

ON THE COHOMOLOGY OF A GALOIS ENTWINING

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RÉSUMÉ. Nous calculons la cohomologie tressée d’une structure tressée (A, C, ψ) telle que A est une C -extension galoisienne plate (au sens de Brzeziński et Hajac) de la sous-algèbre C -invariante B , en l’associant à la cohomologie de Hochschild de B . Ceci complète le calcul de Brzeziński pour le cas spécial où $B = k$ est le corps de base.

ABSTRACT. In this paper we compute the entwined cohomology of an entwined structure (A, C, ψ) such that A is a flat C -Galois extension (in the sense of Brzeziński and Hajac) of the C -invariant subalgebra B , relating it to the Hochschild cohomology of B . This completes the calculation of Brzeziński in the special case in which $B = k$ is the ground field.

1. Introduction. An extension of algebras A/B is said to be *Hopf-Galois over a Hopf algebra* H if A is an H -comodule algebra, $B = A^{\text{co}H}$ is the subalgebra of coinvariants, and a certain Galois condition is satisfied; see [5]. Such an extension can be viewed as a non-commutative principal bundle with structure group H . It turns out, though, that to develop a satisfactory theory of principal bundles in the non-commutative setting the notion of Hopf-Galois extensions is too restrictive; it does not apply, for example, to all the quantum spheres of Podleś [6].

One considers then a more general situation in which the role of the structure group is played by a coalgebra. T. Brzeziński and P. Hajac [2] have proposed a corresponding notion of coalgebra Galois extensions, tightly related to that of entwining structures introduced in [3]. In this context, one can fit the Podleś spheres in coalgebra Galois extensions $SU_q(2)/S_{q,s}^2$ for appropriate coactions of coalgebras on $SU_q(2)$.

In a later paper [1], Brzeziński introduced two cohomology theories for an entwining structure, and, in particular, for coalgebra C -Galois extensions A/B : the entwined cohomology $H_\psi^\bullet(A, -)$ of A with values in A -bimodules, and a C -equivariant version. He computed the entwined cohomology of A when the algebra B of coinvariants in A is the ground field k , and noted that it is essentially trivial.

The purpose of the present note is to record the extension of Brzeziński’s computation of cohomology to the general case of a flat coalgebra Galois extension A/B . We show below that in that situation $H_\psi(A, -)$ coincides with the Hochschild cohomology $HH^\bullet(B, -)$ of the subalgebra B of coinvariants.

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We plan to study the equivariant comohology of the entwined structure corresponding to a coalgebra Galois extension in a future paper.

2. Coalgebra Galois extensions and the theorem. Fix a field k . All spaces and (co)algebras we consider are k -vector spaces and k -(co)algebras, and all unadorned tensor products are taken over k . Most of our statements can be extended to the slightly more general situation in which k is simply a ring, provided one adds appropriate projectivity or flatness hypotheses.

An *entwining structure* is a triple (A, C, ψ) consisting of an algebra A , a coalgebra C and a map $\psi : C \otimes A \rightarrow A \otimes C$ (which we write à la Sweedler, with implicit sums over greek indices, as in $\psi(c \otimes a) = a_\alpha \otimes c^\alpha$) satisfying the following compatibility conditions:

- (1) $(aa')_\alpha \otimes c^\alpha = a_\alpha a'_\beta \otimes c^{\alpha\beta}$;
- (2) $1_\alpha \otimes c^\alpha = 1 \otimes c$;
- (3) $a_\alpha \otimes c^\alpha_1 \otimes c^\alpha_2 = a_{\beta\alpha} \otimes c_1^\alpha \otimes c_2^\beta$;
- (4) $a_\alpha \varepsilon(c^\alpha) = a \varepsilon(c)$.

Entwining structures arise naturally in the following situation. Let C be a coalgebra. Let A be an algebra which is a right C -comodule, and let

$$B = \{b \in A \mid (ba)_0 \otimes (ba)_1 = ba_0 \otimes a_1 \text{ for all } a \in A\}.$$

There is a linear map $\beta : A \otimes_B A \rightarrow A \otimes C$ such that $\beta(a \otimes a') = aa'_0 \otimes a'_1$, which is evidently left A -linear and right C -colinear. When β is bijective, we say that A/B is a *C-Galois extension*. If this is the case, we let $\gamma : C \rightarrow A \otimes_B A$ be the unique map such that $\beta \circ \gamma = \eta \otimes 1$, which we shall write $\gamma(c) = l(c) \otimes r(c)$ with an implicit sum over implicit indices. Then there is a canonical entwining structure (A, C, ψ) associated to the extension A/B in which the map ψ is given by $\psi(c \otimes a) = \beta(\gamma(c)a) = l(c)(r(c)a)_0 \otimes (r(c)a)_1$.

Let (A, C, ψ) be an entwining structure. We shall always consider the space $A \otimes C$ to be endowed with the structure of an A -bimodule with left and right actions given by

- (1) $\lambda \cdot (a \otimes c) = \lambda a \otimes c$;
- (2) $(a \otimes c) \cdot \rho = a \cdot \psi(c \otimes \rho)$.

for each $a, \lambda, \rho \in A$ and $c \in C$.

We note that when A/B is a C -Galois extension, the Galois map $\beta : A \otimes_B A \rightarrow A \otimes C$ is a map of A -bimodules. Indeed, we have

$$\begin{aligned} \beta(a \otimes a') \cdot b &= (aa'_0 \otimes a'_1) \cdot b = aa'_0 l(a'_1)(r(a'_1)b)_0 \otimes (r(a'_1)b)_1 \\ &= a(a'b)_0 \otimes (a'b)_1 = \beta((a \otimes a') \cdot b), \end{aligned}$$

where the third equality follows from the fact, stated as property (iii) in the proof of Theorem 2.7 in [2], that $a_0 l(a_1) \otimes r(a_1) = 1 \otimes a$ for all $a \in A$.

In [1], Brzeziński considered the complex $\text{Bar}_\bullet^\psi(A) = (A \otimes C) \otimes_A \text{Bar}_\bullet(A)$; here $\text{Bar}_\bullet(A)$ is the usual Hochschild resolution of A as an A -bimodule. Since of course A is flat as a left A -module, this complex is acyclic over $A \otimes C$, and since its components are clearly free as A -bimodules, we have in fact a projective resolution of $A \otimes C$ as an A -bimodule.

For each A -bimodule M , Brzeziński [1] defined the cohomology of the entwining structure (A, C, ψ) with values in M to be the graded space $H_\psi^\bullet(A, M)$ obtained by

taking the homology of the cochain complex $\text{Hom}_{A^e}(\text{Bar}_\bullet^\psi(A), M)$. In view of the above observation, we have at once that $H_\psi^\bullet(A, M) = \text{Ext}_{A^e}^\bullet(A \otimes C, M)$.

Observe that with this identification in mind, Proposition 2.3 in [1], stating that $A \otimes C$ is a projective A -bimodule if and only if $H_\psi^1(A, -)$ vanishes identically, becomes immediate.

Proposition 2.6 in [1] and the comments after its proof hint that when A/B is a C -Galois extension, the cohomology of the corresponding entwining structure (A, C, ψ) is related to the Hochschild cohomology of B . In that paper the case where $B = k$ is considered; more generally, we have the following result.

Theorem 2.1. *Let C be a coalgebra. Let A/B be a C -Galois extension, and let (A, C, ψ) be the corresponding entwining structure. Then $H_\psi^0(A, -) \cong H^0(B, -)$ as functors of A -bimodules. Moreover, if A is flat as a (left or right) B -module, then $H_\psi^\bullet(A, -) \cong H^\bullet(B, -)$ as ∂ -functors on the category of A -bimodules.*

Proof. On the one hand, because $\beta : A \otimes_B A \rightarrow A \otimes C$ is an isomorphism of A -bimodules, we have

$$H_\psi^\bullet(A, -) \cong \text{Ext}_{A^e}^\bullet(A \otimes C, -) \cong \text{Ext}_{A^e}^\bullet(A \otimes_B A, -) \quad (1)$$

naturally on A -bimodules. On the other hand, the change-of-rings spectral sequence XVI.§5.(2)₃ constructed in [4], when specialised to the morphism $B^e \rightarrow A^e$, has $E_2^{p,q} \cong \text{Ext}_{A^e}^q(\text{Tor}_p^{B^e}(A^e, B), -)$ and converges to $\text{Ext}_{B^e}^\bullet(B, -)$; note that we know from Corollary IX.§4.4 in [4] that $\text{Tor}_\bullet^{B^e}(A^e, B) \cong \text{Tor}_\bullet^B(A, A)$. Now, since this spectral sequence lives on the first quadrant, the isomorphism (1) and the convergence immediately imply that $H_\psi^0(A, -) \cong H^0(B, -)$. When A is flat as a B -module, the spectral sequence degenerates at once, and this, together with (1), give an isomorphism $\text{Ext}_{A^e}^\bullet(A \otimes_B A, -) \cong \text{Ext}_{B^e}^\bullet(B, -)$. $\square$

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Résumé substantiel en français. On dit qu'une extension d'algèbres A/B est *Hopf-galoisienne relativement à une algèbre de Hopf H* si A est une algèbre H -comodule, $B = A^{\text{co}H}$, et une certaine condition de Galois est satisfaite. Une telle extension peut être vue comme un fibré principal non-commutatif de groupe structural H . Or, il y a des exemples d'extensions d'algèbres (tels certains exemples où interviennent les sphères quantiques $S_{q,s}^2$ de P. Podleś [6]) qu'on voudrait voir comme fibrés non-commutatifs, mais qui ne sont pas des extensions Hopf-galoisiennes. T. Brzeziński et P. Hajac ont proposé dans [2] une notion plus générale d'extension galoisienne relativement à une cogèbre; elle s'applique, par exemple, aux extensions $SU_q(2)/S_{q,s}^2$ des sphères quantiques. Cette notion produit, en particulier, des exemples de paires tressées formées d'une algèbre et d'une cogèbre, comme dans [3].

Dans [1], Brzeziński a introduit deux théories cohomologiques pour une paire tressée et, en particulier, pour une extension galoisienne A/B relativement à une cogèbre C : la cohomologie tressée $H_\psi^\bullet(A, -)$ de A à valeurs dans un A -bimodule, et une version C -équivalente. Il a donné un calcul de la première dans le cas où $B = k$ est le

corps de base : il se trouve que les groupes de cohomologie sont dans ce cas nuls en degrés positifs.

Dans cette note, on généralise ce résultat.

Théorème 2.1. *Soit A/B une extension galoisienne relativement à une cogèbre C , et soit $(A, C)_\psi$ la paire tressée canoniquement associée à l'extension. Alors $H_\psi^0(A, -) \cong H^0(B, -)$ en tant que foncteurs sur la catégorie des A -bimodules. De plus, si A est plat en tant que B -module à gauche ou à droite, alors il existe un isomorphisme $H_\psi^\bullet(A, -) \cong H^\bullet(B, -)$ de ∂ -foncteurs définis sur les A -bimodules.*

Il est clair qu'on retrouve le calcul de Brzeziński lorsque $B = k$.

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