

HOMOTOPY THEORIES FOR POSPACES AND TOPOLOGICAL CATEGORIES

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RÉSUMÉ. Nous définissons différentes théories d'homotopie pour des pospaces et des catégories topologiques. Les outils et résultats développés peuvent être appliqués tant en informatique qu'en théorie des catégories et en topologie.

ABSTRACT. In this paper, I define and compare many homotopy theories for pospaces and topological categories. The tools and results developed here can be applied in computer science, for example in concurrency theory, as well as in abstract category theory and topology.

1. Introduction. In the last years, many mathematicians have been studying pospaces and local pospaces. The more well-known homotopy theory for (local) pospaces is the so-called dihomotopy or directed homotopy. This theory has applications in many subjects, especially in computer science, for example in concurrency theory [3, 4, 5, 6, 7].

In this paper, I investigate different homotopy theories for pospaces and relate these theories to the homotopy theories of topological categories. One can associate to a pospace $(X, \leq)$ a topological category $C(X)$ and, via the classifying space functor, a new space X' which contains the information of the topology and the partial order of the pospace X . Of course, there is a loss of information in this procedure and the comparison between the various homotopy theories intends to detect how much information is lost in this process.

This paper is part of a larger project in which I investigate the homotopy theory of local pospaces from a new point of view. This new insight is related to the theory of global actions and groupoid atlases (see [1, 2]). Global actions were introduced by A. Bak as the algebraic counterpart of topological spaces [1]. The concept of global action gives algebraic objects, such as groups, structures which allow one to develop a homotopy theory in the classical way by defining paths and deformations of their morphisms. Groupoid atlases were introduced in a joint paper with A. Bak, R. Brown and T. Porter [2] and they constitute a natural generalization of the concept of global actions.

In this paper, I analyse the homotopy theory of pospaces and topological categories; and in a subsequent paper, I study the homotopy theory of local pospaces and atlases of topological categories [13].

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Definition 1.1. A *pospace* $(X, \leq)$ is a topological space X together with a partial order $\leq$ defined in X .

Remark 1.2. Sometimes the partial order is required to be *closed*, this means that the subspace $A = \{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$ with the product topology. Note that if $(X, \leq)$ is a pospace with closed partial order, then for every $x \in X$, the sets $\downarrow x = \{y \in X \mid y \leq x\}$ and $\uparrow x = \{y \in X \mid x \leq y\}$ are closed in X . Moreover, the intervals $[x, y] = \downarrow y \cap \uparrow x$ are closed.

In this paper I work with any partial order in X (not necessarily closed).

A *morphism of pospaces* $f : X \rightarrow Y$ is a continuous map which preserves the partial order, that is $f(x) \leq f(x')$ whenever $x \leq x'$.

There is a generalization of the concept of pospaces, which is the notion of local pospaces. Consider the interval $I = [0, 1] \subset \mathbb{R}$ with the usual topology and the usual order. The circle S^1 is the quotient of I as a topological space, by identifying the points 0 and 1, but S^1 is no longer this quotient in the category of pospaces: since any morphism $f : I \rightarrow X$ of pospaces with $f(0) = f(1)$ is constant. It is shown below that S^1 is the quotient of the interval in the category of local pospaces.

Definition 1.3. Let X be a topological space.

- (1) A *local partial order* on X is an open covering $\mathcal{U}$ of X by partially ordered sets $(U, \leq_U)$ such that for every $U, V \in \mathcal{U}$ one has:

$$x \leq_U y \iff x \leq_V y$$

for all $x, y \in U \cap V$.

- (2) A *refinement* $\mathcal{V}$ of a local partial order $\mathcal{U}$ is a refinement of the covering $\mathcal{U}$ which preserves the partial orders, that is for every $V \subseteq U$ with $V \in \mathcal{V}$ and $U \in \mathcal{U}$, we have:

$$x \leq_V y \iff x \leq_U y$$

for all $x, y \in V$.

- (3) Two local partial orders on X are *equivalent* if they have a common refinement.
- (4) A space X together with an equivalence class of local partial orders $\mathcal{U}$ is called a *local pospace* and it is denoted by $(X, \mathcal{U})$.

A *morphism* $f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ between local pospaces (also called *dimap*) is a continuous map $f : X \rightarrow Y$ which locally preserves the order, that means that there are refinements $\tilde{\mathcal{U}}$ of $\mathcal{U}$ and $\tilde{\mathcal{V}}$ of $\mathcal{V}$ such that for every $U \in \tilde{\mathcal{U}}$ and every $V \in \tilde{\mathcal{V}}$ we have

$$x \leq_U y \implies f(x) \leq_V f(y)$$

for all $x, y \in U \cap f^{-1}(V)$.

Recall that I denotes the unit interval $[0, 1]$ with the usual (global) order. A *dipath* in a local pospace $(X, \mathcal{U})$ is a morphism $f : I \rightarrow X$.

Example 1.4. Consider the circle S^1 as a local pospace having $\{U_i^+, U_i^-\}_{i=1,2}$ as open cover, where

$$U_i^+ = \{(x_1, x_2) \in S^1 \mid x_i > 0\} \quad \text{and} \quad U_i^- = \{(x_1, x_2) \in S^1 \mid x_i < 0\}.$$

These open sets are pospaces with the usual anticlockwise structure.

The map $f(t) = e^{2\pi it}$ from the interval I to S^1 is a morphism of local pospaces and $f(0) = f(1)$. In fact, S^1 is the quotient, in the category of local pospaces, obtained from I by identifying the points 0 and 1.

The homotopy theory of local pospaces is investigated in a subsequent paper [13] and it is related to the homotopy theory of topological atlases [2, 10]. In the last section of this paper I explain some of the ideas of that theory.

2. Homotopy theories for pospaces. Given two pospaces X and Y , the product $X \times Y$ is the pospace with the partial order defined by

$$(x, y) \leq (x', y') \iff x \leq x' \text{ and } y \leq y'.$$

Definition 2.1. Let $f, g : X \rightarrow Y$ be morphisms of pospaces. A *strictly directed homotopy* from f to g is a morphism $H : X \times I \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In this case, we write $H : f \xrightarrow{sd} g$, or simply $f \xrightarrow{sd} g$.

Note that a strictly directed homotopy is a homotopy in the classical (topological) sense with the additional two following properties:

- (1) For each instant $t \in I$, the map $H_t : X \rightarrow Y$ defined by $H_t(x) = H(x, t)$ is a morphism of pospaces;
- (2) For each $x \in X$, the map $H_x : I \rightarrow Y$ defined by $H_x(t) = H(x, t)$ is a morphism (dipath).

If $H : f \xrightarrow{sd} g$, then the continuous map $\overline{H} : X \times I \rightarrow Y$ defined by $\overline{H}(x, t) = H(x, 1 - t)$ is a continuous homotopy from g to f and each $\overline{H}_t$ is a morphism of pospaces. However, the maps $\overline{H}_x$ are not morphisms in general because $\overline{H}_x$ reverses the partial order. Therefore $\xrightarrow{sd}$ is not an equivalence relation but we define the equivalence relation generated by it.

Definition 2.2. Let $f, g : X \rightarrow Y$ be morphisms. The map f is said to be *sd-homotopic* to g , ($f \simeq^{sd} g$ in symbols), if there are morphisms $f_0 = f, f_1, \dots, f_r = g : X \rightarrow Y$ such that for each i , we have $f_i \xrightarrow{sd} f_{i+1}$ or $f_{i+1} \xrightarrow{sd} f_i$.

Example 2.3. Let $x, y \in X$ and let $c_x, c_y : Z \rightarrow X$ be the constant morphisms. Note that $c_x \simeq^{sd} c_y$ if and only if there exists a *piecewise directed path* from x to y or vice versa, this means that there are points $x_0 = x, x_1, \dots, x_r = y$ and dipaths from x_i to x_{i+1} or from x_{i+1} to x_i for each i . In particular, not every contractible space in the usual sense is sd-contractible.

A *po-subspace* of a given pospace $(X, \leq)$ will mean a subspace of X with the induced partial order.

It is well-known that the convex sets of $\mathbb{R}^n$ are contractible. It is clear that the convex po-subspaces of $\mathbb{R}^n$ need not be sd-contractible as one can see in the following example.

Example 2.4. Let X be the po-subspace of $\mathbb{R}^2$ defined by

$$X = \{(x, -x) \mid x \in \mathbb{R}\}.$$

Note that there are no piecewise directed path between any pair of different points in X , thus in particular X is not sd-contractible.

Proposition 2.5. *Let X be a convex po-subspace of $\mathbb{R}^n$ and let $f, g : Z \rightarrow X$ be morphisms. Suppose there exists a point $x_0 \in X$ such that $f(z) \leq x_0$ for all z , or $x_0 \leq f(z)$ for all z ; and $g(z) \leq x_0$ for all z , or $x_0 \leq g(z)$ for all z . Then $f \stackrel{sd}{\simeq} g$.*

Proof. I will show the case where $f(z) \leq x_0$ and $g(z) \leq x_0$ for all z . The other three cases are similar.

Define $H : Z \times I \rightarrow X$ by

$$H(z, t) = (1 - t)f(z) + tx_0.$$

It is clear that H is a well-defined (classical) homotopy from f to the constant map x_0 . Since f is a morphism, the maps $H_t : Z \rightarrow X$ are morphisms for all t . The paths $H_z : I \rightarrow X$ are directed paths for all z since $f(z) \leq x_0$.

This way one can find a sd-homotopy $H : f \stackrel{sd}{\Rightarrow} c_{x_0}$ from f to the constant map x_0 .

In the same way one finds an sd-homotopy $g \stackrel{sd}{\Rightarrow} c_{x_0}$. Thus $f \stackrel{sd}{\simeq} g$. $\square$

Corollary 2.6. *Let X be a convex po-subspace of $\mathbb{R}^n$. If there exists an element $x_0 \in X$ such that $x_0 \leq x$ for all $x \in X$, or $x \leq x_0$ for all x , then X is sd-contractible.*

In particular, one obtains the following results.

Corollary 2.7. *The pospace I is sd-contractible.*

Corollary 2.8. *If Z is a compact pospace, all morphisms $f : Z \rightarrow \mathbb{R}^n$ are sd-homotopic. In particular, all dipaths $w : I \rightarrow \mathbb{R}^n$ are sd-homotopic.*

The more well-known notion of homotopy in this context is the concept of dihomotopy.

Definition 2.9. Let $\tilde{I}$ denote the unit interval with the usual topology but with discrete order, that is given $t, s \in I$, we have $t \leq s$ if and only if $t = s$.

Note that a morphism $f : \tilde{I} \rightarrow X$ is simply a continuous map. Moreover, a continuous map $H : X \times \tilde{I} \rightarrow Y$ is a morphism if and only if the maps $H_t : X \rightarrow Y$ are morphisms for all $t \in I$.

Note also that the identity map $1 : \tilde{I} \rightarrow I$ is a morphism which is not an isomorphism of pospaces.

Definition 2.10. Two morphisms $f, g : X \rightarrow Y$ are *dihomotopic* if there exists a morphism $H : X \times \tilde{I} \rightarrow Y$ such that $H_0 = f$ and $H_1 = g$: that is H is a continuous homotopy from f to g with the additional property that for each instant $t \in I$, the map $H_t : X \rightarrow Y$ is a morphism of pospaces.

Note that this defines an equivalence relation in the set of all morphisms from X to Y .

Since the identity $1 : \tilde{I} \rightarrow I$ is a morphism, every sd-homotopy H is in particular a dihomotopy because of the commutative diagram of pospaces

$$\begin{array}{ccc} X \times \tilde{I} & \xrightarrow{H} & Y \\ \downarrow 1 & \nearrow H & \\ X \times I & & \end{array}$$

The notion of dihomotopy is strictly weaker than the notion of sd-homotopy as it is shown in the following example.

Example 2.11. Consider the pospace X of Example 2.4. All constant maps from I to X are dihomotopic but they are not sd-homotopic.

In fact, this example shows also that the pospace $\tilde{I}$ is not sd-contractible. It is important to remark that $\tilde{I}$ is dcontractible. The map $H : \tilde{I} \times \tilde{I} \rightarrow \tilde{I}$, defined by $H(s, t) = st$, is a dihomotopy between the identity of $\tilde{I}$ and the constant map 0.

3. Homotopy theories for topological categories. I begin by fixing some notation and terminology. Given a small category $\mathcal{C}$, let $\mathcal{C}_0$ denote the set of objects and $\mathcal{C}_1$ the set of arrows. Given an arrow f in $\mathcal{C}$, let $s(f)$ denote the source of f and $t(f)$ the target. A *topological category* will mean a category object in the category $\mathcal{Top}$ of topological spaces [11], that is a (small) category $\mathcal{C}$ where $\mathcal{C}_0$ and $\mathcal{C}_1$ are topological spaces and the four structural maps

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{\circ} \mathcal{C}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \\ \xleftarrow{id} \end{array} \mathcal{C}_0$$

are continuous functions.

A *functor* $f : \mathcal{C} \rightarrow \mathcal{D}$ between topological categories is a functor (in the usual sense) such that the induced maps $f_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ and $f_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ are continuous.

It is well-known that a poset $(X, \leq)$ can be seen as a category on the set X with exactly one map $(x, y) : x \rightarrow y$ whenever $x \leq y$. In the same way, a pospace can be considered as a topological category. Explicitly, given a pospace $(X, \leq)$, consider the topological category $\mathcal{C}$ having $\mathcal{C}_0 = X$ as the topological space of objects and $\mathcal{C}_1 = \{(x, y) \mid x \leq y\} \subset X \times X$, with the subspace topology, as the topological space of arrows. This topological category will be denoted $C(X, \leq)$, or simply $C(X)$.

Note that any morphism $f : (X, \leq) \rightarrow (Y, \leq)$ of pospaces induces a (continuous) functor $C(f) : C(X) \rightarrow C(Y)$.

One defines this way an inclusion functor

$$C : \mathcal{PoTop} \rightarrow \mathcal{Topcat}$$

from the category of pospaces to the category of topological categories.

A *discrete category* will mean a topological category where $\mathcal{C}_0$ and $\mathcal{C}_1$ are discrete topological spaces: that is a discrete category is a small category in the usual sense. There are different notions of homotopy for discrete categories (see for example the works of G. Segal, D. Quillen, R. Thomason and G. Hoff [8, 9, 14, 15, 16]). The classical notion of homotopy for categories, which I call *weak homotopy*, is defined via the classifying space functor $B : \mathcal{Cat} \rightarrow \mathcal{Top}$ from the category of discrete categories to that of topological spaces. In fact, this functor is defined in the category $\mathcal{Topcat}$ of topological categories. The classifying space of a topological category is a topological space (although it is not in general a CW-complex, as in the case of discrete categories) and one can simply extend the notion of weak homotopy (see below).

The notion of *strong homotopy* for discrete categories is the symmetric transitive closure of the relation given by: $f \simeq g$ if and only if there exists a natural transformation between them. Clearly this notion can also be extended to topological categories.

In the paper [12], I proved that the category $\mathcal{Cat}$ of small (discrete) categories admits an alternative axiomatic model structure related to the stronger notion of homotopy. This approach to an axiomatic model structure on $\mathcal{Cat}$ has the advantage that all constructions, whether main or subsidiary ones, are done inside this category, which means that a solution of a problem can be followed step by step in the category. In that paper I have also related the homotopy theory obtained with this construction to the homotopy theory of Thomason [16].

The concepts of sd-homotopy and dihomotopy of pospaces can be defined in the context of topological categories. Therefore one has four different definitions of homotopy for continuous functors.

Given a (topological) category $\mathcal{C}$, the *nerve* NC is the simplicial space which in degree n consists of the space of diagrams in $\mathcal{C}$ of the form

$$x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n.$$

The i -th face is obtained by deleting the object x_i (and composing the maps) and the i -th degeneracy is obtained by replacing x_i by $x_i \xrightarrow{1} x_i$ (see [14, 15]). The classifying space BC is the geometric realization of the nerve NC . If $\mathcal{C}$ is discrete, then BC is a CW-complex with exactly one n -cell for each non-degenerate n -simplex in NC , that is such that none of the arrows of the diagram is the identity. In general BC is constructed from NC by taking the disjoint union

$$\bigsqcup_n (\Delta^n \times (NC)_n),$$

where Δ^n is the standard topological n -simplex, and by identifying points via the face and degeneracy maps of NC (see [14, 15]).

It is clear that continuous functors $f : \mathcal{C} \rightarrow \mathcal{D}$ induce continuous maps $Bf : BC \rightarrow BD$.

Example 3.1. The classifying space of the category

$$I(1) : 0 \rightarrow 1$$

(which is the discrete category with two points $\{0, 1\}$ and exactly one arrow from 0 to 1) is the topological interval I . In general, for every $n \in \mathbb{N}$, the classifying space of the category

$$I(n) : 0 \rightarrow 1 \leftarrow 2 \rightarrow 3 \leftarrow \dots \leftarrow n$$

is also the interval I . These categories are called the interval categories (all of them are categorical models for the topological interval).

Example 3.2. Consider the pospace $\tilde{I}$ defined in Definition 2.9. The classifying space $\tilde{I}' = BC(\tilde{I})$, of the associated topological category, turns out to be the interval I as well. Note that the categories $I(n)$ and $C(\tilde{I})$ are very different but their classifying spaces are equal. This shows the loss of information of this procedure. The classifying space $I' = BC(I)$ induced by the pospace I (with usual order) is much more complicated to describe. It is a cell complex having infinitely many n -cells for every $n \in \mathbb{N}$. However this space is contractible, as it is shown below.

Now I recall two notions of homotopy for categories. They were introduced in first place for discrete categories but can be generalized for topological categories.

Definition 3.3. Two functors $f, g : \mathcal{C} \rightarrow \mathcal{D}$ are said to be *weak homotopic* (or *w-homotopic*, for short) if the induced continuous maps $Bf, Bg : B\mathcal{C} \rightarrow B\mathcal{D}$ are homotopic (in the classical topological sense). In this case, we write $f \stackrel{w}{\simeq} g$.

Given two functors $f, g : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation $\nu : f \Rightarrow g$ from f to g is equivalent to a functor $\tilde{\nu} : \mathcal{C} \times I(1) \rightarrow \mathcal{D}$ with $\tilde{\nu}(a, 0) = f(a)$ and $\tilde{\nu}(a, 1) = g(a)$. Since $B(\mathcal{C} \times I(1)) = B\mathcal{C} \times I$, we deduce that natural transformations induce weak homotopies between the functors.

One obtains in this way the first simple corollary of this fact, which was also proven in [14].

Proposition 3.4. *If $f : \mathcal{C} \rightarrow \mathcal{D}$ has an adjoint, then it is a w-homotopy equivalence. Moreover, if $\mathcal{C}$ has an initial or a terminal object, then it is w-contractible.*

Definition 3.5. Two functors $f, g : \mathcal{C} \rightarrow \mathcal{D}$ are *strong homotopic* (or *s-homotopic*, for short) if there are functors $f_0 = f, f_1, \dots, f_r = g$ from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations from f_i to f_{i+1} or vice versa for each i : or equivalently if there exists $r \in \mathbb{N}$ and a functor $H : \mathcal{C} \times I(r) \rightarrow \mathcal{D}$ such that $H(a, 0) = f(a)$ and $H(a, r) = g(a)$ for all objects a of $\mathcal{C}$. In this case, we write $f \stackrel{s}{\simeq} g$.

Since $BI(r) = I$,

$$f \stackrel{s}{\simeq} g \Rightarrow f \stackrel{w}{\simeq} g.$$

This notion of homotopy is strictly stronger than the weak notion (see [12]). In fact, Quillen's Theorems A and B are no longer valid for this kind of homotopy.

As pointed out above, these notions of homotopy can be applied for discrete categories as well as for topological categories in general.

Now I generalize the notions of dihomotopy and sd-homotopy in this context. These notions make only sense if one works with topological categories. For discrete categories, these concepts are trivial.

Denote by $\mathcal{I}$ the category $\mathcal{I} = C(I)$ associated to the pospace I (with usual order), and by $\tilde{\mathcal{I}}$ the one associated to the pospace $\tilde{I}$ (with discrete order) as in Definition 2.9.

Note that a path $\omega : \tilde{\mathcal{I}} \rightarrow \mathcal{C}$ simply consists of a topological path in the space $\mathcal{C}_0$ of objects of $\mathcal{C}$. Note also that the space of maps of the category $\mathcal{I}$ is the topological space

$$\mathcal{I}_1 = \{(s, t) \in I \times I \mid s \leq t\}$$

with the usual topology. Thus a path $\omega : \mathcal{I} \rightarrow \mathcal{C}$ consists of a path in the space $\mathcal{C}_0$ of objects of $\mathcal{C}$ and a singular 2-simplex in the space $\mathcal{C}_1$ of arrows of $\mathcal{C}$.

Definition 3.6. Two functors $f, g : \mathcal{C} \rightarrow \mathcal{D}$ are *dihomotopic* if there exists a functor $H : \mathcal{C} \times \tilde{\mathcal{I}} \rightarrow \mathcal{D}$ with $H(a, 0) = f(a)$ and $H(a, 1) = g(a)$ for all objects a in $\mathcal{C}$. In this case, we write $f \stackrel{d}{\simeq} g$.

Note that this determines an equivalence relation on the set of functors from $\mathcal{C}$ to $\mathcal{D}$. Since $B\tilde{\mathcal{I}} = I$ (by Example 3.2),

$$f \stackrel{d}{\simeq} g \Rightarrow f \stackrel{w}{\simeq} g.$$

It is important to remark that the notions of s-homotopy and dihomotopy are not comparable. For example, on the one hand, the category $I(1)$ is s-contractible (it has

an initial element) and it is not dcontractible. On the other hand, the category $\tilde{\mathcal{I}}$ itself is dcontractible but not s-contractible.

The strongest notion of homotopy is the concept of sd-homotopy.

Definition 3.7. Let $f, g : \mathcal{C} \rightarrow \mathcal{D}$ be functors. A *strictly directed homotopy* from f to g is a functor $H : \mathcal{C} \times \mathcal{I} \rightarrow \mathcal{D}$ such that $H(a, 0) = f(a)$ and $H(a, 1) = g(a)$ for all objects a of $\mathcal{C}$. In this case, we write $H : f \xRightarrow{sd} g$, or simply $f \xRightarrow{sd} g$. The functor f is said to be *sd-homotopic* to g ($f \simeq^{sd} g$ in symbols), if there are functors $f_0 = f, f_1, \dots, f_r = g$ from $\mathcal{C}$ to $\mathcal{D}$ such that for each i , we have $f_i \xRightarrow{sd} f_{i+1}$ or $f_{i+1} \xRightarrow{sd} f_i$.

There exists a (unique) functor from $\tilde{\mathcal{I}}$ to $\mathcal{I}$ which is the identity on the objects. This proves that

$$f \simeq^{sd} g \Rightarrow f \simeq^d g.$$

In addition, there exists a (unique) functor from $I(1)$ to $\mathcal{I}$ taking 0 to 0 and 1 to 1. Thus

$$f \simeq^{sd} g \Rightarrow f \simeq^s g.$$

Theorem 3.8. Let $f, g : X \rightarrow Y$ be morphisms of pospaces and let $C(f), C(g)$ be the induced functors between the associated categories. Then

- (1) $f \simeq^{sd} g$ if and only if $C(f) \simeq^{sd} C(g)$.
- (2) $f \simeq^d g$ if and only if $C(f) \simeq^d C(g)$.

In particular, the category $\mathcal{I}$ is sd-contractible and therefore its classifying space $I' = B\mathcal{I}$ is contractible. The category $\tilde{\mathcal{I}}$ is dcontractible but not sd-contractible.

Proof. It is easy to verify that the functor C commutes with the products, thus (1) and (2) follow from the definitions.

I have shown before that I is an sd-contractible pospace, therefore one deduces from (1) that $\mathcal{I}$ is sd-contractible as well. Since sd-contractible implies w-contractible, it turns out that $B\mathcal{I}$ is a contractible space. Finally, by Example 3.2 and (2), we see that $\tilde{\mathcal{I}}$ is dcontractible but not sd-contractible. $\square$

4. Towards the ‘right’ homotopy theory for topological atlases. If $(X, \mathcal{U})$ is a local pospace, as defined in Definition 1.3, then it can be considered *locally* as a topological category. This induces the idea of an atlas of topological categories, which is a generalization of the concept of groupoid atlas.

An atlas of topological categories on a topological space X consists of a family of topological categories whose object spaces are subspaces of X . These topological categories satisfy certain conditions of compatibility on the non-empty intersections. Moreover, the topology of X is coherent with the topology of the object spaces of the family.

In order to construct an atlas from a local pospace $(X, \mathcal{U})$, one has to replace $\mathcal{U}$ by a maximal equivalent local order containing $\mathcal{U}$. Then, one takes X as the underlying space of the atlas, and the open sets of the maximal cover as the object spaces of the family of topological categories.

In [2] we have developed a homotopy theory (weak and strong) for (discrete) groupoid atlases. Generalizing the techniques developed in that paper, one can define and compare different notions of homotopy for atlases and relate these ideas to the homotopy theories of local pospaces. This is done in [13].

Applications are in both ways: on the one hand, one can use the homotopy theory of local pospaces to investigate concurrent processes and, on the other hand, the homotopy theory of atlases is related to the theory of simplicial complexes, global actions and K-theory [2, 10, 13].

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