

SOLUTIONS OF THE DIOPHANTINE EQUATION

$$au^x + bv^y + cw^z = n!$$

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RÉSUMÉ. Soient $k, a_1, \dots, a_k$ et $u_1, \dots, u_k$ des nombres entiers avec $u_i \geq 2$ pour tout $i \in \{1, \dots, k\}$. En utilisant des bornes pour les formes linéaires en logarithmes p -adiques, nous démontrons que l'équation $\sum_{i=1}^k a_i u_i^{x_i} = n!$ n'a qu'un nombre fini de solutions en entiers positifs pour $k \leq 3$. Une stratégie générale est proposée pour résoudre une telle équation avec k arbitraire. Nous montrons aussi qu'il existe une constante absolue c_0 telle que l'ensemble des entiers $n \leq x$ pour lesquels il existe des nombres premiers $p_1 < \dots < p_k < c_0(\log x)^{1/4}(\log \log x)^{3/4}$, avec la propriété que $n! = \sum_{i=1}^k \varepsilon_i p_i^{x_i}$ pour certains entiers $\varepsilon_i = \{-1, 0, 1\}$ et $x_1, \dots, x_k$, est de cardinal $o(x)$ lorsque $x \rightarrow \infty$.

ABSTRACT. Let $k, a_1, \dots, a_k$ and $u_1, \dots, u_k$ be integers with $u_i \geq 2$ for all $i \in \{1, \dots, k\}$. By using bounds for linear forms in p -adic logarithms, we prove that the Diophantine equation $\sum_{i=1}^k a_i u_i^{x_i} = n!$ has only finitely many non-negative solutions for $k \leq 3$. A modular approach is employed to solve instances of such an equation with arbitrary k . We also prove that there exists an absolute constant c_0 such that the set of $n \leq x$ for which there exist primes $p_1 < p_2 < \dots < p_k < c_0(\log x)^{1/4}(\log \log x)^{3/4}$ with the property that $n! = \sum_{i=1}^k \varepsilon_i p_i^{x_i}$ for some $\varepsilon_i \in \{\pm 1, 0\}$ and integers $x_1, \dots, x_k$, has cardinality $o(x)$ as $x \rightarrow \infty$.

1. Introduction

There are several recent papers in the literature dealing with solutions of the Diophantine equations arising when one tries to write a factorial as a sum of members of some specific sequences. For example, in [10] the authors studied the Diophantine equations of the form $x^y \pm y^x = \prod n_i!$. All the positive integer solutions (x, y, z, n) to the Diophantine equation $x^y + y^z + z^x = n!$ with pairwise coprime x, y and z were determined in [4]. In [11] it was shown that $6!$ is the largest factorial expressible as a sum of three Fibonacci numbers. With the roles of the factorials and Fibonacci numbers reversed, in [9] it was shown that 144 is the largest Fibonacci number which is a sum of two factorials. Not quite related, but in the same spirit, are the papers [5] and [6], in which the authors looked at those positive integers m which can be represented as a sum of powers of some of their prime factors. Some of the interesting examples include

$$870 = 2 \cdot 3 \cdot 5 \cdot 29 = 2^2 + 5^2 + 29^2,$$

and

$$2836295 = 5 \cdot 7 \cdot 11 \cdot 53 \cdot 139 = 5^3 + 7^3 + 11^3 + 53^3 + 139^3,$$

although neither 870 nor 2836295 is a factorial.

In this paper, we look at additive representations of a factorial as a sum of powers of some fixed positive integers. Allowing repetitions, we are led to study integer solutions $(x_1, \dots, x_k, n)$ to a Diophantine equation of the type

$$(1) \quad a_1 u_1^{x_1} + a_2 u_2^{x_2} + \dots + a_k u_k^{x_k} = n!,$$

with some fixed integers $k, a_1, \dots, a_k$ and $u_1, \dots, u_k$. To avoid trivialities, we assume that $u_i \geq 2$ holds for all $i \in \{1, \dots, k\}$.

When faced with such an equation, a first question which arises is to decide whether there are finitely or infinitely many such solutions. The finiteness of the number of solutions is easy to see in case when $k = 1$ for then n is less than the smallest prime number that does not divide $a_1 u_1$. When $k = 2$, the finiteness of the number of solutions can be deduced in the following way. Let p be the smallest prime not dividing $a_1 u_1 a_2 u_2$. Using lower bounds for linear forms in complex logarithms in the left hand side of equation (1), one deduces that $\max\{x_1, x_2\}$ is of order of magnitude $\log n! < n \log n$. Now, on the one hand, linear forms in p -adic logarithms show that the exponent of p in $a_1 u_1^{x_1} + a_2 u_2^{x_2}$ is of order of magnitude at most $\max\{\log x_1, \log x_2\}^2 < 2(\log n)^2$ for large values of n . Since, on the other hand, the exponent of p in $n!$ is at least $\lfloor n/p \rfloor$, we get, by comparing the above two bounds, that $\lfloor n/p \rfloor < 2(\log n)^2$. In particular, n is bounded. This in turn implies that x_1 and x_2 are bounded as well. When $k = 3$, the situation is less transparent. Our first result shows that there are only finitely many solutions to a slightly more general Diophantine equation. Here and in what follows, for a prime p and a non-zero integer m , we write $v_p(m)$ for the exponent at which p appears in the prime factorization of m . We put $v_p(0) = \infty$.

Theorem 1. *Let a, b, c, u, v and w be non-zero integers, with $\min\{u, v, w\} \geq 2$, such that au, bv and cw are pairwise coprime. Let s_n be an increasing sequence of positive integers such that for any prime divisor p of the product uvw we have*

$$\lim_{n \rightarrow \infty} \frac{v_p(s_n)}{\log \log s_n} = \infty.$$

Then the Diophantine equation

$$(2) \quad au^x + bv^y + cw^z = s_n$$

has only finitely many solutions (x, y, z, n) in positive integers. When a, b and c are positive, these solutions can be effectively computed.

The proof of this theorem and its application to equation (1) are given in the next section. The main tool used in its proof is the theory of lower bounds for linear forms in p -adic logarithms. We could not find a way to use linear forms in logarithms or any other transcendental techniques to establish the finiteness of number of solutions of equation (1) when $k > 3$. In Section 3, we show that modular arguments suffice to completely solve particular instances of equation (1) when $k = 3, 4$ or 5 . While looking at the equation (1) reduced modulo suitable primes is always feasible, our choices of moduli depend on the particular specific values of a_i and u_i for $i = 1, \dots, k$. We do not have an argument to the effect that it is always possible to choose such moduli for

all $k \geq 4$ and choices of coefficients a_i and u_i for $i = 1, \dots, k$. We would like to leave this problem for the reader.

Based on our computational experience, we conjecture that if $p_1 < \dots < p_k$ are fixed prime numbers, then

$$\sum_{i=1}^k p_i^{x_i} = n!$$

has only finitely many non-negative integer solutions $(x_1, \dots, x_k, n)$. While we cannot prove this, we can at least show that the set of n such that $n!$ has a representation as a sum of small primes to some integer exponents is of asymptotic density zero. The precise statement is given in the last section of the paper.

2. Proof of Theorem 1

In this section, we prove Theorem 1. Then we specialize it to obtain the finiteness of the number of solutions to equation (1) when $k = 3$.

Proof of Theorem 1. We may assume that $u \leq v \leq w$. Put $t = \max\{x, y, z\}$. The Subspace Theorem yields that if we put $\varepsilon_0 = (\log u)/(\log 2)$, then for all fixed $\varepsilon \in (0, \varepsilon_0)$ the inequality

$$|au^x + bv^y + cw^z| < 2^{\varepsilon t}$$

has only finitely many integer solutions (x, y, z) . In light of this fact, we may assume that

$$(3) \quad |s_n| > 2^{\varepsilon t} \quad \text{holds with } \varepsilon = \varepsilon_0/2.$$

Let ℓ and Λ be defined by

$$\ell = \begin{cases} p & \text{if } t = x, \\ q & \text{if } t = y \text{ and } x < t, \\ r & \text{otherwise,} \end{cases} \quad \text{and} \quad \Lambda = \begin{cases} bv^y + cw^z & \text{if } \ell = p, \\ au^x + cw^z & \text{if } \ell = q, \\ au^x + bv^y & \text{if } \ell = r, \end{cases}$$

where p, q and r are the smallest prime divisors of u, v and w , respectively. Using lower bounds for linear forms in ℓ -adic logarithms (see [2] or [3]), we get

$$(4) \quad v_\ell(\Lambda) \ll \log t.$$

If the inequality $v_\ell(s_n) \ll \log t$ holds for some choice of $\ell \in \{p, q, r\}$, then the condition from the hypothesis of Theorem 1 together with inequality (3) leads to the conclusion that n is bounded, which finishes the proof of this theorem. Thus, we are left with the case in which all three equalities

$$v_p(au^x) = v_p(bv^y + cw^z), \quad v_q(bv^y) = v_q(au^x + cw^z), \quad \text{and} \quad v_r(cw^z) = v_r(au^x + bv^y)$$

hold. The three equalities above together with estimate (4) imply that

$$t = \max\{x, y, z\} \leq \max\{v_\ell(\Lambda) : \ell \in \{p, q, r\}\} \ll \log t,$$

leading to $t \ll 1$. Hence, the Diophantine equation (2) has only finitely many solutions. Since we applied the Subspace Theorem, these solutions are not computable in general. However, when a, b and c are positive, then the inequality (3) always holds. Since the

remaining arguments from the above proof are effective, it follows that all the solutions of the Diophantine equation (2) are computable. $\square$

Theorem 1 has the following corollary.

Corollary 1. *Assume that a, b, c, u, v and w are as in Theorem 1 and let $P(X) \in \mathbb{Z}[X]$ be a polynomial such that $P(0) = 0$. Then the Diophantine equation*

$$au^x + bv^y + cw^z = P(n!)$$

has only finitely many solutions (x, y, z, n) in positive integers.

Proof. Put $s_n = P(n!)$. Since $P(0) = 0$, it follows that $n!$ divides $P(n!)$. Given any fixed prime number p , we have

$$v_p(P(n!)) \geq v_p(n!) \geq \lfloor n/p \rfloor,$$

while $s_n \ll n!^d < n^{nd}$, where d is the degree of $P(X)$. Thus,

$$\frac{v_p(s_n)}{\log \log s_n} \geq \frac{\lfloor n/p \rfloor}{\log(nd) + \log \log n} \rightarrow \infty$$

when $n \rightarrow \infty$. The corollary now follows from Theorem 1. $\square$

If the constant term of the polynomial P is non-zero, the above proof fails because if p is any fixed prime, then $v_p(P(n!))$ remains bounded for large n . The same phenomenon occurs, for instance, when comparing the Diophantine equation $n! = y^2$ (whose solutions in non-negative integers are $(n, y) = (0, 1), (1, 1)$ only), to the Diophantine equation $n! + 1 = y^2$, for which it is not known whether there are only finitely many solutions. The reader should also consult [1] for other polynomial-factorial examples of Diophantine equations.

3. Explicit results

When dealing with a concrete instance of equation (1), one can use other ideas than those invoked in the above proofs. The easiest one is to exploit the particularities of the problem at hand. For instance, it is clear that $(2, 0, 0, 3)$ is a solution for every equation

$$(5) \quad u^x + v^y + w^z = n!$$

with $u = 2$, while $(1, 1, 0, 3)$ verifies each equation of the type $2^x + 3^y + w^z = n!$.

A more systematic approach is based on reducing the given equation modulo suitable positive integers. In this way, we can solve completely some simple cases of equation (1).

Theorem 2. *The only solutions of equation*

$$(6) \quad 2^x + 3^y + 5^z = n!$$

are

$$2^1 + 3^1 + 5^0 = 3!, \quad 2^2 + 3^0 + 5^0 = 3!, \quad \text{and} \quad 2^4 + 3^1 + 5^1 = 4!.$$

Proof. We verify that the only solutions of (1) for $1 \leq n \leq 5$ are

$$2^1 + 3^1 + 5^0 = 3!, \quad 2^2 + 3^0 + 5^0 = 3!, \quad \text{and} \quad 2^4 + 3^1 + 5^1 = 4!.$$

Next we suppose $n \geq 6$. Looking modulo 2 we get that x must be positive. If $x = 1$, then

$$3^y + 5^z \equiv 14 \pmod{16}$$

and, since, modulo 16, we have $3^y \in \{1, 3, 9, 11\}$ and $5^z \in \{1, 5, 9, 13\}$, we infer that

$$(y, z) \equiv (0, 3) \text{ or } (2, 1) \pmod{4}.$$

But, modulo 3, if $(x, y, z) \equiv (1, 0, 1) \pmod{2}$, then $2^x + 3^y + 5^z \not\equiv 0 \pmod{3}$. Thus, there is no solution with $x = 1$.

If $x = 2$, then

$$3^y + 5^z \equiv 12 \pmod{16},$$

and we see that this implies

$$(y, z) \equiv (1, 2) \text{ or } (3, 0) \pmod{4}.$$

Modulo 5, this leads again to a contradiction. Thus, there is no solution with $x = 2$.

If $x \geq 3$, then

$$3^y + 5^z \equiv 0 \pmod{8},$$

and we see that

$$(y, z) \equiv (1, 1) \pmod{2}.$$

For $x \geq 4$, we have

$$(y, z) \equiv (1, 3) \text{ or } (3, 1) \pmod{4}.$$

In both cases, y and z are both odd. Modulo 3 we have

$$2^x + 5^z \equiv 0 \pmod{3};$$

hence,

$$x + z \equiv 1 \pmod{2}.$$

Since z is odd, x must be even. Modulo 5, we have

$$2^x + 3^y \equiv \pm 1 \pm 3 \not\equiv 0 \pmod{5}.$$

We thus conclude that there is no solution for $n \geq 6$ and the theorem follows. $\square$

Corollary 2. *The Diophantine equation (5) corresponding to $(u, v, w) = (3, 5, 8)$ has no integral solutions.*

Looking more closely at the proof of Theorem 2 one observes that it is actually a proof of a stronger result: the congruence

$$2^x + 3^y + 5^z \equiv 0 \pmod{240}$$

has no integer solution (x, y, z) . This can also be verified with a computer. For the computer verification, it may be interesting to notice the following observation, whose proof is immediate.

Lemma 3. *Let a and m be integers greater than 1. Write $m = Am'$, where $\gcd(m', a) = 1$ and the sets of prime divisors of a and A coincide. Then*

$$\{a^n \pmod{m} : n \geq 0\} = \{a^n \pmod{m} : 0 \leq n < \text{ord}(a, m') + s\},$$

where

$$s = \max_{p|a} \left\lceil \frac{v_p(A)}{v_p(a)} \right\rceil.$$

It is possible to construct classes of examples for which an equation similar to (1) has no solution for n large enough. Suppose, for example, that for given integers $u_1, u_2, \dots, u_k$, all greater than 1, there exists some integer $m > k$ such that $u_i \equiv \varepsilon_i \pmod{m}$, where $\varepsilon_i \in \{0, 1\}$ for all i and $\sum \varepsilon_i \neq 0$. Then the equation

$$u_1^{x_1} + \dots + u_k^{x_k} = n!$$

has no solution with $n \geq m$. The same applies when k is odd and $u_i \equiv \pm 1 \pmod{m}$ for all $i = 1, \dots, k$. For example, the equation $6^x + 11^y + 19^z = n!$ has no solution for $n \geq 5$.

A computer-aided search reveals several equations of type (3) without small solutions (that is, $n < 30$), such as those corresponding to

$$(u, v, w) = (3, 7, 10), (3, 8, 11), (5, 6, 7), (5, 6, 11), (5, 7, 8).$$

Then taking the moduli $m = 280, 280, 560 \times 33, 560$, and 560 , respectively, we find that there is no solution in any of these cases.

By elementary means, one can prove that the equations below have only the indicated solutions. Their interest is due either to the comparatively large number of solutions, or to the somewhat rare value of the component n . Moreover, for some of them, the summands are not coprime. In each case we indicate a modulus m for which the congruence $u^x + v^y + w^z \equiv 0 \pmod{m}$ has no solution.

(u, v, w)	(x, y, z, n)	m
(3, 6, 9)	(2, 1, 1, 4), (1, 2, 2, 5)	80
(3, 9, 12)	(1, 1, 1, 4), (3, 2, 1, 5)	80
(4, 7, 7)	(1, 0, 0, 3), (2, 1, 0, 4), (2, 0, 1, 4), (3, 2, 1, 5), (3, 1, 2, 5)	80
(4, 7, 11)	(1, 0, 0, 3), (2, 1, 0, 4), (4, 3, 2, 6)	280
(4, 8, 12)	(1, 0, 0, 3), (1, 1, 1, 4), (3, 3, 2, 6)	$3 \times 5 \times 7 \times 11 \times 13$
(5, 9, 14)	(0, 1, 1, 4), (2, 2, 1, 5), (4, 2, 1, 6)	280
(6, 10, 14)	(1, 2, 1, 5), (4, 3, 3, 7)	$9 \times 25 \times 11$
(8, 12, 14)	(3, 1, 2, 6)	$3 \times 5 \times 7 \times 13$
(9, 10, 11)	(1, 2, 1, 5)	$8 \times 3 \times 11$

We do not claim that our choices of m are optimal.

The next result gives the complete lists of additive representations of a factorial as a sum of powers of four or five small prime numbers.

Theorem 3. *The equation*

$$(7) \quad 2^x + 3^y + 5^z + 7^t = n!$$

has only the solution $(x, y, z, t, n) = (0, 1, 0, 0, 3)$. Similarly, the equation

$$(8) \quad 3^x + 5^y + 7^z + 11^t = n!$$

has only the solutions $(x, y, z, t, n) = (1, 0, 0, 0, 3)$ and $(0, 1, 1, 1, 4)$. Finally, the equation

$$2^x + 3^y + 5^z + 7^s + 11^t = n!$$

has only the solutions: $(x, y, z, s, t) = (1, 0, 0, 0, 0, 3), (1, 1, 0, 1, 1, 4), (1, 1, 0, 1, 1, 4), (1, 2, 0, 0, 1, 4), (1, 2, 1, 1, 0, 4), (2, 0, 0, 1, 1, 4), (2, 1, 1, 0, 1, 4), (3, 1, 0, 0, 1, 4), (3, 1, 1, 1, 0, 4), (3, 2, 1, 0, 0, 4), (4, 0, 1, 0, 0, 4), (1, 4, 2, 0, 1, 5), (3, 3, 2, 2, 1, 5), (4, 4, 1, 1, 1, 5), (5, 1, 2, 2, 1, 5), (5, 3, 0, 2, 1, 5), (5, 4, 1, 0, 0, 5), (6, 0, 1, 2, 0, 5), (1, 4, 4, 0, 1, 6), (3, 3, 4, 2, 1, 6), (3, 5, 1, 3, 2, 6), (3, 5, 3, 3, 0, 6), (5, 1, 4, 2, 1, 6), (7, 1, 3, 3, 2, 6), (7, 5, 1, 3, 0, 6), (9, 4, 1, 0, 2, 6), (9, 4, 3, 0, 0, 6).$

Proof. Assume that (x, y, z, t, n) is a solution to equation (7) with $n \geq 4$. Modulo 2, one concludes immediately that $x = 0$. Reduction modulo 3 yields either

$$1 + 5^z + 1 \equiv 0 \pmod{3} \quad \text{or} \quad 1 + 1 + 5^z + 1 \equiv 0 \pmod{3},$$

so that we get that y is positive and z is even. Then it would follow $3^y + 7^t \equiv 6 \pmod{8}$, which is impossible. Therefore, equation (7) has solutions only when $n = 3$.

A similar argument works for equation (8). Assume again that $n \geq 4$. If $x > 0$, modulo 3 we get $1 + 5^y + 11^t \equiv 0 \pmod{3}$, so y and t are both even. Modulo 8, we then get $2 + 3^x + 7^z \equiv 0 \pmod{8}$, which never happens. So, $x = 0$. Now modulo 3, we get $2 + 5^y + 11^t \equiv 0 \pmod{3}$, implying that both y and t are odd. Then, assuming that $n \geq 5$ and reducing modulo 5, we get $1 + 1 + 7^z \equiv 0 \pmod{5}$ and therefore $z \equiv 3 \pmod{4}$. To proceed, assume that $n \geq 7$. Reducing equation (8) modulo 7, we get $1 + 5^y + 11^t \equiv 0 \pmod{7}$, whence one readily concludes that the only possibility is $(y, t) \equiv (1, 3) \pmod{6}$. Thus, $5^y \equiv 5 \pmod{9}$ and $11^t \equiv 8 \pmod{9}$, so

$$1 + 5^y + 7^z + 11^t \equiv 1 + 5 + 8 + 7^z \equiv 0 \pmod{9},$$

giving $7^z \equiv 4 \pmod{9}$, or $z \equiv 5 \pmod{6}$. Since also $z \equiv 3 \pmod{4}$, we get $z \equiv 11 \pmod{12}$. Thus, $7^z \equiv 2 \pmod{13}$, $5^y \equiv \pm 5 \pmod{13}$ and $11^t \equiv \pm 5 \pmod{13}$ because $(y, t) \equiv (1, 3) \pmod{6}$. For $n \geq 13$, we therefore get

$$1 + 2 \pm 5 \pm 5 \equiv 0 \pmod{13},$$

which is possible only if both ± 1 are equal to 1. This gives $y \equiv 1 \pmod{12}$ and $t \equiv 3 \pmod{12}$.

If $n \geq 19$, then since $7^3 \equiv 1 \pmod{19}$ and $11^3 \equiv 1 \pmod{19}$, we get $5^y \equiv 6 \pmod{19}$, which is false for $y \equiv 1 \pmod{12}$. Hence, equation (8) can have solutions only when $n \leq 18$. A computer check finds the solutions given in the statement of Theorem 3.

Finally, for the last equation above we checked that it has no solutions for $n \geq 61$ by using the modulus $m = 2^4 \times 3^2 \times 5^2 \times 7 \times 11 \times 13 \times 31 \times 61$. The verification of the small cases took two hours on a home computer. $\square$

4. The asymptotic behaviour

With regard to the conjecture that if $p_1 < \dots < p_t$ are fixed prime numbers, then

$$\sum_{i=1}^t p_i^{x_i} = n!$$

has only finitely many non-negative integer solutions $(x_1, \dots, x_t, n)$, we have the following result.

Theorem 4. *There exists an absolute constant c_0 such that the set of $n \leq x$, for which there exist primes*

$$p_1 < p_2 < \dots < p_t < c_0(\log x)^{1/4}(\log \log x)^{3/4}$$

with the property that

$$n! = \sum_{i=1}^t \varepsilon_i p_i^{x_i}$$

for some $\varepsilon_i \in \{\pm 1, 0\}$ and integers $x_1, \dots, x_t$, has cardinality $o(x)$ as $x \rightarrow \infty$.

Proof. Let s be positive integer which is a function of x to be determined later, and assume that $n \in (x/\log x, x)$ has the property that

$$n! = \sum_{i=1}^t \varepsilon_i p_i^{x_i}$$

for some $\varepsilon_i \in \{\pm 1, 0\}$ and integers $x_1, \dots, x_t$, where $p_1 < \dots < p_t \leq p_s$ are t distinct primes not exceeding p_s . Since p_i are distinct primes, we have that $x_i \geq 0$. We fix $t \leq s$, the numbers $\varepsilon_i \in \{\pm 1, 0\}$ and the primes $p_1 < \dots < p_t$. We write $X_i = p_i^{x_i}$ for $i = 1, \dots, t$.

We first assume that $\max\{X_i : i = 1, \dots, t\} < x^{2x}$. To fix ideas, we assume that X_1 is the larger of all variables. Let $q = p_{s+1}$. Then

$$v_q \left(\sum_{i=1}^t X_i \right) = v_q(n!) = \left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{q^2} \right\rfloor + \dots > \frac{n}{2q} \geq \frac{n}{4p_s \log q} \geq \frac{2x \log x}{8p_s (\log x)^2 \log q},$$

therefore

$$\begin{aligned} \left\| \sum_{i=1}^t X_i \right\|_q &= q^{-v_q \left(\sum_{i=1}^t X_i \right)} < \exp \left(-\frac{2x \log x}{8(\log x)^2 p_s} \right) \\ &< \max\{X_i : i = 1, \dots, t\}^{-\delta} = X_1^{-\delta}, \end{aligned}$$

where $\delta = 1/(8(\log x)^2 p_s)$. Here and in what follows, for a prime number q and an integer m we write $\|m\|_q = q^{-v_q(m)}$ for the normalized q -adic valuation of m . We also write $\|m\|_\infty = |m|$ for the archimedean valuation at infinite place. Let $\mathcal{S}$ be the set of valuations corresponding to the primes $p_1, \dots, p_s, p_{s+1}$ and ∞ . For each $(i, \mu) \in \{1, \dots, t\} \times \mathcal{S}$ we let the linear forms defined for $\mathbf{x} = (x_1, \dots, x_t) \in \mathbf{Q}^t$ be

$L_{i,\mu}(\mathbf{x}) = x_i$ for all (i, μ) except when $(i, \mu) = (1, p_{s+1})$, when we take $L_{1,p_{s+1}}(\mathbf{x}) = \sum_{i=1}^t \varepsilon_i x_i$. Note that when $\mathbf{x} = (X_1, \dots, X_t)$, then

$$\begin{aligned} \prod_{i=1}^t \prod_{\mu \in \mathcal{S}} \|L_{i,\mu}(\mathbf{x})\|_{\mu} &= \prod_{i=1}^t \prod_{\mu \in \mathcal{S} \setminus \{p_{s+1}\}} \|X_i\|_{\mu} \prod_{i=2}^t \|X_i\|_{p_{s+1}} \left\| \sum_{i=1}^t \varepsilon_i X_i \right\|_{p_{s+1}} \\ &\leq \left\| \sum_{i=1}^t \varepsilon_i X_i \right\|_{p_{s+1}} \leq X_1^{-\delta}, \end{aligned}$$

where in the above we used the fact that all $X_1, \dots, X_t$ are $\mathcal{S}$ -units. A quantitative version of the Subspace Theorem due to Evertse (see [7]) tells us that the solutions of the above inequality satisfy

$$(9) \quad \sum_{i=1}^t a_i X_i = 0$$

for some vectors $\mathbf{a} = (a_1, \dots, a_t)$ not all zero, with integer components, and that the number of such vectors $\mathbf{a}$ does not exceed

$$\begin{aligned} \left(2^{60(s+2)^2} \delta^{-7(s+2)} \right)^{s+2} &= \exp \left(60 \log 2 (s+2)^3 + 7(s+2)^2 \log (8p_s (\log x)^2) \right) \\ (10) \quad &= \exp \left((60 \log 2 + o(1)) s^3 \right) \end{aligned}$$

as $x \rightarrow \infty$, assuming that $\log \log x = o(s)$, which will be the case. Each one of the equations (9) is an $\mathcal{S}$ -unit equation. Replacing t by a smaller number, if needed, we may assume that $a_i \neq 0$ for all $i = 1, \dots, t$. Let $I_1 \cup \dots \cup I_\ell$ be a partition of $\{1, \dots, t\}$ such that each of the subequations

$$(11) \quad \sum_{i \in I_j} a_i X_i = 0$$

is non-degenerate in the sense that no proper subsum of it vanishes. Let $s_j = \#I_j$. Then another theorem of Evertse (see [8]) concerning the number of solutions of non-degenerate $\mathcal{S}$ -unit equations tells us that equation (11) has at most $(2^{35} s_j)^{s_j^3(s_j+1)}$ projective solutions $[X_i : i \in I_j]$. We show that it can have at most this number of solutions $X = (X_i : i \in I_j)$ also. If not, there are two distinct solutions of (11), say $X = (X_i : i \in I_j)$ and $X' = (X'_i : i \in I_j)$, such that $X_{i_1}/X'_{i_1} = X_{i_2}/X'_{i_2}$ for all $i_1 \neq i_2 \in I_j$. Since $X_i = p_i^{x_i}$, $X'_i = p_i^{x'_i}$ for all $i \in I_j$ and the primes p_i are distinct, it follows, by unique factorization, that $X = X'$, which is a contradiction. Thus, the total number of solutions to equation (9) is at most

$$\begin{aligned} (12) \quad \sum_{\{1, \dots, t\} = I_1 \cup \dots \cup I_\ell} \prod_{j=1}^{\ell} (2^{35} s_j)^{s_j^3(s_j+1)} &< 2^{s^2} (2^{35} s)^{s^3(s+1)} \\ &= \exp \left((1 + o(1)) s^4 \log s \right). \end{aligned}$$

Here, 2^{s^2} is an upper bound for the number of ways of partitioning $\{1, \dots, t\}$ (a better inequality is valid from what is known about the maximal order of the Bell numbers but we will not be needing it).

Denote by M the number of possibilities for the vector $X = (X_1, \dots, X_t)$. The number M is obtained by summing up over all the possible values of t (s of them), of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_t$ (at most 3^s of them), over the set of primes $\{p_1, \dots, p_t\}$ (at most 2^s of them), over the number of t possibilities to choose i such that $X_i = \max\{X_1, \dots, X_t\}$ (here, we only made the convention that $i = 1$ but to account for all solutions we need to consider all the other possibilities for i also), over the number of possible subspaces which is bounded as in (10), and over the number of solutions in each subspace which is bounded as in (12). Putting everything together, we get that for fixed x and s , we have

$$M \leq \exp\left((1 + o(1))s^4 \log s\right).$$

Imposing that this number is less than or equal to $x/\log x$ and solving for s we get that

$$(1 + o(1))s^4 \log s = (1 + o(1)) \log x,$$

or

$$s = (c_1 + o(1))(\log x)^{1/4}/(\log \log x)^{1/4},$$

where $c_1 = \sqrt{2} = 1.4142\dots$. With the Prime Number Theorem, we get that the statement holds for all $c_0 < c_1/4$. In particular, it holds for $c_0 = 0.3$.

In case $\max\{X_i : i = 1, \dots, t\} > x^{2x}$, we have that

$$\max\{X_i : i = 1, \dots, t\} > n!^2.$$

So, assuming again that $X_1 = \max\{X_i : i = 1, \dots, s\}$, we then get that

$$(13) \quad \left\| \sum_{i=1}^t \varepsilon_i X_i \right\|_{\infty} < X_1^{1/2}.$$

In this case, instead of the “small” q -adic linear form that we used in the previous case, we use the “small” form in the infinite valuation (13) to arrive again, via the Subspace Theorem, to the conclusion that $X = (X_1, \dots, X_t)$ satisfies one of a number of equations of the form (9), the totality of which is bounded by the amount shown in (10), and the previous argument finishes the job. This completes the proof of this theorem. $\square$

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