'_m, s_m]$ for $m \geq 2$. Thus, by Corollary 1.4,

$$\hat{\phi} \in \bigcap_{m=2}^{\infty} \mathcal{K}[s'_m, s_m] = \mathcal{K}[\sigma', \sigma].$$

$\square$

Theorem 4.5. Let $\gamma \in \text{BV}[0, \ell]$ be admissible and let Φ be given by (1) with $\lambda \geq 0$ and $\phi(z) = e^{\zeta z} \varphi(z)$, where $\varphi(z) = \mathcal{L}\{d\gamma\}$ and ζ is complex. Then, ϕ is given by (2) with $\kappa = \Re(\zeta) + \ell/2$ given by (6), and where

$$(14) \quad \begin{cases} \Re(\zeta) \geq -\ell/2 - 2\lambda\sigma & \iff \Phi \in \mathcal{K}(-\infty, \sigma], \\ \Re(\zeta) \leq -\ell/2 - 2\lambda\sigma' & \iff \Phi \in \mathcal{K}[\sigma', \infty), \\ -\ell/2 - 2\lambda\sigma \leq \Re(\zeta) \leq -\ell/2 - 2\lambda\sigma' & \iff \Phi \in \mathcal{K}[\sigma', \sigma], \end{cases}$$

whenever σ and σ' are RIA and LIA of φ , respectively.

Proof. Put $\hat{\phi}(z) = e^{-\ell z/2} \varphi(z)$, as in Lemma 4.4. Since ϕ is given by (2), one gets that $\hat{\phi}(z) = e^{(-\zeta - \ell/2)z} \phi(z)$ has the product form given by

$$\hat{\phi}(z) = C e^{(\xi - \zeta - \ell/2)z} z^\eta \prod_j \left(1 - \frac{z}{z_j}\right) e^{z/z_j},$$

where C, ξ are complex and η is a non negative integer. By Lemma 4.4, $\hat{\phi} \in \mathcal{K}[\sigma', \sigma]$. Hence, by the converse of Corollary 1.7,

$$\Re(\xi - \zeta - \ell/2) + \sum_j \Re(1/z_j) = \kappa - \Re(\zeta) - \ell/2 = 0.$$

Therefore, it follows that $\kappa = \Re(\zeta) + \ell/2$. Now, the theorem follows directly from Corollary 1.7. $\square$

Corollary 4.6. Let $\gamma \in \text{BV}[0, \ell]$ be real valued on $[0, \ell]$ and let $\phi(z) = e^{\zeta z} \varphi(z)$, where $\varphi = \mathcal{L}\{d\gamma\}$ and ζ is complex. If γ is increasing (resp. decreasing) and convex up (resp. down) on $[0, \ell]$, then $\Re(\zeta) \geq -\ell/2$ implies $\phi \in \mathcal{K}(-\infty, 0]$. Similarly, if γ is decreasing (resp. increasing) and convex up (resp. down) on $[0, \ell]$, then $\Re(\zeta) \leq -\ell/2$ implies $\phi \in \mathcal{K}[0, \infty)$.

Proof. If γ is increasing and convex up on $[0, \ell]$, then $\delta_\gamma(0) = 0$. This is not necessarily the case for $\delta_\gamma(\ell)$, albeit $\gamma(\ell-) \leq \gamma(\ell)$. Modify the end terms of $\mathcal{S}_n$ by defining, for $n \geq 2$,

$$\hat{\varphi}_n(z) = \sum_{m=0}^{n-1} \hat{\mu}_m e^{\tau_m z}$$

where, for $m = 1, 2, \dots, n-2$, we have $\hat{\mu}_0 = \mu_1, \hat{\mu}_m = \mu_m$ and

$$\hat{\mu}_{n-1} = \begin{cases} \hat{\mu}_{n-2} & \text{if } \delta_\gamma(\ell) = 0, \\ \mu_{n-1} & \text{if } \delta_\gamma(\ell) \neq 0. \end{cases}$$

A standard construction yields admissible functions $\hat{\gamma}_n \in \text{BV}[0, \ell]$ with $\hat{\varphi}_n = \mathcal{L}\{d\hat{\gamma}_n\}$, for $n \geq 2$. A routine argument then shows $\hat{\varphi}_n \rightarrow \mathcal{L}\{d\gamma\}$ as $n \rightarrow \infty$, uniformly on compacta. Since γ is increasing and convex up, we get $\hat{\mu}_m > 0$, for $m = 0, 2, \dots, n-1$ and, for $n \geq 2$,

$$\rho_n = \max \left\{ \frac{\hat{\mu}_m}{\hat{\mu}_{m+1}} \mid m = 0, 1, \dots, n-1 \right\} \leq 1.$$

By Lemma 3.2, $\sigma = 0$ is a RIA of $\hat{\varphi}_n$, for $n \geq 2$. If $\Re(\zeta) \geq -\ell/2$, then

$$\hat{\phi}_n(z) = e^{\zeta z} \hat{\varphi}_n \in \mathcal{K}(-\infty, 0]$$

for $n \geq 2$, by Theorem 4.5. It follows that $\phi \in \mathcal{K}(-\infty, 0]$. If γ is increasing and convex down on $[0, \ell]$, then define $\hat{\gamma} \in \text{BV}[0, \ell]$ by $\hat{\gamma}(\tau) = -\gamma(\tau)$, for $0 \leq \tau \leq \ell$. Then, $\hat{\gamma}$ is increasing and convex up on $[0, \ell]$ with $\mathcal{L}\{d\hat{\gamma}\} = -\varphi(z)$. The result in this case now follows by the argument above. If γ is decreasing and convex up on $[0, \ell]$, then $\delta_\gamma(\ell) = 0$. Define $\tilde{\gamma} \in \text{BV}[0, \ell]$ by $\tilde{\gamma}(\tau) = \gamma(\ell - \tau)$, for $0 \leq \tau \leq \ell$. Putting

$$\tilde{\phi}(z) = e^{-(\zeta+\ell)z} \tilde{\varphi}(z),$$

where

$$\tilde{\varphi} = \mathcal{L}\{d\tilde{\gamma}\} = -e^{\ell z} \varphi(-z)$$

yields $\tilde{\phi}(z) = -\phi(-z)$. Since $\tilde{\gamma}$ is increasing and convex up, we have that

$$\Re(-(\zeta + \ell)) \geq -\ell/2 \implies \tilde{\phi} \in \mathcal{K}(-\infty, 0].$$

This last condition is equivalent to the consequent, by (10). If γ is increasing and convex down on $[0, \ell]$, then a similar argument yields the result. $\square$

5. Membership conditions for Laplace Lebesgue integrals

If $\gamma \in \text{BV}[0, \ell]$, then the *Laplace integral* of γ is defined by the Lebesgue integral

$$\mathcal{L}\{\gamma\}(z) = \int_0^\ell e^{z\tau} \gamma(\tau) d\tau.$$

Definitions of the LIA and RIA for the Laplace integral of γ are identical to the definitions for the Laplace integral of $d\gamma$.

Corollary 5.1. *Let $\gamma \in \text{BV}[0, \ell]$ be non negative on $[0, \ell]$, and let $\phi(z) = e^{\zeta z} \varphi(z)$, where $\varphi = \mathcal{L}\{\gamma\}$ and ζ is complex.*

- (i) *If γ is increasing on $[0, \ell]$ and $\Re(\zeta) \geq -\ell/2$, then $\phi \in \mathcal{K}(-\infty, 0]$,*
- (ii) *If γ is decreasing on $[0, \ell]$ and $\Re(\zeta) \leq -\ell/2$, then $\phi \in \mathcal{K}[0, \infty)$.*

Proof. Let $\Upsilon \in \text{BV}[0, \ell]$ be given by

$$(15) \quad \Upsilon(\tau) = \int_0^\tau \gamma(t) dt,$$

for $0 \leq \tau \leq \ell$, so that $\varphi = \mathcal{L}\{d\Upsilon\}$. Since γ is non negative on $[0, \ell]$, we have that Υ is increasing on $[0, \ell]$. Moreover, if γ is increasing on $[0, \ell]$, then Υ is concave up on $[0, \ell]$. By Corollary 4.6, it follows that $\phi \in \mathcal{K}(-\infty, 0]$. If γ is non negative and decreasing on $[0, \ell]$, then $-\gamma$ is non positive and increasing on $[0, \ell]$. Thus, $\Upsilon_- = -\Upsilon$ is decreasing and convex up on $[0, \ell]$, with $\varphi = -\mathcal{L}\{d\Upsilon_-\}$. The result for this case now follows from Corollary 4.6. $\square$

For extending Theorem 4.5 to the Lebesgue integral, denote the subclass

$$\text{BV}_\circ[0, \ell] = \{\gamma \in \text{BV}[0, \ell] \mid \gamma(0) = \gamma(\ell) = 0\}.$$

If $\gamma \in \text{BV}_\circ[0, \ell]$, then integration by parts yields the relation $\mathcal{L}\{d\gamma\}(z) = -z\mathcal{L}\{\gamma\}(z)$. If γ is q -times differentiable on $[0, \ell]$, then denote $\gamma^p = \gamma^{(p)}$ for $p = 0, 1, \dots, q$. Let $\text{AC}_\circ[0, \ell]$ denote the subclass of functions in $\text{BV}_\circ[0, \ell]$ that are absolutely continuous on $[0, \ell]$, and define the subclass:

$$\text{BV}_\circ^q[0, \ell] = \{\gamma \mid \gamma^q \in \text{BV}_\circ[0, \ell] \text{ and } \gamma^p \in \text{AC}_\circ[0, \ell], 0 \leq p \leq q-1, \}.$$

It follows from the definition that $BV^0_\circ[0, \ell] = BV_\circ[0, \ell]$.

Lemma 5.2. *Suppose $\gamma \in BV^q_\circ[0, \ell]$. Then*

$$\mathcal{L}\{d\gamma^q\}(z) = (-z)^{q+1} \mathcal{L}\{\gamma\}(z), \quad q \geq 0.$$

Proof. If $q = 0$, then $\gamma = \gamma^0 \in BV_\circ[0, \ell]$ and $\mathcal{L}\{d\gamma\}(z) = -z\mathcal{L}\{\gamma\}(z)$, as noted above. Assume that the result holds for functions in $BV^q_\circ[0, \ell]$, and take γ in $BV^{q+1}_\circ[0, \ell] \subseteq BV^q_\circ[0, \ell]$. By hypothesis,

$$\mathcal{L}\{d\gamma^q\}(z) = (-z)^{q+1} \mathcal{L}\{\gamma\}(z).$$

Expressing γ^q as a Lebesgue integral

$$\gamma^q(\tau) = \int_0^\tau \gamma^{q+1}(t) dt$$

yields $\mathcal{L}\{d\gamma^q\} = \mathcal{L}\{\gamma^{q+1}\}$, i.e., a Steiltjes integral as a Lebesgue integral. Integration by parts gives $\mathcal{L}\{d\gamma^{q+1}\}(z) = -z\mathcal{L}\{\gamma^{q+1}\}(z)$. Combining these relations gives

$$\mathcal{L}\{d\gamma^{q+1}\}(z) = -z\mathcal{L}\{\gamma^{q+1}\}(z) = -z\mathcal{L}\{d\gamma^q\}(z) = (-z)^{q+2} \mathcal{L}\{\gamma\}(z).$$

The result follows by induction. $\square$

Let $\gamma \in BV^q_\circ[0, \ell]$ with γ^q admissible, and let $\varphi^q = \mathcal{L}\{\gamma^q\}$ and $\varphi = \mathcal{L}\{\gamma\}$. In view of Lemma 5.2, σ and σ' are RIA and LIA of φ^q if and only if σ and σ' are RIA and LIA of φ , respectively. By Lemma 4.1, if $\delta_{\gamma^q}(0) = |\gamma^q(0+)|$ and $\delta_{\gamma^q}(\ell) = |\gamma^q(\ell-)|$, then the inequations (resp. equations)

$$\int_0^{\ell-} e^{t\tau} d|\gamma^q|(t) \underset{(\text{=})}{\leq} \delta_{\gamma^q}(\ell) e^{\ell\tau} \quad \text{and} \quad \int_{0+}^{\ell} e^{t\tau} d|\gamma^q|(t) \underset{(\text{=})}{\leq} \delta_{\gamma^q}(0)$$

have solutions (resp. unique solutions) denoted by σ and σ' , which are thus RIA and LIA of φ , respectively.

Corollary 5.3. *Let $\gamma \in BV^q_\circ[0, \ell]$ with γ^q admissible, and let Φ be given by (1) with $\lambda \geq 0$ and $\phi(z) = e^{\zeta z} \varphi(z)$, where $\varphi(z) = \mathcal{L}\{\gamma\}$ and ζ is complex. Then, ϕ is given by (2) with $\kappa = \Re(\zeta) + \ell/2$ given by (6), and where equivalences (14) hold whenever σ and σ' are RIA and LIA of φ , respectively.*

Proof. The results of Theorem 4.5 apply directly to $\phi^q(z) = e^{\zeta z} \varphi^q(z)$, where $\varphi^q = \mathcal{L}\{\gamma^q\}$. By Lemma 5.2, $\varphi(z) = (-z)^{-(q+1)} \varphi^q(z)$, from which these results translate over to ϕ . $\square$

6. Membership conditions for Dirichlet series

Finite sums having the form

$$(16) \quad \varphi(z) = \sum_{m=0}^n \mu_m e^{\tau_m z},$$

where the μ_m are complex for $m = 0, 1, \dots, n$, and $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = \ell$ is a partition of $[0, \ell]$, are formally related to Dirichlet series by the transformation

$z \rightarrow -z$. Expressions given by (16) are considered below since they are readily applied to the preceding results.

The proof of the next lemma is a routine calculation of the Stieltjes integral of a step function.

Lemma 6.1. *Let $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = \ell$ be a partition of $[0, \ell]$. For $m = 0, 1, \dots, n$, let μ_m be complex, and let φ be given by (16). Suppose $\gamma \in \text{BV}[0, \ell]$ is the step function*

$$(17) \quad \gamma(\tau) = \begin{cases} 0 & \text{if } \tau = \tau_0, \\ \gamma_n & \text{if } \tau = \tau_n, \\ \gamma_0 & \text{if } \tau_0 < \tau < \tau_1, \\ \gamma_m & \text{if } \tau_m \leq \tau < \tau_{m+1}, \quad m = 1, \dots, n-1. \end{cases}$$

where $\gamma_m = \sum_{k=0}^m \mu_k$, for $m = 0, 1, \dots, n$. Then $\varphi = \mathcal{L}\{d\gamma\}$. In particular, if $\mu_0\mu_n \neq 0$, then the step function given by (17) is admissible.

In view of Lemma 4.1, where $\delta_\gamma(0) = |\mu_0|$ and $\delta_\gamma(\ell) = |\mu_n|$, the inequations (resp. equations)

$$\sum_{m=0}^{n-1} |\mu_m| e^{\tau_m \tau} \leq |\mu_n| e^{\tau_n \tau} \quad \text{and} \quad \sum_{m=1}^n |\mu_m| e^{\tau_m \tau} \leq |\mu_0|$$

(=) (=)

have solutions (resp. unique solutions) σ and σ' , which are RIA and LIA of φ , respectively. The next result follows directly from Theorem 4.5 and Lemma 6.1.

Corollary 6.2. *Let Φ be given by (1) with $\lambda \geq 0$ and $\phi(z) = e^{\zeta z} \varphi(z)$, where ζ is complex and $\varphi = \mathcal{L}\{d\gamma\}$, and where γ is given by (17) and $\mu_0\mu_n \neq 0$. Then, ϕ is given by (2) with $\kappa = \Re(\zeta) + \ell/2$ given by (6), and where equivalences (14) hold, whenever σ and σ' are RIA and LIA of φ , respectively.*

Expressions (16) are special cases of the Laplace integral of a step function as in the next lemma.

Lemma 6.3. *Let $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = \ell$ be a partition of $[0, \ell]$. For $m = 0, 1, \dots, n$, let μ_m be complex, and let φ be given by (16). Suppose $\gamma \in \text{BV}[0, \ell]$ is the step function*

$$(18) \quad \gamma(\tau) = \begin{cases} 0 & \text{if } \tau = \tau_0, \tau_n, \\ \gamma_0 & \text{if } \tau_0 < \tau < \tau_1, \\ \gamma_m & \text{if } \tau_m \leq \tau < \tau_{m+1}, \quad m = 1, \dots, n-1. \end{cases}$$

where $\gamma_m = \sum_{k=0}^m \mu_k$, for $m = 0, 1, \dots, n-1$. Then $\varphi(z) = -z\mathcal{L}\{\gamma\}(z)$.

Proof. The condition $\gamma_n = 0$ in Lemma 6.1 gives $\gamma \in \text{BV}_o[0, \ell]$. This yields

$$\mathcal{L}\{d\gamma\}(z) = -z\mathcal{L}\{\gamma\}(z)$$

as claimed. $\square$

Corollary 6.4. *Let Φ be given by (1) with $\lambda \geq 0$ and $\phi(z) = e^{\zeta z} z\varphi(z)$, where ζ is complex and $\varphi = \mathcal{L}\{\gamma\}$, and where γ is given by (18) and $\mu_0\mu_n \neq 0$. Then, ϕ is given*

by (2) with $\kappa = \Re(\zeta) + \ell/2$ given by (6), and where equivalences (14) hold, whenever σ and σ' are RIA and LIA of φ , respectively.

Proof. The proof follows directly from Corollary 6.2 and Lemma 6.3. $\square$

7. Laplace integrals with only imaginary zeros

A classical result due to Schur ([14] or [7]) states that if $p \in \mathcal{K}(-\infty, 0]$ is a real polynomial, then the polynomial

$$\hat{p}(z) = \frac{p(z) - p^-(z)}{z}$$

is real and is even with $\hat{p} \in \mathcal{K}[0, 0]$, where $p^-(z) = p(-z)$. This notation used for the transformation $z \rightarrow -z$ is needed below for several results.

Lemma 7.1. *Let $\Phi \in \mathcal{K}(-\infty, \sigma]$ be a real entire function. Then, the entire function*

$$\hat{\Phi}(z) = \frac{\Phi_\sigma(z) - \Phi_\sigma^-(z)}{z}$$

is real and is even, with $\hat{\Phi} \in \mathcal{K}[0, 0]$.

Proof. By Corollary 2.3, $g_{n,\sigma} \in \mathcal{K}(-\infty, 0]$ so that $\hat{g}_{n,\sigma} \in \mathcal{K}[0, 0]$, for $n \geq 1$. By (9), $g_{n,\sigma}$ converges on compacta to $\hat{\Phi}$, so that $\hat{g}_{n,\sigma}$ converges on compacta to $\hat{\Phi}$. It follows that $\hat{\Phi} \in \mathcal{K}[0, 0]$. $\square$

If $\Gamma \in \text{BV}[-\ell, \ell]$, then the *Laplace integrals* of $d\Gamma$ and Γ are defined by the integrals

$$\mathcal{L}\{d\Gamma\}(z) = \int_{-\ell}^{\ell} e^{z\tau} d\Gamma(\tau) \quad \text{and} \quad \mathcal{L}\{\Gamma\}(z) = \int_{-\ell}^{\ell} e^{z\tau} \Gamma(\tau) d\tau,$$

respectively.

Theorem 7.2. *Let $\gamma \in \text{BV}[0, \ell]$ be real valued. For σ real, define $\Gamma \in \text{BV}[-\ell, \ell]$ by*

$$\Gamma(\tau) = \int_{-\ell}^{\tau} [e^{\sigma t} d\tilde{\gamma}(t) + e^{-\sigma t} d\tilde{\gamma}^-(t)],$$

where $\tilde{\gamma} \in \text{BV}[-\ell, \ell]$ is the extension of γ given by

$$(19) \quad \tilde{\gamma}(t) = \begin{cases} 0 & \text{if } -\ell \leq t < 0, \\ \gamma(t) & \text{if } 0 \leq t \leq \ell. \end{cases}$$

If σ is a RIA of $\varphi = \mathcal{L}\{d\gamma\}$, then $\mathcal{L}\{d\Gamma\} \in \mathcal{K}[0, 0]$ and $\mathcal{L}\{\Gamma\} \in \mathcal{K}[0, 0]$.

Proof. A calculation shows that

$$\mathcal{L}\{d\Gamma\}(z) = \int_{-\ell}^{\ell} e^{\tau z} [e^{\sigma t} d\tilde{\gamma}(t) + e^{-\sigma t} d\tilde{\gamma}^-(t)] = \varphi_\sigma(z) - \varphi_\sigma^-(z) = z\hat{\varphi}(z).$$

By Lemma 7.1, $\mathcal{L}\{d\Gamma\} \in \mathcal{K}[0, 0]$. Since $\Gamma(-\ell) = \Gamma(\ell) = 0$, integration by parts yields

$$\mathcal{L}\{\Gamma\}(z) = -z\mathcal{L}\{d\Gamma\}(z)$$

and thus, $\mathcal{L}\{\Gamma\} \in \mathcal{K}[0, 0]$. $\square$

In view of Corollary 4.6, Theorem 7.2 applies directly to the following special case.

Corollary 7.3. *Let $\gamma \in \text{BV}[0, \ell]$ be increasing (resp. decreasing) and convex up (resp. down) on $[0, \ell]$. Let $\Gamma \in \text{BV}[-\ell, \ell]$ be defined by*

$$(20) \quad \Gamma(\tau) = \int_{-\ell}^{\tau} d[\tilde{\gamma}(t) + \tilde{\gamma}^-(t)]$$

with $\tilde{\gamma}$ given by (19). Then, $\mathcal{L}\{d\Gamma\} \in \mathcal{K}[0, 0]$ and $\mathcal{L}\{\Gamma\} \in \mathcal{K}[0, 0]$.

Corollary 7.4. *Suppose $\Gamma \in \text{BV}[-\ell, \ell]$ is non negative, even, and concave down on $[-\ell, \ell]$. Then, $\mathcal{L}\{\Gamma\} \in \mathcal{K}[0, 0]$.*

Proof. Let Γ be given by (20) with $\gamma \in \text{BV}[0, \ell]$ decreasing and convex down on $[0, \ell]$. The result follows from Corollary 7.3. $\square$

Corollary 7.5. *Let $\gamma \in \text{BV}[0, \ell]$ be real valued. For σ real, define $\Gamma \in \text{BV}[-\ell, \ell]$ by*

$$\Gamma(\tau) = \int_{-\ell}^{\tau} [e^{\sigma t} \tilde{\gamma}(t) - e^{-\sigma t} \tilde{\gamma}^-(t)] dt,$$

where $\tilde{\gamma}$ is given by (19). Suppose σ is a RIA of $\varphi = \mathcal{L}\{\gamma\}$. Then, $\mathcal{L}\{d\Gamma\} \in \mathcal{K}[0, 0]$ and $\mathcal{L}\{\Gamma\} \in \mathcal{K}[0, 0]$.

Proof. Note that $\Upsilon \in \text{BV}[0, \ell]$, given by (15), satisfies the conditions of Theorem 7.2 with $\mathcal{L}\{\gamma\} = \mathcal{L}\{\Upsilon\}$. $\square$

The proof of the next corollary now follows directly from Corollary 7.5.

Corollary 7.6. *Let $\gamma \in \text{BV}[0, \ell]$ be non negative and increasing on $[0, \ell]$. Let $\Gamma \in \text{BV}[-\ell, \ell]$ be defined by*

$$(21) \quad \Gamma(\tau) = \int_{-\ell}^{\tau} [\tilde{\gamma}(t) - \tilde{\gamma}^-(t)] dt,$$

where $\tilde{\gamma}$ is given by (19). Then, $\mathcal{L}\{d\Gamma\} \in \mathcal{K}[0, 0]$ and $\mathcal{L}\{\Gamma\} \in \mathcal{K}[0, 0]$.

Corollary 7.7. *Suppose $\Upsilon \in \text{BV}[-\ell, \ell]$ is increasing (resp. decreasing) and odd on $[-\ell, \ell]$. Then, $\mathcal{L}\{\Upsilon\} \in \mathcal{K}[0, 0]$.*

Proof. Suppose first that Υ is increasing and odd. Let $\Upsilon = \Gamma'$, where Γ is given by (21) with $\gamma \in \text{BV}[0, \ell]$ non negative and increasing on $[0, \ell]$. The result now follows from Corollary 7.6, since $\mathcal{L}\{\Upsilon\} = \mathcal{L}\{d\Gamma\}$. If, now, Υ is decreasing and odd, then the function given by $\Upsilon_- = -\Upsilon$ is increasing and odd with $\mathcal{L}\{\Upsilon\} = -\mathcal{L}\{\Upsilon_-\}$. $\square$

8. Application to the Hausdorff moment problem

For $\gamma \in \text{BV}[0, 1]$ and $m \geq 0$, the value

$$(22) \quad \mu_m = \int_0^1 \tau^m d\gamma(\tau)$$

is called the m -th moment of γ , and $\mu = \{\mu_m\}_{m=0}^{\infty}$ is called a *moment sequence* for γ . Conversely, given a sequence $\mu = \{\mu_m\}_{m=0}^{\infty}$, the problem of finding $\gamma \in \text{BV}[0, 1]$ with γ non decreasing on $[0, 1]$ such that (22) holds is known as the *Hausdorff moment problem* (HMP) for μ (see [16]).

For $n \geq 0$ and $m \geq 0$, define the backward differences

$$\nabla^i \mu_m = \begin{cases} \mu_m & \text{if } n = 0, \\ \nabla^{n-1} \mu_m - \nabla^{n-1} \mu_{m+1} & \text{if } n > 0. \end{cases}$$

Hence,

$$\nabla^n \mu_m = \sum_{k=0}^n (-1)^k \binom{n}{k} \mu_{m+k},$$

for $m \geq 0$ and $n \geq 0$. Denoting the entire function

$$\varphi_m(z) = \sum_{k=0}^{\infty} \mu_{m+k} \frac{z^k}{k!},$$

the n -th Appell polynomial associated with φ_m is defined by

$$\rho_{m,n}(w) = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \mu_{m+k} w^{n-k},$$

for $m \geq 0$ and $n \geq 0$. These polynomials are generated by

$$e^{wz} \varphi_m(z) = \sum_{n=0}^{\infty} \rho_{m,n}(w) z^n,$$

so the differences are generated by

$$e^{-z} \varphi_m(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \nabla^n \mu_m}{n!} z^n,$$

for $m \geq 0$. The sequence μ is called *completely monotone* if it satisfies the condition $\nabla^n \mu_m \geq 0$, for $m \geq 0$ and $n \geq 0$. Hausdorff's classic result (see [16]) states that this condition is necessary and sufficient for the existence of a solution to the HMP. For $n \geq 0$, define the triangular array

$$\mu_n(p, q) = \binom{n}{k} \nabla^q \mu_p,$$

for all $p + q = n$. Clearly, μ is completely monotone if and only if $\mu_n(p, q) \geq 0$, for all $p + q = n$ and $n \geq 0$. Necessary and sufficient conditions for the solution γ to the HMP for μ to be increasing and convex up on $[0, 1]$ are given in the following theorem by Diaconis and Freedman (see [5]).

Theorem 8.1. *Let γ be non decreasing on $[0, 1]$ and absolutely continuous with γ' non decreasing on $[0, 1]$. Necessary and sufficient conditions for μ to be a moment sequence for $\gamma \in \text{BV}[0, 1]$ are that μ is completely monotone with*

$$\mu_n(p, q) \leq \mu_n(p + 1, q - 1),$$

for $p + q = n$ and $n \geq 0$.

Denote the sequence $\mu^m = \{\mu_{m+k}\}_{k=0}^{\infty}$, for $m \geq 0$. The proof of the next lemma follows from a routine induction argument.

Lemma 8.2. *Suppose the sequence μ is completely monotone and suppose also that $\mu_n(p, q) \leq \mu_n(p+1, q-1)$, for $p+q = n$ and $n \geq 1$. Then, for $m \geq 0$, the sequence μ^m is completely monotone with $\mu_n^m(p, q) \leq \mu_n^m(p+1, q-1)$, for $p+q = n$ and $n \geq 1$*

Denote the function $\phi_m(z) = e^{\zeta z} \varphi_m(z)$, where ζ is complex, for $m \geq 0$.

Theorem 8.3. *Suppose μ is completely monotone with $\mu_n(p, q) \leq \mu_n(p+1, q-1)$, for $p+q = n$ and $n \geq 1$. Then, $\Re(\zeta) \geq -1/2$ implies $\phi_m \in \mathcal{K}(-\infty, 0]$, for $m \geq 0$.*

Proof. By Lemma 8.2, μ^m satisfies the conditions of Theorem 8.1, for $m \geq 0$. Thus, μ^m is a moment sequence for some $\gamma_m \in \text{BV}[0, 1]$ so that $\varphi_m = \mathcal{L}\{d\gamma_m\}$, where γ_m is increasing and convex up on $[0, 1]$, for $m \geq 0$. The result now follows directly from Corollary 5.1. $\square$

Corollary 8.4. *Let $\ell > 0$ and put $\hat{\mu} = \{\mu_m/\ell^m\}_{m=0}^\infty$. Suppose $\hat{\mu}$ is completely monotone with $\hat{\mu}_n(p, q) \leq \hat{\mu}_n(p+1, q-1)$, for $p+q = n$ and $n \geq 1$. Then, we have that $\Re(\zeta) \geq -\ell/2$ implies $\phi_m \in \mathcal{K}(-\infty, 0]$ for $m \geq 0$.*

Proof. For $m \geq 0$, put

$$\hat{\varphi}_m(z) = \sum_{k=0}^{\infty} \frac{\mu_{m+k} z^k}{\ell^{m+k} k!}$$

and $\hat{\phi}_m(z) = e^{\zeta z/\ell} \hat{\varphi}_m(z)$. By Theorem 8.3, $\Re(\zeta/\ell) \geq -1/2$ yields $\hat{\phi}_m \in \mathcal{K}(-\infty, 0]$, for $m \geq 0$. Since $\phi_m(z) = \ell^m \hat{\phi}_m(\ell z)$, one has that $\hat{\phi}_m \in \mathcal{K}(-\infty, 0]$ implies ϕ_m is in $\mathcal{K}(-\infty, 0]$ for $m \geq 0$. $\square$

The n -th Jensen polynomial associated with φ_m is defined by

$$g_{m,n}(w) = \sum_{k=0}^n \binom{n}{k} \mu_{m+k} w^k,$$

for $m \geq 0$ and $n \geq 0$. The Appell and Jensen polynomials associated with Laplace integrals of convex measures are a natural source of Hurwitz polynomials, as in the next corollary.

Corollary 8.5. *Let $\ell > 0$ and put $\hat{\mu} = \{\mu_m/\ell^m\}_{m=0}^\infty$. Suppose $\hat{\mu}$ is completely monotone with $\hat{\mu}_n(p, q) \leq \hat{\mu}_n(p+1, q-1)$, for $p+q = n$ and $n \geq 1$. Then, one has $\rho_{m,n} \in \mathcal{K}(-\infty, -\ell/2]$, for $m \geq 0$ and $n \geq 0$. Moreover, the zeros of $g_{m,n}$ are all in the disc $|z + 1/\ell| \leq 1/\ell$.*

Proof. By Corollary 8.4, if $\Re(\zeta) > -\ell/2$, then $\phi_m \in \mathcal{K}(-\infty, 0]$ and therefore, $\phi_m^{(n)} \in \mathcal{K}(-\infty, 0]$, for $m \geq 0$ and $n \geq 0$. Thus, the sequence $\psi_k(\zeta) = \phi_m^{(n)}(1/k)$, for $k \geq 1$, is non zero for $\Re(\zeta) > -\ell/2$. Since $\psi_k(\zeta) \rightarrow n! \rho_{m,n}(\zeta)$ as $k \rightarrow \infty$ uniformly on compacta, one has, by Hurwitz theorem, that $\rho_{m,n}$ is non zero in the half plane $\Lambda(-\ell/2, \infty)$, for $m \geq 0$ and $n \geq 0$. The rest of the proof follows from the relation $g_{m,n}(w) = n! w^n \rho_{m,n}(w^{-1})$. $\square$

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