

# **A THEOREM ON THE BOUNDEDNESS OF SOLUTIONS OF FIFTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAY**

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**RÉSUMÉ.** Nous établissons des conditions suffisantes pour que toutes les solutions d'une équation différentielle non-linéaire du cinquième ordre à retard soient bornées. Ce nouveau résultat s'obtient au moyen d'une fonctionnelle de Lyapunov.

**ABSTRACT.** In this paper, some sufficient conditions are established to ensure the boundedness of all solutions of a nonlinear differential equation of fifth order with delay. By introducing a Lyapunov functional, we obtain a new result on the subject.

## **1. Introduction**

Especially since the 1960's, qualitative behaviors of solutions of ordinary nonlinear differential equations of higher order without delay or with delay have been discussed extensively. In particular, during this period, many research papers have been published and important results on the stability, the instability, the boundedness of solutions, the existence of periodic solutions and so forth of certain ordinary nonlinear differential equations of fifth order without delay have been obtained. For some papers dealing with the stability and boundedness of solutions of fifth order nonlinear differential equations without delay, the readers can refer to the papers of Abou-El Ela and Sadek [1, 2], Burganskaja [3], Chukwu [6, 7], Sadek [19], Sinha [20], Tunç [21-23], Yu [30] and the references cited in these papers. In these papers, the authors have considered Lyapunov functions [16]. In particular, their results show that Lyapunov's [16] second method has been very successfully applied to higher order nonlinear differential equations without delay to discuss the qualitative behaviors of the solutions.

In spite of existence of many papers on the stability and boundedness of solutions of fifth order nonlinear differential equations without delay, it seems that the study of stability and boundedness of solutions of fifth order nonlinear differential equations with delay, which will be considered here, has not yet received any attention in the literature. This work is a first attempt to obtain sufficient conditions on the boundedness of solutions of fifth order nonlinear differential equation with delay; see also [26], for a different work on the subject. Perhaps, the possible difficulties in this case are due to the construction of Lyapunov functionals for the differential equations of higher order with delay.

How to construct those Lyapunov functionals? So far, no authors have discussed them. In fact, there is no general methods to construct Lyapunov functionals. It is indeed well-known that it is more difficult to construct Lyapunov functionals for higher order differential equations with delay than Lyapunov functions for those equations without delay. However, in recent years, many good books were published on differential equations with delay, for example, one can refer to the books of Burton [4], Èl'sgol'ts [8], Èl'sgol'ts and Norkin [9], Hale [10], Hale and Lunel [11], Kolmanovskii and Myshkis [12], Kolmanovskii and Nosov [13], Krasovskii [14], Mohammed [17], Yoshizawa [29] and the references cited in these sources. There also exist many works on the qualitative behaviors of differential equations of first and second order with delay and without delay; see, for instance, the books of Kolmanovskii and Myshkis [12], Kolmanovskii and Nosov [13] and the papers of Burton and Zhang [5], Liu and Huang [15], Tunç ([24, 25]), Tunç and Şevli [27], C. Tunç and E. Tunç [28] and the references listed in these sources.

Here, we are interested in obtaining a result on the boundedness of solutions of the fifth order nonlinear differential equation with delay

$$(1) \quad \begin{aligned} p(t) = & x^{(5)}(t) + f_1(t, x''(t), x'''(t))x^{(4)}(t) + \alpha_2 x'''(t) + f_3(x''(t-r)) \\ & + f_4(x'(t-r)) + \alpha_5 x(t) \end{aligned}$$

or its equivalent system

$$(2) \quad \begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ z'(t) = w(t), \\ w'(t) = u(t), \\ u'(t) = -f_1(t, z(t), w(t))u(t) - \alpha_2 w(t) - f_3(z(t)) - f_4(y(t)) \\ \quad - \alpha_5 x(t) + \int_{t-r}^t f'_3(z(s))w(s)ds + \int_{t-r}^t f'_4(y(s))z(s)ds + p(t), \end{cases}$$

where  $f_1, f_3, f_4$  and  $p$  are continuous functions for the arguments displayed explicitly,  $r$  is a positive constant (that is,  $r$  is a fixed delay) and  $\alpha_2$  and  $\alpha_5$  are positive constants. It is assumed that the derivatives  $f'_3(z)$  and  $f'_4(y)$  exist and are continuous for all  $z$  and  $y$ ; throughout the paper  $x(t), y(t), z(t), w(t)$  and  $u(t)$  are abbreviated as  $x, y, z, w$  and  $u$ , respectively. All solutions considered are also assumed to be real valued. We also use Lyapunov's second (or direct) method to investigate the boundedness properties of solutions of the differential equation with delay (1).

## 2. Preliminaries

In order to obtain our main result, we will give some important basic information for the general non-autonomous differential system with delay; see also Burton [4], Èl'sgol'ts [8], Èl'sgol'ts and Norkin [9], Hale [10], Hale and Lunel [11], Kolmanovskii and Myshkis [12], Kolmanovskii and Nosov [13], Krasovskii [14], Mohammed [17] and Yoshizawa [29].

Now, we consider the general non-autonomous differential system with delay

$$(3) \quad \dot{x} = f(t, x_t), \quad x_t(\theta) = x(t + \theta), \quad \text{with } -r \leq \theta \leq 0 \quad \text{and} \quad t \geq 0,$$

where  $f : [0, \infty) \times C_H \rightarrow \mathbb{R}^n$  is a continuous mapping,  $f(t, 0) = 0$ , and we suppose that  $f$  takes closed bounded sets into bounded sets of  $\mathbb{R}^n$ . Here  $(C, \|\cdot\|)$  is the Banach space of continuous function  $\phi : [-r, 0] \rightarrow \mathbb{R}^n$  with supremum norm,  $r > 0$  and  $C_H$  is the open  $H$ -ball in  $C$ , that is  $C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : |\phi| < H\}$ . Standard existence theory (see Burton [4, p. 312]) shows that if  $\phi \in C_H$  and  $t \geq 0$ , then there is at least one continuous solution  $x(t, t_0, \phi)$  such that, on  $[t_0, t_0 + \alpha)$ , the equation (3) is satisfied for  $t > t_0$ ,  $x_t(t, \phi) = \phi$  and  $\alpha$  is a positive constant. If there is a closed subset  $B \subset C_H$  for which the solution remains in  $B$ , then  $\alpha = \infty$ . Further, the symbol  $|\cdot|$  will denote the norm in  $\mathbb{R}^n$  with  $|x| = \max_{1 \leq i \leq n} |x_i|$ .

The next definition is taken from [4, p. 260].

**Definition.** Let  $V(t, \phi)$  be a continuous functional, defined for  $t \geq 0$ , and let  $\phi \in C_H$ . The derivative of  $V$  along the solutions of (3) will be denoted by  $\dot{V}_{(3)}$  and is defined by the relation

$$\dot{V}_{(3)}(t, \phi) = \limsup_{h \rightarrow 0} \frac{V(t+h, x_{t+h}(t_0, \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where  $x(t_0, \phi)$  is the solution of (3) with  $x_{t_0}(t_0, \phi) = \phi$ .

**Example.** We consider the second order nonlinear differential equation with delay

$$\begin{aligned} x''(t) + \varphi(t, x(t-r(t)), x'(t-r(t))) + (1+e^{-t})f(x(t-r(t))) \\ = \frac{1}{1+t^2+x^2(t)+x^2(t-r(t))+x'^2(t)+x'^2(t-r(t))} \end{aligned}$$

where  $0 \leq r(t) \leq \gamma_0$ ,  $\gamma_0$  is a positive constant which will be determined later,  $t \in \mathbb{R}^+$ ,  $\mathbb{R}^+ = [0, \infty)$ , the functions  $\varphi$  and  $f$  are continuous in their respective arguments and the derivative  $f'(x) = \frac{d}{dx}f(x)$  exists and is continuous.

Instead of the above equation, we take into consideration the equivalent system:

$$(4) \quad \begin{cases} x' = y; \\ y' = -\varphi(t, x(t-r(t)), y(t-r(t))) - (1+e^{-t})f(x) \\ \quad + (1+e^{-t}) \int_{t-r(t)}^t f'(x(s))y(s)ds \\ \quad + \frac{1}{1+t^2+x^2+x^2(t-r(t))+y^2+y^2(t-r(t))}. \end{cases}$$

We also assume that there are positive constants  $\alpha_1, \alpha_2$  and  $L$  such that the following conditions are satisfied:

- (i)  $\frac{\varphi(t, x(t-r(t)), y(t-r(t)))}{y} \geq \alpha_1$ , for all  $t, x, x(t-r(t)), y \neq 0$ , and  $y(t-r(t))$ ;
- (ii)  $\frac{f(x)}{x} \geq \alpha_2$ , for  $x \neq 0$ , and  $|f'(x)| \leq L$ , for all  $x$ .

To show the boundedness of all solutions of the system (4), we introduce the Lyapunov functional

$$2V_0(t, x_t, y_t) = 2(1 + e^{-t}) \int_0^x f(s)ds + y^2 + 2\lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds,$$

where  $\lambda$  is a positive constant which will be determined later. It is clear that the functional  $V_0(t, x_t, y_t)$  is positive definite:

$$V_0(t, 0, 0) = 0$$

and

$$\begin{aligned} V_0(t, x_t, y_t) &= (1 + e^{-t}) \int_0^x \frac{f(s)}{s} s ds + \frac{1}{2}y^2 + \lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds \\ &\geq \int_0^x \alpha_2 s ds + \frac{1}{2}y^2 + \lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds \\ &= \frac{1}{2}\alpha_2 x^2 + \frac{1}{2}y^2 \geq k_1(x^2 + y^2), \end{aligned}$$

where  $k_1 = \min \left\{ \frac{1}{2}\alpha_2, \frac{1}{2} \right\}$ .

Now, let  $(x(t), y(t))$  be an arbitrary solution of the system (4). Evaluating the derivative of the functional  $V_0(t, x_t, y_t)$  with respect to  $t$  along the solution  $(x(t), y(t))$ , we get

$$\begin{aligned} \frac{d}{dt} V_0(t, x_t, y_t) &= -\varphi(t, x(t-r(t)), y(t-r(t)))y \\ &\quad + (1 + e^{-t})y \int_{t-r(t)}^t f'(x(s))y(s)ds \\ &\quad + \frac{y}{1 + t^2 + x^2 + x^2(t-r(t)) + y^2 + y^2(t-r(t))} \\ &\quad + \lambda y^2 r(t) - \lambda(1 - r'(t)) \int_{t-r(t)}^t y^2(s)ds - e^{-t} \int_0^x f(s)ds. \end{aligned}$$

By using the assumptions (i), (ii), the inequality  $2|ab| \leq a^2 + b^2$  and the fact that

$$\frac{y}{1 + t^2 + x^2 + x^2(t-r(t)) + y^2 + y^2(t-r(t))} \leq \frac{|y|}{1 + t^2} \leq \frac{1}{1 + t^2} + \frac{y^2}{1 + t^2},$$

it follows that

$$\begin{aligned} \frac{d}{dt} V_0(t, x_t, y_t) &\leq -\alpha_1 y^2 + L y^2 r(t) + L \int_{t-r(t)}^t y^2(s)ds + \lambda y^2 r(t) \\ &\quad + \frac{1}{1+t^2} + \frac{y^2}{1+t^2} - \lambda(1 - r'(t)) \int_{t-r(t)}^t y^2(s)ds \\ &= -[\alpha_1 - (L + \lambda)r(t)] y^2 + \frac{1}{1+t^2} + \frac{y^2}{1+t^2} \\ &\quad + (L - \lambda(1 - r'(t))) \int_{t-r(t)}^t y^2(s)ds. \end{aligned}$$

Let  $r'(t) \leq \beta$ , where  $\beta$  is a constant with  $0 < \beta < 1$ . Hence

$$\begin{aligned} \frac{d}{dt} V_0(t, x_t, y_t) &\leq -[\alpha_1 - (L + \lambda)\gamma_0] y^2 + \frac{1}{1+t^2} + \frac{y^2}{1+t^2} \\ &\quad + (L - \lambda(1 - \beta)) \int_{t-r(t)}^t y^2(s) ds \end{aligned}$$

for  $0 \leq r(t) \leq \gamma_0$ .

Let us choose  $\lambda = \frac{L}{(1-\beta)}$ . Then

$$\frac{d}{dt} V_0(t, x_t, y_t) \leq -[\alpha_1 - (L + \lambda)\gamma_0] y^2 + \frac{1}{1+t^2} + \frac{y^2}{1+t^2},$$

hence

$$\frac{d}{dt} V_0(t, x_t, y_t) \leq -\alpha y^2 + \frac{1}{1+t^2} + \frac{y^2}{1+t^2}$$

for some constant  $\alpha > 0$  provided that  $\gamma_0 < \frac{\alpha_1}{L+\lambda}$ .

Observe that

$$\frac{d}{dt} V_0(t, x_t, y_t) \leq \frac{1}{1+t^2} + \frac{y^2}{1+t^2}.$$

Clearly, from

$$k_1 (x^2 + y^2) \leq V_0(t, x_t, y_t),$$

we get

$$y^2 \leq k_1^{-1} V_0(t, x_t, y_t).$$

This last inequality implies that

$$\frac{d}{dt} V_0(t, x_t, y_t) \leq \frac{1}{1+t^2} + \frac{1}{k_1(1+t^2)} V_0(t, x_t, y_t).$$

Integrating this inequality from 0 to  $t$ , using the fact that  $\frac{1}{1+t^2} \in L^1(0, \infty)$  and the Gronwall-Reid-Bellman inequality, (see [18]), one can obtain

$$\begin{aligned} V_0(t, x_t, y_t) &\leq V_0(0, x_0, y_0) + \frac{\pi}{2} + \frac{1}{k_1} \int_0^t \frac{1}{1+s^2} (V_0(s, x_s, y_s)) ds \\ &\leq (V_0(0, x_0, y_0) + \frac{\pi}{2}) \exp \left( \frac{1}{k_1} \int_0^t \frac{1}{1+s^2} ds \right) \\ &= (V_0(0, x_0, y_0) + \frac{\pi}{2}) \exp \left( \frac{\pi}{2k_1} \right) \\ &= k_2 < \infty, \end{aligned}$$

where  $k_2$  is the strictly positive constant  $(V_0(0, x_0, y_0) + \frac{\pi}{2}) \exp \left( \frac{\pi}{2k_1} \right)$ .

Now, in view of the above inequalities, we have

$$x^2 + y^2 \leq k_1^{-1} V_0(t, x_t, y_t) \leq k_3$$

where  $k_3 = k_1^{-1} k_2$ . Thus, for solutions of the system (4), we conclude that

$$|x| \leq k_3 \quad \text{and} \quad |y| \leq k_3$$

for all  $t \geq 0$ . That is,

$$|x| \leq k_3, \quad \text{and} \quad |x'| \leq k_3$$

for all  $t \geq 0$ . This confirms that all the solutions of the considered equation are bounded.

### 3. Main result

The following result is established.

**Theorem 1.** *In addition to the basic assumptions imposed on the functions  $f_1, f_3, f_4$  and  $p$  appearing in equation (1), assume that there are positive constants  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \varepsilon, \varepsilon_0, \varepsilon_1, \delta, M, H$  and  $H_1$  such that the following conditions are satisfied for every  $(t, x, y, z, w, u)$  in the set  $\Omega$  of all tuples  $(t, x, y, z, w, u) \in \mathbb{R}^6$  such that  $0 \leq t < \infty, |x| < H_1, |y| < H_1, |z| < H_1, |w| < H_1, |u| < H_1$  and  $H_1 \leq H$ :*

(i) Firstly,

$$\left\{ \begin{array}{l} 0 < \alpha_1 \alpha_2 - \alpha_3, \\ 0 < (\alpha_1 \alpha_2 - \alpha_3) \alpha_3 - (\alpha_1 \alpha_4 - \alpha_5) \alpha_1, \\ \delta_0 := (\alpha_3 \alpha_4 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3) - (\alpha_1 \alpha_4 - \alpha_5)^2 > 0, \\ \Delta_1 := \frac{(\alpha_3 \alpha_4 - \alpha_2 \alpha_5)(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - (\alpha_1 f'_4(y) - \alpha_5) \geq 2\varepsilon \alpha_2, \\ \Delta_2 := \frac{\alpha_3 \alpha_4 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} - \frac{\gamma(\alpha_1 \alpha_4 - \alpha_5)}{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)} - \frac{\varepsilon}{\alpha_1} > 0, \end{array} \right.$$

where

$$\gamma := \begin{cases} \frac{f_4(y)}{y} & \text{if } y \neq 0, \\ f'_4(0) & \text{if } y = 0. \end{cases}$$

(ii) Moreover,  $\varepsilon_0 \leq f_1(t, z, w) - \alpha_1 \leq \varepsilon_1$ , where the constant  $\varepsilon_1$  satisfies

$$\varepsilon_1 \leq \min \left\{ \frac{\varepsilon \alpha_2 (\alpha_1 \alpha_4 - \alpha_5)^2}{64 \alpha_4^2 (\alpha_1 \alpha_2 - \alpha_3)^2}, \frac{\varepsilon}{16 \alpha_1^2}, \frac{\varepsilon \alpha_4}{16 \delta^2} \right\}.$$

(iii)  $f_3(0) = 0$  and, for  $z \neq 0$ ,

$$\frac{f_3(z)}{z} \geq \alpha_3, \quad |f'_3(z)| \leq M \quad \text{and} \quad \left[ \frac{f_3(z)}{z} - \alpha_3 \right]^2 \leq \min \left\{ \frac{\varepsilon^2 \alpha_2 \alpha_4}{4 \delta^2}, \frac{\varepsilon \alpha_2 \varepsilon_0}{4} \right\}.$$

(iv)  $f_4(0) = 0$  and, for  $y \neq 0$ ,

$$\frac{f_4(y)}{y} \geq \alpha_4, \quad \frac{\partial}{\partial y} f_4(y) - \frac{f_4(y)}{y} < \frac{\alpha_5 \delta_0}{\alpha_4^2 (\alpha_1 \alpha_2 - \alpha_3)} \quad \text{and} \quad (f'_4(y) - \alpha_4)^2 < \frac{\varepsilon^2 \alpha_2}{4}.$$

(v)  $|p(t)| \leq q(t)$ , where  $q(t)$  is a non-negative and continuous function in  $t$  satisfying

$$\int_{t_0}^t q(s) ds \leq A < \infty,$$

for all  $t \geq t_0 \geq 0$ .

Then, there exists a finite positive constant  $K$  such that the solution  $x(t)$  of the equation (1) defined by the initial functions

$x(t) = \phi(t)$ ,  $x'(t) = \phi'(t)$ ,  $x''(t) = \phi''(t)$ ,  $x'''(t) = \phi'''(t)$  and  $x^{(4)}(t) = \phi^{(4)}(t)$  satisfies the inequalities

$$|x(t)| \leq K, \quad |x'(t)| \leq K, \quad |x''(t)| \leq K, \quad |x'''(t)| \leq K \quad \text{and} \quad |x^{(4)}(t)| \leq K$$

for all  $t \geq t_0$ , where  $\phi \in C^4([t_0 - r, t_0], \mathbb{R})$ , provided that

$$r < \min \left\{ \frac{\varepsilon \alpha_4}{\delta(M + \varepsilon_2)}, \frac{3\varepsilon \alpha_2}{16} \left[ \mu + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (\varepsilon_2 + M) \right]^{-1}, \frac{\varepsilon}{2\rho + \alpha_1(\varepsilon_2 + M)}, \frac{9\varepsilon_0}{8(\varepsilon_2 + M)} \right\}.$$

**Proof.** The proof of this theorem depends on a scalar differentiable Lyapunov functional  $V_t = V(t, x_t, y_t, z_t, w_t, u_t)$ . The idea of Lyapunov's method is to impose some conditions on the functional  $V_t$  and on its time derivative  $\frac{d}{dt} V(t, x_t, y_t, z_t, w_t, u_t)$  which imply boundedness of all solutions of the equation (1). We define the Lyapunov's functional  $V_t$  by

(5)

$$\begin{aligned} 2V_t = & u^2 + 2\alpha_1 u w + \frac{2\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} u z + 2\delta u y + \left[ \alpha_1^2 + \alpha_2 - \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right] w^2 \\ & + 2 \left[ \alpha_3 + \frac{\alpha_1 \alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \delta \right] w z + 2\alpha_1 \delta w y + 2w f_4(y) \\ & + 2\alpha_5 w x + 2\alpha_1 \int_0^z f_3(\tau) d\tau + \left[ \frac{\alpha_2 \alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_4 - \alpha_1 \delta \right] z^2 \\ & + 2\delta \alpha_2 y z + 2\alpha_1 z f_4(y) - 2\alpha_5 z y + 2\alpha_1 \alpha_5 z x + \frac{2\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \int_0^y f_4(\eta) d\eta \\ & + (\delta \alpha_3 - \alpha_1 \alpha_5) y^2 + \frac{2\alpha_4 \alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} y x + \delta \alpha_5 x^2 + 2\mu \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \\ & + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds, \end{aligned}$$

where  $\mu$  and  $\rho$  are some positive constants which will be determined later in the proof and  $\delta$  is a positive constant satisfying

$$(6) \quad \delta := \frac{\alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} + \varepsilon.$$

Now, the Lyapunov functional  $V_t = V(t, x_t, y_t, z_t, w_t, u_t)$  defined in (5) can be rearranged in the form:

$$\begin{aligned} (7) \quad 2V_t = & \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right]^2 + \frac{\alpha_4 \delta_0}{(\alpha_1 \alpha_4 - \alpha_5)^2} \left( z + \frac{\alpha_5}{\alpha_4} y \right)^2 \\ & + \frac{\alpha_4(\alpha_1 \alpha_4 - \alpha_5)}{\gamma(\alpha_1 \alpha_2 - \alpha_3)} \left[ \frac{\alpha_5(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} x + \frac{\gamma(\alpha_1 \alpha_2 - \alpha_3)}{(\alpha_1 \alpha_4 - \alpha_5)} y + \frac{\gamma \alpha_1}{\alpha_4} z + \frac{\gamma}{\alpha_4} w \right]^2 \\ & + \Delta_2 (w + \alpha_1 z)^2 + \frac{\varepsilon}{\alpha_1} w^2 + 2\varepsilon \left( \frac{\alpha_3 \alpha_4 - \alpha_2 \alpha_5}{\alpha_1 \alpha_4 - \alpha_5} \right) y z \\ & + 2\mu \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds + \sum_{i=1}^3 V_i, \end{aligned}$$

where

$$\begin{cases} V_1 &:= \delta \alpha_5 x^2 - \frac{\alpha_4 \alpha_5^2 (\alpha_1 \alpha_2 - \alpha_3)}{\gamma (\alpha_1 \alpha_4 - \alpha_5)} x^2, \\ V_2 &:= \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \left[ 2 \int_0^y f_4(\eta) d\eta - y f_4(y) \right] \\ &\quad + \left[ \delta \alpha_3 - \alpha_1 \alpha_5 - \frac{\alpha_5^2 \delta_0}{\alpha_4 (\alpha_1 \alpha_4 - \alpha_5)^2} - \delta^2 \right] y^2, \\ V_3 &:= \alpha_1 \left[ 2 \int_0^z f_3(\tau) d\tau - \alpha_3 z^2 \right]. \end{cases}$$

By using the fact that  $\delta = \frac{\alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} + \varepsilon$  and the assumptions (i) and (iv), we obtain

$$V_1 = \frac{\alpha_5^2 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} x^2 + \varepsilon \alpha_5 x^2 - \frac{\alpha_4 \alpha_5^2 (\alpha_1 \alpha_2 - \alpha_3)}{\gamma (\alpha_1 \alpha_4 - \alpha_5)} x^2 \geq \varepsilon \alpha_5 x^2.$$

It is also evident that

$$(8) \quad y f_4(y) = \int_0^y f_4(\eta) d\eta + \int_0^y \eta f_4'(\eta) d\eta.$$

Hence, the assumptions (i), (iv), together with (6), imply

$$\begin{aligned} V_2 &= \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \left[ 2 \int_0^y f_4(\eta) d\eta - y f_4(y) \right] \\ &\quad + \left[ \frac{\alpha_5 \delta_0}{\alpha_4 (\alpha_1 \alpha_4 - \alpha_5)} - \varepsilon \left( \varepsilon + \frac{2\alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right) \right] y^2 \\ &= \int_0^y \frac{2\alpha_5 \delta_0}{\alpha_4 (\alpha_1 \alpha_4 - \alpha_5)} - \frac{\alpha_4 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \left( f_4'(\eta) - \frac{f_4(\eta)}{\eta} \right) \\ &\quad - 2\varepsilon \left( \varepsilon + \frac{2\alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right) \eta d\eta \\ &\geq \int_0^y \left[ \frac{\alpha_5 \delta_0}{\alpha_4 (\alpha_1 \alpha_4 - \alpha_5)} - 2\varepsilon \left( \varepsilon + \frac{2\alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right) \right] \eta d\eta \\ &\geq \frac{\alpha_5 \delta_0}{4\alpha_4 (\alpha_1 \alpha_4 - \alpha_5)} y^2 \end{aligned}$$

provided that

$$\frac{\alpha_5 \delta_0}{4\alpha_4 (\alpha_1 \alpha_4 - \alpha_5)} \geq \varepsilon \left[ \varepsilon + \frac{2\alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} - \alpha_3 \right],$$

which we now assume.

We get

$$V_3 = 2\alpha_1 \int_0^z \left[ \frac{f_3(\tau)}{\tau} - \alpha_3 \right] \tau d\tau \geq 0$$

by (iii).

Substituting these estimates, that is, the above inequalities involving  $V_1, V_2$  and  $V_3$ , into (7), we have

$$(9) \quad \begin{aligned} 2V_t \geq & \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2} \left( z + \frac{\alpha_5}{\alpha_4} y \right)^2 \\ & + \Delta_2 (w + \alpha_1 z)^2 + \varepsilon \alpha_5 x^2 + \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} y^2 + \frac{\varepsilon}{\alpha_1} w^2 \\ & + 2\varepsilon \left( \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz + 2\mu \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds. \end{aligned}$$

Now, it is evident, from the first six terms in (9), that there exist sufficiently small positive constants  $D_i$ , for  $i = 1, 2, 3, 4, 5$ , such that

$$(10) \quad \begin{aligned} 2V_t \geq & D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 w^2 + D_5 u^2 + 2\varepsilon \left( \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz \\ & + 2\mu \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds. \end{aligned}$$

Consider the sum

$$V_4 =: \frac{D_2}{2} y^2 + 2\varepsilon \left( \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz + \frac{D_3}{2} z^2$$

made of some terms in (10). Obviously,  $V_4$  represents a quadratic form. Thus, it easily follows that  $V_4$  is positive semi-definite if the symmetric matrix

$$\begin{bmatrix} \frac{D_2}{2} & \varepsilon \left( \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) \\ \varepsilon \left( \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) & \frac{D_3}{2} \end{bmatrix}$$

is positive semi-definite. Hence  $V_4 \geq 0$  provided that

$$\varepsilon \leq \frac{1}{2} \left( \frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_3\alpha_4 - \alpha_2\alpha_5} \right) \min \{D_2, D_3\} = D_6,$$

for  $D_6 > 0$ . So, we get from (10) that

$$(11) \quad \begin{aligned} 2V_t \geq & D_1 x^2 + \frac{D_2}{2} y^2 + \frac{D_3}{2} z^2 + D_4 w^2 + D_5 u^2 + 2\mu \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \\ & + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds. \end{aligned}$$

Therefore, the existence of a continuous function  $W_1(|\phi(0)|)$ , with  $W_1(|\phi(0)|) \geq 0$  and satisfying the inequality  $W_1(|\phi(0)|) \leq V(t, \phi)$ , is easily verified, since the integrals

$$\mu \int_{-r}^0 \int_{t+s}^t z^2(\theta) d\theta ds \quad \text{and} \quad \rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds$$

are non-negative. Differentiating the functional  $V(t, x_t, y_t, z_t, w_t, u_t)$  with respect to  $t$  along a solution  $(x(t), y(t), z(t), w(t), u(t))$  of the system (2), we have the following,

in which we let  $\frac{d}{dt}V_t = \frac{d}{dt}V(t, x_t, y_t, z_t, w_t, u_t)$ :

(12)

$$\begin{aligned}
\frac{d}{dt}V_t &= -[f_1(t, z, w) - \alpha_1]u^2 - \left[ \alpha_1\alpha_2 - \left( \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right) \right] w^2 \\
&\quad - \left[ \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \frac{f_3(z)}{z} - (\delta\alpha_2 + (\alpha_1 f'_4(y) - \alpha_5)) \right] z^2 \\
&\quad - \left[ \delta \frac{f_4(y)}{y} - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] y^2 - \alpha_1 [f_1(t, z, w) - \alpha_1]wu \\
&\quad - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} [f_1(t, z, w) - \alpha_1]zu - \delta [f_1(t, z, w) - \alpha_1]yu \\
&\quad - \left[ \frac{f_3(z)}{z} - \alpha_3 \right] zu - \delta \left[ \frac{f_3(z)}{z} - \alpha_3 \right] yz + [f'_4(y) - \alpha_4]zw \\
&\quad + u \int_{t-r}^t f'_3(z(s))w(s)ds + \alpha_1 w \int_{t-r}^t f'_3(z(s))w(s)ds \\
&\quad + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^t f'_3(z(s))w(s)ds + \delta y \int_{t-r}^t f'_3(z(s))w(s)ds \\
&\quad + u \int_{t-r}^t f'_4(y(s))z(s)ds + \alpha_1 w \int_{t-r}^t f'_4(y(s))z(s)ds \\
&\quad + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^t f'_4(y(s))z(s)ds + \delta y \int_{t-r}^t f'_4(y(s))z(s)ds + \mu z^2 r \\
&\quad - \mu \int_{t-r}^t z^2(s)ds + \rho w^2 r - \rho \int_{t-r}^t w^2(s)ds \\
&\quad + \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right] p(t).
\end{aligned}$$

Evidently,

$$f_1(t, z, w) - \alpha_1 \geq \varepsilon_0$$

and

$$\begin{aligned}
&\alpha_1\alpha_2 - \left( \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right) \\
&\quad = \alpha_1\alpha_2 - \alpha_3 + \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \varepsilon - \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} = \varepsilon
\end{aligned}$$

by (i) and (6).

Now, it follows by the assumptions (i), (iii), (iv) of the theorem and (6) that

$$\begin{aligned}
&\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \frac{f_3(z)}{z} - (\delta\alpha_2 + (\alpha_1 f'_4(y) - \alpha_5)) \\
&\quad \geq \frac{\alpha_3\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\delta\alpha_2 + \alpha_1 f'_4(y) - \alpha_5) \\
&\quad = \frac{(\alpha_1\alpha_2 - \alpha_3)(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1 f'_4(y) - \alpha_5) - \varepsilon\alpha_2 \\
&\quad > \varepsilon\alpha_2
\end{aligned}$$

and

$$\delta \cdot \frac{f_4(y)}{y} - \frac{\alpha_4 \alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \geq \delta \alpha_4 - \frac{\alpha_4 \alpha_5 (\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} = \varepsilon \alpha_4.$$

Subject to the assumptions of the theorem and the inequality  $2|ab| \leq a^2 + b^2$ , we obtain the following inequalities:

$$\left\{ \begin{array}{l} u \int_{t-r}^t f_3'(z(s))w(s)ds \leq \frac{M}{2}ru^2 + \frac{M}{2} \int_{t-r}^t w^2(s)ds, \\ \alpha_1 w \int_{t-r}^t f_3'(z(s))w(s)ds \leq \frac{\alpha_1 M}{2}rw^2 + \frac{\alpha_1 M}{2} \int_{t-r}^t w^2(s)ds, \\ \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z \int_{t-r}^t f_3'(z(s))w(s)ds \leq \frac{\alpha_4 M(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} rz^2 \\ \quad + \frac{\alpha_4 M(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} \int_{t-r}^t w^2(s)ds, \\ \delta y \int_{t-r}^t f_3'(z(s))w(s)ds \leq \frac{\delta M}{2}ry^2 + \frac{\delta M}{2} \int_{t-r}^t w^2(s)ds, \\ u \int_{t-r}^t f_4'(y(s))z(s)ds \leq \frac{\varepsilon_2}{2}ru^2 + \frac{\varepsilon_2}{2} \int_{t-r}^t z^2(s)ds, \\ \alpha_1 w \int_{t-r}^t f_4'(y(s))z(s)ds \leq \frac{\alpha_1 \varepsilon_2}{2}rw^2 + \frac{\alpha_1 \varepsilon_2}{2} \int_{t-r}^t z^2(s)ds, \\ \delta y \int_{t-r}^t f_4'(y(s))z(s)ds \leq \frac{\delta \varepsilon_2}{2}ry^2 + \frac{\delta \varepsilon_2}{2} \int_{t-r}^t z^2(s)ds, \\ \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z \int_{t-r}^t f_4'(y(s))z(s)ds \leq \frac{\alpha_4 \varepsilon_2(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} rz^2 \\ \quad + \frac{\alpha_4 \varepsilon_2(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} \int_{t-r}^t z^2(s)ds, \end{array} \right.$$

where  $\varepsilon_2 = \alpha_4 + \frac{\sqrt{\alpha_2 \varepsilon}}{2}$ .

Hence, combining these inequalities with (12), we obtain the following inequality:

(13)

$$\begin{aligned} \frac{d}{dt} V_t \leq & - \left[ \frac{\varepsilon \alpha_4}{2} - \left( \frac{\delta(M + \varepsilon_2)}{2} \right) r \right] y^2 - \left( \frac{3\varepsilon \alpha_2}{16} - \left[ \mu + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (\varepsilon_2 + M) \right] r \right) z^2 \\ & - \left[ \frac{\varepsilon}{2} - \left( \rho + \frac{\alpha_1(\varepsilon_2 + M)}{2} \right) r \right] w^2 - \left[ \frac{9\varepsilon_0}{16} - \left( \frac{\varepsilon_2 + M}{2} \right) r \right] u^2 \\ & - \left( \rho - \left[ \frac{M(1 + \alpha_1 + \delta)}{2} + \frac{\alpha_4 M(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} \right] \right) \int_{t-r}^t w^2(s)ds \\ & - \left( \mu - \left[ \frac{\varepsilon_2}{2} \left( (1 + \alpha_1 + \delta) + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right) \right] \right) \int_{t-r}^t z^2(s)ds \\ & + \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right] p(t) - \sum_{k=5}^{10} V_k, \end{aligned}$$

where

$$\left\{ \begin{array}{l} V_5 = \frac{1}{4} [f_1(t, z, w) - \alpha_1] u^2 + \left[ \frac{f_3(z)}{z} - \alpha_3 \right] zu + \frac{\varepsilon \alpha_2}{4} z^2, \\ V_6 = \frac{\varepsilon \alpha_2}{4} z^2 + \delta \left[ \frac{f_3(z)}{z} - \alpha_3 \right] yz + \frac{\varepsilon \alpha_4}{4} y^2, \\ V_7 = \frac{\varepsilon}{4} w^2 - [f'_4(y) - \alpha_4] zw + \frac{\varepsilon \alpha_2}{4} z^2, \\ V_8 = \frac{1}{16} [f_1(t, z, w) - \alpha_1] u^2 + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} [f_1(t, z, w) - \alpha_1] zu + \frac{\varepsilon \alpha_2}{16} z^2, \\ V_9 = \frac{1}{16} [f_1(t, z, w) - \alpha_1] u^2 + \alpha_1 [f_1(t, z, w) - \alpha_1] wu + \frac{\varepsilon}{4} w^2, \\ V_{10} = \frac{1}{16} [f_1(t, z, w) - \alpha_1] u^2 + \delta [f_1(t, z, w) - \alpha_1] yu + \frac{\varepsilon \alpha_4}{4} y^2. \end{array} \right.$$

Since

$$\left[ \frac{f_3(z)}{z} - \alpha_3 \right]^2 \leq \frac{\varepsilon \alpha_2 (f_1(t, z, w) - \alpha_1)}{4},$$

we have

$$\begin{aligned} V_5 &\geq \frac{1}{4} [f_1(t, z, w) - \alpha_1] u^2 - \left[ \frac{f_3(z)}{z} - \alpha_3 \right] |z| \cdot |u| + \frac{\varepsilon \alpha_2}{4} z^2 \\ &\geq \frac{\varepsilon_0}{4} u^2 - \frac{\sqrt{\varepsilon \alpha_2 \varepsilon_0}}{2} |z| \cdot |u| + \frac{\varepsilon \alpha_2}{4} z^2 \\ &= \left[ \frac{\sqrt{\varepsilon_0}}{2} |u| - \frac{\sqrt{\varepsilon \alpha_2}}{2} |z| \right]^2 \geq 0. \end{aligned}$$

Now, obviously,  $V_6$  represents a quadratic form and ones can rearrange the terms in  $V_6$  so that

$$V_6 = [z, y] \begin{bmatrix} \frac{\varepsilon \alpha_2}{4} & \frac{\delta}{2} \left( \frac{f_3}{z} - \alpha_3 \right) \\ \frac{\delta}{2} \left( \frac{f_3}{z} - \alpha_3 \right) & \frac{\varepsilon \alpha_4}{4} \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}.$$

Subject to the basic information related to the positive semi-definiteness of quadratic forms, we find that  $V_6 \geq 0$  provided

$$\left[ \frac{f_3(z)}{z} - \alpha_3 \right]^2 \leq \frac{\varepsilon^2 \alpha_2 \alpha_4}{4 \delta^2}.$$

By a similar discussion, the terms in  $V_7$  can be rearranged so that

$$V_7 = [w, z] \begin{bmatrix} \frac{\varepsilon}{4} & -\frac{(f'_4 - \alpha_4)}{2} \\ -\frac{(f'_4 - \alpha_4)}{2} & \frac{\varepsilon \alpha_2}{4} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}.$$

Hence, an elementary computation yields that  $V_7 \geq 0$  if  $(f'_4(y) - \alpha_4)^2 < \frac{\varepsilon^2 \alpha_2}{4}$ .

Now,  $V_8$  also represents a quadratic form. Thus, one can write

$$V_8 = [u, z] \begin{bmatrix} \frac{1}{16} (f_1 - \alpha_1) & \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (f_1 - \alpha_1) \\ \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (f_1 - \alpha_1) & \frac{\varepsilon \alpha_2}{16} \end{bmatrix} \begin{bmatrix} u \\ z \end{bmatrix},$$

and conclude that  $V_8 \geq 0$  provided

$$f_1(t, z, w) - \alpha_1 < \frac{\varepsilon \alpha_2 (\alpha_1 \alpha_4 - \alpha_5)^2}{64 \alpha_4^2 (\alpha_1 \alpha_2 - \alpha_3)^2}.$$

Similarly,

$$V_9 = [u, w] \begin{bmatrix} \frac{1}{16} (f_1 - \alpha_1) & \frac{\alpha_1}{2} (f_1 - \alpha_1) \\ \frac{\alpha_1}{2} (f_1 - \alpha_1) & \frac{\varepsilon}{4} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix},$$

and we have  $V_9 \geq 0$ , since  $(f_1 - \alpha_1) < \frac{\varepsilon}{16 \alpha_1^2}$  by (ii).

Finally,

$$V_{10} = [u, y] \begin{bmatrix} \frac{1}{16} (f_1 - \alpha_1) & \frac{\delta}{2} (f_1 - \alpha_1) \\ \frac{\delta}{2} (f_1 - \alpha_1) & \frac{\varepsilon \alpha_4}{4} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix},$$

and an elementary calculation gives that  $V_{10} \geq 0$  since  $(f_1 - \alpha_1) < \frac{\varepsilon \alpha_4}{16 \delta^2}$  by (ii).

Now, since  $V_5 \geq 0$ ,  $V_6 \geq 0$ ,  $V_7 \geq 0$ ,  $V_8 \geq 0$ ,  $V_9 \geq 0$  and  $V_{10} \geq 0$ , it is easily seen from (11) that, by letting  $\frac{d}{dt} V_t = \frac{d}{dt} V(t, x_t, y_t, z_t, w_t, u_t)$  as before, we have

$$\begin{aligned} \frac{d}{dt} V_t &\leq - \left[ \frac{\varepsilon \alpha_4}{2} - \left( \frac{\delta(M + \varepsilon_2)}{2} \right) r \right] y^2 \\ &\quad - \left( \frac{3\varepsilon \alpha_2}{16} - \left[ \mu + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (\varepsilon_2 + M) \right] r \right) z^2 \\ &\quad - \left[ \frac{\varepsilon}{2} - \left( \rho + \frac{\alpha_1(\varepsilon_2 + M)}{2} \right) r \right] w^2 - \left[ \frac{9\varepsilon_0}{16} - \left( \frac{\varepsilon_2 + M}{2} \right) r \right] u^2 \\ (14) \quad &\quad - \left( \mu - \left[ \frac{\varepsilon_2}{2} \left( (1 + \alpha_1 + \delta) + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right) \right] \right) \int_{t-r}^t z^2(s) ds \\ &\quad - \left( \rho - \left[ \frac{M}{2} (1 + \alpha_1 + \delta) + \frac{\alpha_4 M (\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} \right] \right) \int_{t-r}^t w^2(s) ds \\ &\quad + \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right] p(t). \end{aligned}$$

If we choose

$$\mu = \frac{\varepsilon_2}{2} \left( (1 + \alpha_1 + \delta) + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} \right)$$

and

$$\rho = \frac{M}{2} (1 + \alpha_1 + \delta) + \frac{\alpha_4 M (\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)},$$

then the expression (14) implies that

$$\begin{aligned}
 \frac{d}{dt} V_t \leq & - \left[ \frac{\varepsilon \alpha_4}{2} - \left( \frac{\delta(M+\varepsilon_2)}{2} \right) r \right] y^2 \\
 & - \left( \frac{3\varepsilon \alpha_2}{16} - \left[ \mu + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (\varepsilon_2 + M) \right] r \right) z^2 \\
 & - \left[ \frac{\varepsilon}{2} - \left( \rho + \frac{\alpha_1(\varepsilon_2 + M)}{2} \right) r \right] w^2 - \left[ \frac{9\varepsilon_0}{16} - \left( \frac{\varepsilon_2 + M}{2} \right) r \right] u^2 \\
 & + \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right| |p(t)|.
 \end{aligned}
 \tag{15}$$

Therefore, we can conclude from (15) that

$$\frac{d}{dt} V_t \leq \left| u + \alpha_1 w + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} z + \delta y \right| |p(t)|$$

for some constant  $\tau > 0$  provided that

$$r < \min \left\{ \frac{\varepsilon \alpha_4}{\delta(M+\varepsilon_2)}, \frac{3\varepsilon \alpha_2}{16} \left[ \mu + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} (\varepsilon_2 + M) \right]^{-1}, \frac{\varepsilon}{2\rho + \alpha_1(\varepsilon_2 + M)}, \frac{9\varepsilon_0}{8(\varepsilon_2 + M)} \right\}.$$

Hence, it is clear that

$$\begin{aligned}
 \frac{d}{dt} V_t & \leq \left( |u| + \alpha_1 |w| + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5} |z| + \delta |y| \right) q(t) \\
 & \leq D_7 (|y| + |z| + |w| + |u|) q(t),
 \end{aligned}
 \tag{16}$$

where  $D_7 = \max \left\{ 1, \alpha_1, \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)}{\alpha_1 \alpha_4 - \alpha_5}, \delta \right\}$ .

Since  $|y| < 1 + y^2$ ,  $|z| < 1 + z^2$ ,  $|w| < 1 + w^2$  and  $|u| < 1 + u^2$ , it follows from the inequality (16) that

$$\frac{d}{dt} V_t \leq D_7 [4 + 4(y^2 + z^2 + w^2 + u^2)] q(t).
 \tag{17}$$

By (11), evidently,

$$D_8 (x^2(t) + y^2(t) + z^2(t) + w^2(t) + u^2(t)) \leq V(t, x_t, y_t, z_t, w_t, u_t).
 \tag{18}$$

for some positive constant  $D_8$ .

Thus, by substituting (18) into (17), we obtain

$$\begin{aligned}
 \frac{d}{dt} V_t & \leq D_7 \left[ 4 + \frac{4}{D_8} V_t \right] q(t) \\
 & = 4D_7 q(t) + \frac{4D_7}{D_8} V_t q(t).
 \end{aligned}
 \tag{19}$$

Now, integrating (19) from  $t_0$  to  $t$ , using  $\int_{t_0}^t q(s) ds \leq A < \infty$  and the Gronwall-Reid-Bellman inequality, (see [18]), we obtain

$$\begin{aligned}
 V_t & \leq V_{t_0} + 4D_7 A + \frac{4D_7}{D_8} \int_{t_0}^t (V_s) q(s) ds \\
 & \leq [V_{t_0} + 4D_7 A] \exp \left( 4D_7 D_8^{-1} \int_{t_0}^t q(s) ds \right) \\
 & \leq [V_{t_0} + 4D_7 A] \exp (4D_7 D_8^{-1} A) < \infty.
 \end{aligned}
 \tag{20}$$

From (18) and (20), we get the inequalities

$$|x(t)| \leq K, \quad |x'(t)| \leq K, \quad |x''(t)| \leq K, \quad |x'''(t)| \leq K \quad \text{and} \quad |x^{(4)}(t)| \leq K$$

for all  $t \geq t_0$ . Hence, the proof of the theorem is complete.  $\square$

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