

ON INTEGER-VALUED ENTIRE FUNCTIONS AT THE GAUSSIAN INTEGERS SATISFYING ADDITIONAL CONGRUENCE CONDITIONS

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RÉSUMÉ. Dans cette note nous étendons le résultat de Gramain sur les fonctions entières prenant en tout entier de Gauss une valeur entière de Gauss au cas de fonctions ayant cette propriété ainsi que leurs premières dérivées et satisfaisant certaines conditions de congruence.

ABSTRACT. In this note we extend the result of Gramain [4] on integer-valued entire functions at the Gaussian integers to functions that together with their first $s - 1$ derivatives take integral values and in addition satisfy certain congruence conditions.

1. Introduction and main result

In 1915 Pólya [8] showed, that among all entire transcendental functions mapping the set of positive rational integers $\mathbb{N}$ into the set of rational integers $\mathbb{Z}$, the function 2^z has least growth. Since then, a lot of various problems on integer-valued entire functions were studied. In the following decades the field of the so-called integer-valued entire functions grew and a lot of various problems were attacked. The research reached from functions that take integral values at other sequences than the integers (for example at a geometric progression q^n or at the Gaussian integers), and functions that take together with their first $s - 1$ (or all) derivatives integral values to such integer-valued functions whose values satisfy additional congruence conditions. For an overview we redirect the reader to [3]. For more recent results, see [11, 12] and the literature cited there.

In this note we study entire functions that map together with their first $s - 1$ derivatives the set of Gaussian integers $\mathbb{Z}[i]$ into $\mathbb{Z}[i]$ and fulfill certain additional congruence conditions.

For an entire function f , we denote by $|f|_r$ the maximum of $|f(z)|$ on the circle $|z| = r$. The results we obtained in [13] for functions taking together with their first $s - 1$ derivatives integral values at the positive integers or a geometric progression are the following:

Theorem 1.1. *a) There exists a sequence $(\gamma_s)_{s \in \mathbb{N}}$ of positive real numbers with $\gamma_s/s \rightarrow 0.78592968 \dots$ if $s \rightarrow +\infty$ and the following property holds: if $s \geq 2$ and $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function such that*

$$f^{(\sigma)}(n) \in \mathbb{Z}$$

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for all $n \in \mathbb{N}_0$ and $\sigma = 0, \dots, s-1$, and there exists $\gamma < \gamma_s$ such that

$$\log |f|_r \leq \gamma r$$

for all sufficiently large r , then f is a polynomial function.

b) Let $q \in \mathbb{Z}$ with $|q| > 1$. There exists a sequence $(\gamma_s)_{s \in \mathbb{N}}$ of positive real numbers with $\gamma_s/s \rightarrow 1/4$ if $s \rightarrow +\infty$ and the following property holds: if $s \geq 2$ and $f : \mathbb{C} \rightarrow \mathbb{C}$ is an entire function such that

$$f^{(\sigma)}(q^n) \in \mathbb{Z}$$

for all $n \in \mathbb{N}_0$ and $\sigma = 0, \dots, s-1$, and there exists $\gamma < \gamma_s$ such that

$$\log |f|_r \leq \gamma \frac{(\log r)^2}{\log |q|}$$

for all sufficiently large r , then f is a polynomial function.

Perelli and Zannier [7] proved in 1981 the following result on integer-valued entire functions (in the sense of Pólya) that satisfy additional congruence conditions.

Theorem 1.2. *Let f be an entire function such that:*

- (a) $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$,
- (b) for all $n \in \mathbb{N}$ and all primes $p \in \mathbb{Z}$, we have

$$f(n+p) \equiv f(n) \pmod{p},$$

- (c) $\log |f|_r \leq \gamma r + o(r)$ for $r \rightarrow \infty$ with a constant $\gamma < \log(1+e)$.

Then f is a polynomial function.

And Bézivin [2] established in 1998 the following q -analogue.

Theorem 1.3. *Let $q \in \mathbb{Z}$ with $|q| > 1$. Let f be an entire function such that*

- (a) $f(q^n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$,
- (b) for all $n, m \in \mathbb{N}$, with $m > 0$, we have

$$f(q^{n+m}) \equiv f(q^n) \pmod{q^m - 1},$$

- (c) $\log |f|_r \leq \gamma \frac{(\log r)^2}{\log |q|} + o\left((\log r)^2\right)$ for $r \rightarrow \infty$ with a constant $\gamma < \left(4 - \frac{12}{\pi^2}\right)^{-1}$.

Then f is a polynomial function.

The main result of this note is as follows.

Theorem 1.4. *Let s be a positive integer and f be an entire transcendental function, that is not a polynomial function, such that for all $\sigma = 0, \dots, s-1$ the following two conditions are satisfied:*

- (a) $f^{(\sigma)}(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$,
- (b) for all $\zeta \in \mathbb{Z}[i]$ and all Gaussian primes $\mathfrak{p} \in \mathbb{Z}[i]$, we have

$$f^{(\sigma)}(\zeta + \mathfrak{p}) - f^{(\sigma)}(\zeta) \in \mathfrak{p}\mathbb{Z}[i].$$

Then

$$\limsup_{r \rightarrow \infty} \frac{\log |f|_r}{r^2} \geq \frac{\pi s}{2e} e^{\frac{1}{\pi s}}.$$

Remark 1.5. The proof of the theorem, which follows the lines of Gramain's proof in [4], will show that the Gaussian primes $\mathfrak{p}$ that divide a prime $p \equiv 3 \pmod{4}$ have no influence on the calculus and that we can therefore replace (b) by the weaker assumption

(b') for all $\zeta \in \mathbb{Z}[i]$ and all Gaussian primes $\mathfrak{p} \in \mathbb{Z}[i]$ dividing a prime $p \equiv 1 \pmod{4}$, we have

$$f^{(\sigma)}(\zeta + \mathfrak{p}) - f^{(\sigma)}(\zeta) \in \mathfrak{p}\mathbb{Z}[i].$$

Remark 1.6. The proof also shows that without condition (b) we can still conclude (see page 106) that

$$\limsup_{r \rightarrow \infty} \frac{\log |f|_r}{r^2} \geq \frac{\pi s}{2e}.$$

So, for larger s , the impact of the additional congruence condition (b) on the growth bound becomes smaller.

2. Proof of Theorem 1.4

Theorem 1.4 is an immediate consequence of the following three propositions.

Proposition 2.1. *Let f be an entire function such that for all $\sigma = 0, \dots, s-1$ the following two conditions are satisfied:*

(a) $f^{(\sigma)}(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$,

(b) for all $\zeta \in \mathbb{Z}[i]$ and all Gaussian primes $\mathfrak{p} \in \mathbb{Z}[i]$, we have

$$f^{(\sigma)}(\zeta + \mathfrak{p}) - f^{(\sigma)}(\zeta) \in \mathfrak{p}\mathbb{Z}[i].$$

Further we assume that there exists $\gamma > 0$ such that

$$(1) \quad \log |f|_r \leq \gamma r^2$$

holds for all sufficiently large r . If

$$\gamma < \frac{s\pi}{2e} e^{\frac{1}{s\pi}},$$

then f satisfies in $\mathbb{C}$ two functional equations

$$\begin{cases} \sum_{k=0}^K P_k(z) f(z+k) = 0, \\ \sum_{\ell=0}^L Q_\ell(z) f(z+i\ell) = 0, \end{cases}$$

where the P_k and Q_ℓ are polynomials with coefficients in $\mathbb{Z}[i]$ and $P_K Q_L \neq 0$.

We will prove this proposition in the next section. The following proposition is a special case of Lemma 5 in Gramain [4].

Proposition 2.2. *Let f be an entire function satisfying two functional equations*

$$\begin{cases} \sum_{k=0}^K P_k(z) f(z+k) = 0, \\ \sum_{\ell=0}^L Q_\ell(z) f(z+i\ell) = 0, \end{cases}$$

where the P_k and Q_ℓ are polynomials with coefficients in $\mathbb{C}$ and $P_K Q_L \neq 0$. Then there exists a constant $C_1 > 0$, depending only on the degrees of the polynomials P_k and Q_ℓ , such that

$$\log |f|_r \leq C_1 r \log r$$

holds for all sufficiently large r .

Now we can state the third proposition which is due to Waldschmidt [10].

Proposition 2.3. *There exists a constant $C_2 > 0$ having the following property: if f is an entire function such that $f(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$ and*

$$\log |f|_r \leq C_2 \frac{r^2}{\log r}$$

holds for all sufficiently large r , then f is a polynomial function.

3. Proof of Proposition 2.1

Let $\zeta_0 = 0, \zeta_1, \zeta_2, \dots$ be the elements of $\mathbb{Z}[i]$ ordered by increasing absolute value and argument.

Before we start with the proof we recall two results which are proved in [4].

Lemma 3.1. *There exists a constant C_2 such that for all $n \in \mathbb{N}_0$, we have*

$$\left| |\zeta_n| - \sqrt{\frac{n}{\pi}} \right| \leq C_2.$$

Lemma 3.2. *There exists a constant C_1 such that for all $z \in \mathbb{C}$ with $|z| = \theta |\zeta_n|$ and $n \geq 2$, we have*

$$\left| \log \left| \prod_{j=0}^n (z - \zeta_j) \right| - \frac{1}{2} n \log n - nw(\theta) \right| \leq C_1 \max \{1, \theta\} \sqrt{n} \log n,$$

(here $\prod^*$ signifies that the product is taken over all factors with absolute value ≥ 1) where

$$w(\theta) = \begin{cases} \log \theta - \frac{1}{2} \log \pi & \text{if } \theta \geq 1, \\ \frac{\theta^2}{2} - \frac{1}{2} - \frac{1}{2} \log \pi & \text{if } \theta \leq 1. \end{cases}$$

Now let f be an entire function like in Proposition 2.1. We will prove that f satisfies two functional equations of the above type.

Let α be 1 or i , N be sufficiently large, and $\theta > 1$ be a real parameter which will be chosen later. Let $H := \left[N^{\frac{5}{6}} \right] + 1$ and $K := \left[N^{\frac{1}{3}} \right] + 1$. There are three steps.

(I) There exist elements $a_{h,k} \in \mathbb{Z}[i]$ ($0 \leq h \leq H-1$; $0 \leq k \leq K-1$) with

$$0 < \max_{h,k} |a_{h,k}| \leq \exp \left(2s \frac{\gamma}{\pi} N^{\frac{5}{6}} + o \left(N^{\frac{5}{6}} \right) \right)$$

and such that the entire function F defined by

$$F(z) := \sum_{h,k} a_{h,k} z^h f(z + \alpha k)$$

satisfies

$$F^{(\sigma)}(\zeta_n) = 0$$

for all $n = 0, \dots, N-1$ and $\sigma = 0, \dots, s-1$.

We will apply Siegel's Lemma (see for instance *Hilfssatz 31* in [9]) to prove the claim of this step.

We consider the system

$$0 = F^{(\sigma)}(\zeta_n) = \sum_{h=0}^{H-1} \sum_{k=0}^{K-1} a_{h,k} \sum_{\kappa=0}^{\min\{\sigma,h\}} \sigma! \binom{h}{\kappa} (\zeta_n)^{h-\kappa} \frac{f^{(\sigma-\kappa)}(\zeta_n + \alpha k)}{(\sigma - \kappa)!}$$

for $n = 0, \dots, N-1$ and $\sigma = 0, \dots, s-1$. This is a system of sN linear equations in the $HK \geq N^{7/6}$ unknowns $a_{h,k}$.

By Cauchy's Theorem we have

$$\frac{f^{(\sigma-\kappa)}(\zeta_n + \alpha k)}{(\sigma - \kappa)!} = \frac{1}{2\pi i} \int_{|\xi - (\zeta_n + \alpha k)|=1} \frac{f(\xi)}{(\xi - (\zeta_n + \alpha k))^{\sigma-\kappa+1}} d\xi.$$

Hence by Lemma 3.1, our choice of K and the assumption on $\log |f|_r$ we get

$$\left| \frac{f^{(\sigma-\kappa)}(\zeta_n + \alpha k)}{(\sigma - \kappa)!} \right| = \exp \left(\frac{\gamma}{\pi} N + o(N) \right).$$

By this, together with our choice of H and again Lemma 3.1, the absolute values of the coefficients of the system above are bounded by

$$\exp \left(\frac{\gamma}{\pi} N + o(N) \right).$$

For sufficiently large N , the exponent in Siegel's Lemma can be estimated by

$$\frac{sN}{HK - sN} \leq \frac{sN}{N^{7/6} - sN} \leq \frac{sN}{\frac{1}{2}N^{7/6}} = 2sN^{-1/6}$$

and therefore the claim of the first step follows by applying Siegel's Lemma.

(II) For all $M \geq N$, we have

$$F^{(\sigma)}(\zeta_m) = 0$$

whenever $m = 0, \dots, M-1$ and $\sigma = 0, \dots, s-1$.

We prove this by induction.

The case $M = N$ is the first step. We assume that the claim of the second step is already proven for M . By Cauchy's Theorem we have for $\sigma \in \{0, \dots, s-1\}$

$$F^{(\sigma)}(\zeta_M) = \frac{\prod_{m=0}^{M-1} (\zeta_M - \zeta_m)^s}{2\pi i} \int_{|\xi|=\theta|\zeta_M|} \frac{F(\xi) d\xi}{(\xi - \zeta_M)^{\sigma+1} \prod_{m=0}^{M-1} (\xi - \zeta_m)^s}.$$

By Lemmas 3.1 and 3.2 we get

$$\left| F^{(\sigma)}(\zeta_M) \right| \leq |F|_{\theta|\zeta_M|} \exp(-sM \log \theta + o(M)).$$

We have by (1) and the first step

$$|F|_{\theta|\zeta_M|} \leq \exp\left(\frac{\gamma}{\pi} \theta^2 M + o(M)\right)$$

and hence

$$(2) \quad \left| F^{(\sigma)}(\zeta_M) \right| \leq \exp\left(\left(\frac{\gamma}{\pi} \theta^2 - s \log \theta\right) M + o(M)\right).$$

We are now looking for a lower bound for $|F^{(\sigma)}(\zeta_M)|$ in case $F^{(\sigma)}(\zeta_M) \neq 0$.

If we just use that f and its first $s-1$ derivatives take integer values at the Gaussian integers, ignoring the additional congruence conditions, we get $|F^{(\sigma)}(\zeta_M)| \geq 1$ if $F^{(\sigma)}(\zeta_M) \neq 0$. By putting $\theta := e^{\frac{1}{2}}$, we get a contradiction between this lower bound and the upper bound (2) for all sufficiently large M , if $\gamma < s\pi/(2e)$. Hence we have $F^{(\sigma)}(\zeta_M) = 0$.

Taking the additional congruence conditions into account we can get a better lower bound.

For every polynomial P with coefficients in $\mathbb{Z}[i]$ and every Gaussian prime $\mathfrak{p}$ we have $P(\zeta_M - \mathfrak{p}) - P(\zeta_M) \in \mathfrak{p}\mathbb{Z}[i]$ and also $f^{(\sigma)}(\zeta - \mathfrak{p}) - f^{(\sigma)}(\zeta) \in \mathfrak{p}\mathbb{Z}[i]$ for all $\zeta \in \mathbb{Z}[i]$. Hence $F^{(\sigma)}(\zeta_M - \mathfrak{p}) - F^{(\sigma)}(\zeta_M) \in \mathfrak{p}\mathbb{Z}[i]$.

Since we know that $F^{(\sigma)}(\zeta_M - \mathfrak{p}) = 0$ for every Gaussian prime $\mathfrak{p}$ such that $|\zeta_M - \mathfrak{p}| < |\zeta_M|$, we get

$$F^{(\sigma)}(\zeta_M) \in \mathfrak{p}\mathbb{Z}[i]$$

for every Gaussian prime $\mathfrak{p}$ such that $|\zeta_M - \mathfrak{p}| < |\zeta_M|$.

If p is a prime such that $p \equiv 3 \pmod{4}$ and $p < 2 \cdot \max\{|\operatorname{Re}(\zeta_M)|, |\operatorname{Im}(\zeta_M)|\}$, then there is a Gaussian prime

$$\mathfrak{p}_p \in \{p, -p, ip, -ip\}$$

with $|\zeta_M - \mathfrak{p}_p| < |\zeta_M|$.

If p is a prime such that $p \equiv 1 \pmod{4}$, then p can be written as a sum of two squares, say $p = n^2 + m^2$. If $\sqrt{p} < 2 \cdot \max\{|\operatorname{Re}(\zeta_M)|, |\operatorname{Im}(\zeta_M)|\}$ the union of the disks of radius $|\zeta_M|$ with centers $\pm\zeta_M, \pm i\zeta_M, \pm\overline{\zeta_M}$ and $\pm i\overline{\zeta_M}$ covers the disk with center 0 and radius $\sqrt{p}$. Hence, there is a Gaussian prime

$$\mathfrak{p}_p \in \{n + im, n - im, -n + im, -n - im, m + in, m - in, -m + in, -m - in\}$$

with $|\zeta_M - \mathfrak{p}_p| < |\zeta_M|$.

Therefore

$$F^{(\sigma)}(\zeta_M) \in \left(\prod_{\substack{p \equiv 3 \pmod{4} \\ p < 2 \cdot \max\{|\operatorname{Re}(\zeta_M)|, |\operatorname{Im}(\zeta_M)|\}}} \mathfrak{p}_p \prod_{\substack{p \equiv 1 \pmod{4} \\ \sqrt{p} < 2 \cdot \max\{|\operatorname{Re}(\zeta_M)|, |\operatorname{Im}(\zeta_M)|\}}} \mathfrak{p}_p \right) \mathbb{Z}[i].$$

If $\pi_1(x)$ resp. $\pi_3(x)$ denotes the number of primes $p \leq x$ with $p \equiv 1 \pmod{4}$ resp. $p \equiv 3 \pmod{4}$ then we have by the prime number theorem for primes in arithmetic progressions (see for instance Theorem 7-25 in [6]) $\pi_1(x) \sim \pi_3(x) \sim \frac{x}{2 \log x}$ for $x \rightarrow \infty$.

Now by formula (22.5.2) in Theorem 421 of [5] we get

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \log p = \frac{x}{2} + o(x)$$

and the same asymptotic holds if we sum on the left hand side over the primes $p \equiv 3 \pmod{4}$. Therefore

$$\sum_{\substack{p \leq 4M/(2\pi) \\ p \equiv 1 \pmod{4}}} \log p = \frac{M}{\pi} + o(M)$$

and

$$\sum_{\substack{p \leq 2\sqrt{M/(2\pi)} \\ p \equiv 3 \pmod{4}}} \log p = O(\sqrt{M}).$$

Hence, if we now assume that $F^{(\sigma)}(\zeta_M) \neq 0$, we get

$$\left| F^{(\sigma)}(\zeta_M) \right| \geq \left| \prod_{\substack{p \equiv 3 \pmod{4} \\ p < 2\sqrt{\frac{M}{2\pi}}}} \mathfrak{p}_p \prod_{\substack{p \equiv 1 \pmod{4} \\ \sqrt{p} < 2\sqrt{\frac{M}{2\pi}}}} \mathfrak{p}_p \right| = \exp\left(\frac{M}{2\pi} + o(M)\right).$$

Now we choose $\theta := \exp\left(\frac{s\pi-1}{2s\pi}\right)$. We get a contradiction between the upper and the lower bound if $\gamma < \frac{s\pi}{2e} e^{\frac{1}{s\pi}}$ and N (and therefore M) is sufficiently large.

(III) F vanishes identically, this means that we have $F(z) = 0$ for all $z \in \mathbb{C}$.

The claim of the third step follows from the uniqueness theorem below, which is also an interesting fact in itself.

Lemma 3.3 (Uniqueness Theorem). *Let F be an entire function with $F^{(\sigma)}(\xi) = 0$ for all $\xi \in \mathbb{Z}[i]$ and $\sigma = 0, \dots, s-1$. If $\log |F|_r \leq \gamma r^2 + o(r^2)$ for $r \rightarrow \infty$ with a constant $\gamma < \frac{s\pi}{2}$, then F vanishes identically.*

Proof. Suppose that F does not vanish identically. Jensen's formula (see for instance [1], p. 205-206) says, that if $F(z) = cz^h + O(z^{h+1})$, then for $\rho > 0$, we have

$$\sum_a \log \left(\frac{\rho}{|a|} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log |F(\rho e^{it})| dt - \log |c| - h \log \rho,$$

where the sum on the left hand side extends over all zeros a of F with $0 < |a| < \rho$, each zero appearing accordingly to its multiplicity.

We set $\rho := |\zeta_n|$. By Lemma 3.1 we get for $n \rightarrow +\infty$

$$\frac{1}{2\pi} \int_0^{2\pi} \log |F(\rho e^{it})| dt - \log |c| - h \log \rho \leq \frac{\gamma}{\pi} n + o(n).$$

Besides this, we have again by Lemma 3.1 and Stirling's formula in the weak form, that is $\log n! = n \log n - n + o(n)$, that

$$\sum_a \log \left(\frac{\rho}{|a|} \right) \geq \frac{s}{2} n + o(n).$$

Hence $\gamma \geq \frac{s\pi}{2}$. $\square$

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$$\log |F|_r \geq \sum_a \log \left(\frac{\rho}{|a|} \right) + \log |c| + h \log \rho.$$

One advantage of this approach is that it can be transferred to the p -adic situation where no analogues to Jensen's formula is at hand.

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