

FUNDAMENTAL GROUP OF INCIDENCE ALGEBRAS FROM AN ALGEBRAIC POINT OF VIEW

MARÍA ANDREA GATICA AND ANDREA ALEJANDRA REY

RÉSUMÉ. Nous calculons le groupe fondamental algébrique de certaines familles d'algèbres d'incidence. Nous utilisons les méthodes développées dans [4, 8] et un lemme de réduction de [2].

ABSTRACT. In this paper we compute the fundamental group of some families of incidence algebras. We use methods developed in [4, 8] and a reduction result from [2].

The aim of this paper is to calculate the fundamental group of some incidence algebras using different techniques. Our motivation comes from the property, for an algebra, to be described in terms of its fundamental group, and the relation between the fundamental group of an algebra and its first Hochschild cohomology group.

The paper is divided into four sections. In the first section we recall the definitions and notations that will be used throughout this paper.

The second section is devoted to introducing useful results developed by Bustamante [4] and Reynaud [8] which allow us to compute the fundamental groups of particular classes of incidence algebras. The most important tool we use is a reduction lemma from [2] relating the fundamental group with the existence of some weak crowns. This result allows to reduce the number of points in the considered algebra. The concept of weak crowns has been also defined in [2].

We introduce the families of algebras on which we are going to work in the third section and finally, in the last section, we compute the announced fundamental groups.

1. Definitions and Notation.

1.1. Algebras and quivers. Throughout this paper, k will denote a fixed algebraically closed field and all considered algebras will be incidence algebras. Recall that if $(\Sigma, \leq)$ is a finite poset with n elements, its *incidence algebra* is the subalgebra of the algebra $M_n(k)$ of square matrices over k with elements $(x_{ij}) \in M_n(k)$ satisfying $x_{ij} = 0$ if $i \not\leq j$.

A *quiver* Q is a quadruple (Q_0, Q_1, s, t) consisting of two sets: Q_0 , the set of *vertices*, and Q_1 , the set of *arrows*; and two maps $s, t : Q_1 \rightarrow Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$. Thus, one may think of a quiver as being a directed graph. A *relation* in a quiver Q from a vertex x to a vertex y is a linear combination $\rho = \sum_{i=1}^m \lambda_i w_i$ such that, for each $1 \leq i \leq m$, λ_i is

Reçu le 29 décembre 2004 et, sous forme définitive, le 5 avril 2005.

a non-zero scalar and w_i is a path of length at least two from x to y . A set of relations in Q generates an admissible ideal I in the path algebra kQ of Q , that is $I \subset kQ_1^2$ and $(kQ_1)^m \subset I$ for some $m \in \mathbb{N}$. The pair (Q, I) is then called a *bound quiver*. It is well-known that for every basic algebra A there exists a surjective k -algebra morphism $v : kQ \rightarrow A$ whose kernel I_v is admissible, where Q is the ordinary quiver of A . Thus we have $A \simeq kQ/I_v$. The bound quiver (Q, I_v) is called a *presentation* of A .

Observe that an algebra $A = kQ/I$ can be also seen as a k -category with objects A_0 being the elements of Q_0 and morphisms $A(x, y)$ being the quotient of $kQ(x, y)$, the k -vector space generated by all the paths in Q from x to y , modulo the subspace $I(x, y) = I \cap kQ(x, y)$.

1.2. Modules. Let $A = kQ/I$ be an algebra. The advantage of considering algebras as quivers with relations is that this allows us to describe their finitely generated modules in terms of representations of the corresponding quivers. A (Q, I) finite dimensional representation is given by $M = (M(x)_{x \in Q_0}, M(\alpha)_{\alpha \in Q_1})$, where for each $x \in Q_0$, $M(x)$ is a finite dimensional k -vector space, and for each $\alpha \in Q_1$, $M(\alpha)$ is a linear transformation from $V(s(\alpha))$ to $V(t(\alpha))$. The representation M is said to be *bound by I* if, whenever $\rho = \sum_{i=1}^m \lambda_i \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_{t_i}}$ is a relation in I , we have $\sum_{i=1}^m \lambda_i M(\alpha_{i_1}) M(\alpha_{i_2}) \dots M(\alpha_{i_{t_i}}) = 0$.

A sum of two representations $M = (M(x), M(\alpha))$ and $N = (N(x), N(\alpha))$ is the representation $(M(x) \oplus N(x), M(\alpha) \oplus N(\alpha))$. A representation $M = (M(x), M(\alpha))$ is said to be *indecomposable* if it is not isomorphic to the sum of two non-zero representations.

It is well-known that if $A = kQ/I$ then the category $\text{mod } A$ of finitely generated left A -modules is equivalent to the category $\text{mod}(Q, I)$ of all bound finite-dimensional representations of (Q, I) . Therefore we may identify a module M with the corresponding representation $(M(x), M(\alpha))_{x \in Q_0, \alpha \in Q_1}$ where $M(x) = \overline{e_x} M$ and $M(\alpha)(x) = \overline{\alpha} x$ (recall that $\overline{e_x}$ and $\overline{\alpha}$ are the classes in A of e_x and α respectively).

For each $x \in Q_0$ we denote by S_x the corresponding simple A -module, by P_x (or I_x) the projective cover (or injective envelope, respectively) of S_x and by $\text{rad } P_x$ (or $\text{soc } I_x$) the radical (or socle) of the module P_x (or I_x , respectively).

If $A = kQ/I$ is an incidence algebra then it is easy to describe the representations of the indecomposable projective (or injective) modules P_x (or I_x , respectively) for each $x \in Q_0$. In fact, $P_x(y) = k$ if $y \leq x$, and it vanishes otherwise; and $P_x(\alpha) = id_k$ if $s(\alpha) \leq x$, and it is zero otherwise. It follows now by definition that the module $\text{rad } P_x$ can be described as a module by $\text{rad } P_x(y) = P_x(y)$ if $x \neq y$ and $\text{rad } P_x(x) = 0$; and $\text{rad } P_x(\alpha) = id_k$ if $x > s(\alpha)$, and it vanishes otherwise.

The injective envelope I_x of S_x and the socle of I_x are constructed dually.

1.3. The fundamental group. Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^m \lambda_i w_i$ in $I(x, y)$ is called *minimal* if $m \geq 2$ and for every non-empty proper subset $J \subset \{1, 2, \dots, m\}$ we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse. A *walk* from x to y in Q is a formal composition $\alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \dots \alpha_t^{\epsilon_t}$ (where $\alpha_i \in Q_1$ and $\epsilon_i = \pm 1$, for $1 \leq i \leq t$) starting at x and ending at y . We denote by e_x the trivial path at x .

Let $\sim$ be the smallest equivalence relation on the set of all walks in Q such that:

- (1) If $\alpha : x \rightarrow y$ is an arrow in Q , then $\alpha^{-1} \alpha \sim e_x$ and $\alpha \alpha^{-1} \sim e_y$;

- (2) If $\rho = \sum_{i=1}^m \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all $1 \leq i, j \leq m$;
- (3) If $u \sim v$, then $www' \sim vvv'$ whenever these compositions make sense.

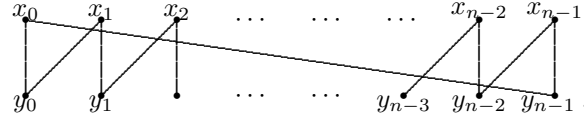
We denote by $[u]$ the equivalence class of a walk u . Let $x_0 \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x_0)$ of equivalence classes of all the closed walks starting and ending at x_0 has a group structure defined by the operation $[u][v] = [uv]$. Clearly the group $\pi_1(Q, I, x_0)$ does not depend on the choice of the base point x_0 . So, we denote it simply by $\pi_1(Q, I)$ and we call it the *fundamental group* of (Q, I) (see [1, 6, 7]).

It follows from the definition that the fundamental group $\pi_1(Q, I)$ depends essentially on I , thus it is not an invariant of A . Bardzell and Marcos [3] proved that if the algebra is *constricted* (that is, for an arrow $\alpha \in Q_1$, $\dim_k t(\alpha)As(\alpha) \leq 1$) the fundamental group does not depend on the presentation. Thus, the notation $\pi_1(A)$ is not ambiguous for constricted algebras. Constricted algebras include incidence algebras.

The fundamental group of an incidence algebra is also the fundamental group of a finite simplicial complex. Indeed, let $A = A(\Sigma)$ be an incidence algebra, and $|\Sigma|$ be the chain complex of the poset Σ (that is, $|\Sigma|$ is the simplicial complex whose i -simplices are the chains $x_0 < x_1 < \dots < x_i$ in Σ); then we have that $\pi_1(A) \simeq \pi_1(|\Sigma|)$ (see [4, 8]). Conversely, if K is a finite simplicial complex, and Σ is the set of its non-degenerated simplices ordered by inclusion, then $\pi_1(|K|) \simeq \pi_1(A(\Sigma))$ (see [4]).

2. Reduction techniques. In this section, we recall the main results we use in the computations and we prove some useful lemmata. First of all, let us recall the following definitions from [2].

Definition 2.1. Let $A = A(\Sigma)$ be the incidence algebra of a finite poset Σ . Let Γ be a full subcategory of A generated by the points $\{x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}\}$ (with $n \geq 2$) and of the form



We say that:

- (1) Γ is a *weak crown* if for each i , the convex hull of $\{x_i, y_i\}$ in Σ intersects those of $\{x_i, y_{i-1}\}$ and $\{x_{i+1}, y_i\}$, and of no other $\{x_h, y_l\}$ (here, and in the sequel, we agree to set $y_{-1} = y_{n-1}$ and $x_0 = x_n$), and the convex hulls of three distinct $\{x_h, y_l\}$ do not intersect.
- (2) A weak crown Γ is a *crown* if the intersection of the convex hulls of $\{x_i, y_i\}$ and of $\{x_i, y_{i-1}\}$ is x_i , and the intersection of the convex hulls of $\{x_{i+1}, y_i\}$ and of $\{x_i, y_i\}$ is y_i for each i such that $0 \leq i \leq n-1$. In this case we say that the crown has *width* n .

Definition 2.2. Let Γ be a weak crown. A point x not in Γ is said to *suspend* Γ if x is a proper predecessor of at least two non-comparable points of Γ and no proper successor of x is a predecessor of the same points of Γ . A suspending point x is said to be a *top* of Γ if x is a proper predecessor of all the maximal points of Γ . We define dually points which *sustain* Γ .

For more details and examples see [2].

Theorem 2.3. *Assume that A is an incidence algebra and that x is a source in A such that $\text{rad } P_x$ is indecomposable, and x tops no weak crown. Let B be the full subcategory of A generated by all objects except x . Then $\pi_1(B) \simeq \pi_1(A)$.*

Proof. This is Lemma 5.3 of [2]. $\square$

Remark 2.4. The convex hull of a weak crown Γ contains a crown as a full subcategory (this is Lemma 3.2 of [2]).

Given an A -module M , we denote by $\text{supp } M$ the *support* of M , that is the full subcategory of A generated by the objects $x \in A_0$ such that $M(x) \neq 0$.

The next lemma will be useful to check the hypotheses of Theorem 2.3, which allows to compute of the fundamental group of some algebras considered in this paper.

Lemma 2.5. *Let $A = kQ/I$ be an algebra and M an A -module satisfying the following conditions:*

- (1) *For any $x \in \text{supp } M$, $M(x) = k$;*
- (2) *If $\alpha \in Q_1$ and $s(\alpha), e(\alpha) \in \text{supp } M$ then $M(\alpha) = \text{id}_k$.*

Then, M is indecomposable whenever $\text{supp } M$ is connected.

Proof. Suppose that $M = M_1 \oplus M_2$ with M_1, M_2 both not trivial. Then, $\text{supp } M = \text{supp } M_1 \sqcup \text{supp } M_2$. Let $x, y \in \text{supp } M$ such that $x \in \text{supp } M_1$ and $y \in \text{supp } M_2$. Since $\text{supp } M$ is connected, there exists a walk from x to y , and therefore we have an arrow $\alpha \in Q_1$ such that $s(\alpha) \in \text{supp } M_1$ and $t(\alpha) \in \text{supp } M_2$ (or conversely). Thus, by hypothesis, the map $M(\alpha)$ is the identity on k . Since the corresponding maps $M_1(\alpha) : k \rightarrow 0$ in M_1 and $M_2(\alpha) : 0 \rightarrow k$ in M_2 are both zero, we get $M_1(\alpha) \oplus M_2(\alpha) = 0 \neq \text{id}_k = M(\alpha)$. Thus, $M_1 = 0$ or $M_2 = 0$. $\square$

Remark 2.6. It follows directly from the previous lemma that if $A = kQ/I$ is an incidence algebra and the set $\text{supp } \text{rad } P_x$ ($x \in Q_0$) is connected, then $\text{rad } P_x$ is indecomposable.

For the following theorem we say that a k -category Λ is a *schurian almost triangular category* if it is schurian (for all $x, y \in \Lambda_0$, $\dim_k \Lambda(x, y) \leq 1$) and the following fact holds: $\Lambda(x, y) = k$ implies $\Lambda(y, x) = 0$. We mean by a *maximal tree* T in a connected simplicial complex S , a subcomplex of S which is a tree, and contains every vertex of S (see [4]).

Theorem 2.7. *Let $A = kQ/I$ be a schurian almost triangular category and T be a maximal tree in the quiver Q . Then $\pi_1(A) \simeq F/K$, where F is the free group with basis Q_1 , and K is the normal subgroup of F generated by the elements of the following two types:*

- (1) γ , for every arrow γ in T ;
- (2) $(\gamma_{pq_1} \gamma_{q_1 q_2} \cdots \gamma_{q_e q})(\beta_{ps_1} \beta_{s_1 s_2} \cdots \beta_{s_d q})^{-1}$, whenever $\gamma_{pq_1} \gamma_{q_1 q_2} \cdots \gamma_{q_e q}$ and $\beta_{ps_1} \beta_{s_1 s_2} \cdots \beta_{s_d q}$ are two non-zero parallel paths in (Q, I) .

Proof. This follows from Lemma 2.1 and Theorem 2.2 of [4]. $\square$

Consider a connected quiver Q and a point $x_0 \in Q_0$. We denote by $P(Q)$ the set of walks in Q and by $P_{x_0}(Q)$ the set of walks starting and ending at x_0 . Moreover, consider an arrow $\tilde{\alpha} : x_0 \rightarrow y_0$ in Q_1 with $x_0 \neq y_0$ and let Q' be the quiver obtained by identifying the vertices x_0 and y_0 , that means:

- The vertices of Q' are the vertices of Q , but where x_0 and y_0 are identified to a unique vertex called z_0 ;
- The arrows of Q' are the arrows of Q , except for the arrow $\tilde{\alpha}$. Each arrow has the same source and target as in Q , but an arrow parallel to $\tilde{\alpha}$ is now a loop in Q' .

Let p be the projection from $P(Q)$ on $P(Q')$:

$$p(x) = \begin{cases} x & \text{if } x \in Q_0, x \neq x_0 \text{ and } x \neq y_0, \\ z_0 & \text{if } x = x_0 \text{ or } x = y_0. \end{cases}$$

$$p(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in Q_1 \text{ and } \alpha \neq \tilde{\alpha}, \\ z_0 & \text{if } \alpha = \tilde{\alpha}. \end{cases}$$

Theorem 2.8. *With the above notations, the morphism p induces an isomorphism:*

$$P_{x_0}(Q)/\sim \simeq P_{x_0}(Q')/\sim'$$

where $\sim$ is the equivalence relation defined in Section 1.3 and $\sim'$ denotes the new relations after the identification of x_0 and y_0 .

Proof. This is Theorem II.1.1 of [8]. $\square$

This theorem gives us a method to simplify the computation of the fundamental group. Let Q be a quiver and I an admissible ideal of kQ . Applying Theorem 2.8, $|Q_0| - 1$ times, we get that the fundamental group $\pi_1(Q, I)$ is isomorphic to the fundamental group of a quiver with only one vertex. This procedure gives a description of $\pi_1(Q, I)$ by generators and relations.

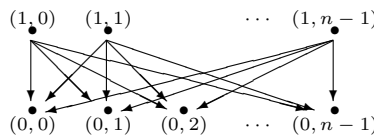
From now on, F_r will denote the free group generated by r elements.

Corollary 2.9. *Let Q be a finite connected quiver and $A = kQ$. Then $\pi_1(A) \simeq F_r$, where $r = |Q_1| - |Q_0| + 1$.*

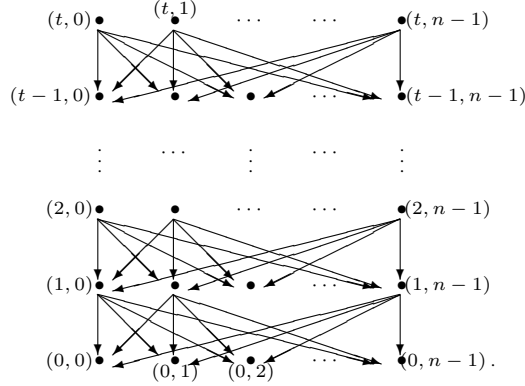
Proof. Suppose that $|Q_0| = n$. Let $Q_0 = \{x_1, \dots, x_n\}$. Since Q is connected, there exists $i_1 \in \{1, \dots, n-1\}$ and an arrow $\alpha^1 \in Q_1$, $\alpha^1 : x_n \rightarrow x_{i_1}$ or $\alpha^1 : x_{i_1} \rightarrow x_n$. We take $\tilde{\alpha} = \alpha^1$ and delete the vertex x_n . In the same way, there exists $i_2 \in \{1, \dots, n-1\} \setminus \{i_1\}$ and an arrow $\alpha^2 \in Q_1$, $\alpha^2 : x_{i_2} \rightarrow x_{i_1}$ or $\alpha^2 : x_{i_1} \rightarrow x_{i_2}$. We take now $\tilde{\alpha} = \alpha^2$ and delete the vertex x_{i_1} . We continue this procedure until we delete all the vertices except one that we call simply x (the procedure finishes because Q is finite). The remaining arrows are loops based on x . Notice that we have taken out $n-1$ arrows. So, the number of remaining arrows is $|Q_1| - (n-1)$ and this proves the result. $\square$

3. Incidence algebras. Now, we describe the four classes of incidence algebras in which we are interested in this article.

(I) Let ${}^n_t U$, with $n \geq 2$ and $t \geq 1$, be the incidence algebra obtained by superposing t algebras of the following type:



where the partial order is defined by $(0, x) < (1, y)$ for all x, y such that $0 \leq x, y \leq n - 1$. So, the ordered quiver associated to ${}^n_t U$ is

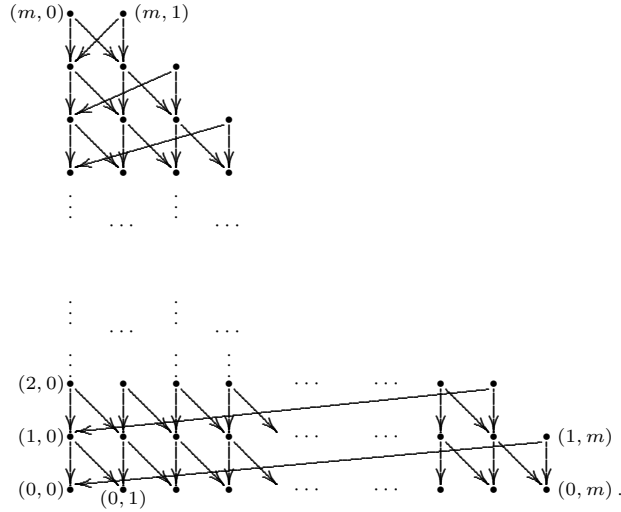


(II) Let A_m^* , with $m \in \mathbb{N}$ and $m \geq 2$, be the incidence algebra associated to the poset P_m^* which is a superposition of crowns of different width. More precisely, P_m^* is defined in the following way:

$$P_m^* = \{(l, t) \in [m] \times [m] : 0 \leq t \leq m + 1 - l, \text{ if } l \neq 0; \text{ or } 0 \leq t \leq m, \text{ if } l = 0\}$$

where we use the notation $[m] = \{0, 1, \dots, m\}$.

The partial order is defined by $(l, t) < (l + 1, t)$ and $(l, t) < (l + 1, t - 1)$, where $(l, -1) = (l, m + 1 - l)$, and therefore A_m^* has the following quiver:



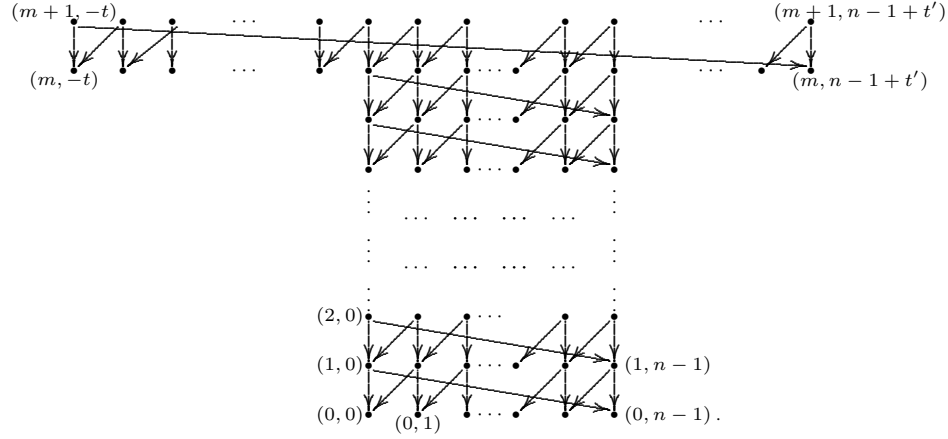
(III) Let $A_{n,m}^{t,t'}$ be the incidence algebra associated to the poset $P_{n,m}^{t,t'}$ which is the superposition of m crowns of the same width n plus a bigger crown at the top. More precisely, $P_{n,m}^{t,t'}$ is defined in the following way: for $n \geq 2$, and $t, t' \in \mathbb{N} \cup \{0\}$ (t and t' not simultaneously zero),

$$P_{n,m}^{t,t'} = P_{n,m} \cup \{(m, p) : -t \leq p < 0\} \cup \{(m, p) : n - 1 < p \leq n - 1 + t'\} \\ \cup \{(m + 1, p) : -t \leq p \leq n - 1 + t'\}$$

with $P_{n,m} = [m] \times [n-1]$.

The partial order is defined by $(l, l') < (l+1, l')$ and $(l, l') < (l+1, l'+1)$, where $(l, n) = (l, 0)$, $0 \leq l \leq m$, and $(m+1, n+t') = (m+1, -t)$.

This means that the incidence algebra associated to $P_{n,m}^{t,t'}$ has the following ordered quiver:



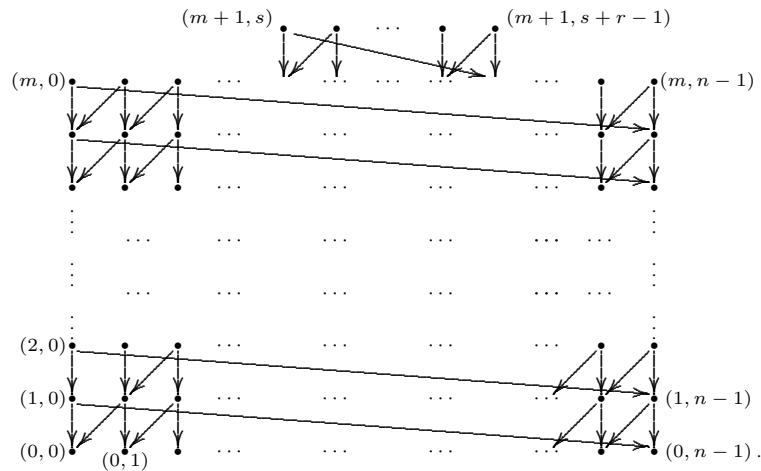
(IV) Let $A_{n,m}^r$ be the incidence algebra whose associated poset $P_{n,m}^r$ is the superposition of m crowns of the same width n plus a smaller crown of width r at the top with $2 \leq r < n$. Since the smaller crown lies completely over the stack of crowns, there exists s such that $0 \leq s \leq n-2$ and $s+r \leq n$.

That is, $P_{n,m}^r$ is defined as follows:

$$P_{n,m}^r = P_{n,m} \cup \{(m, p) : -t \leq p < 0\} \cup \{(m, p) : n-1 < p \leq n-1+t'\} \\ \cup \{(m+1, p) : s \leq p \leq s+(r-1)\}$$

with $P_{n,m} = [m] \times [n-1]$.

The partial order is defined by $(l, l') < (l+1, l')$ and $(l, l') < (l+1, l'+1)$, where $(l, n) = (l, 0)$, $0 \leq l \leq m$, and $(m+1, s+r) = (m+1, s)$. Thus, the incidence algebra associated to $P_{m,n}^r$ has the following ordered quiver:



4. The fundamental group. First of all, we introduce some notation for the first class of algebras ${}^n_t U$ we have defined. For all i, j, k such that $0 \leq i, j \leq n-1$ and $1 \leq k \leq t$, let $\alpha_{i,j}^k$ be the arrow from (k, i) to $(k-1, j)$.

Theorem 4.1. *For all $n \in \mathbb{N}$, $n \geq 2$, we have:*

$$\pi_1({}^n_t U) = \begin{cases} F_{(n-1)^2} & \text{if } t = 1, \\ 1 & \text{if } t > 1. \end{cases}$$

Proof. First, we consider the case $t = 1$. Since the algebra ${}^n_1 U$ has no relations we can compute its fundamental group using Corollary 2.9. We have that $|Q_0| = 2n$ and $|Q_1| = n^2$. So, $\pi_1({}^n_1 U)$ is the free group generated by $n^2 - 2n + 1 = (n-1)^2$ elements.

Suppose now $t > 1$. Here we use the technique described in Theorem 2.7. Let T be the maximal tree that contains the arrows

$$\{\alpha_{i,i}^k : 0 \leq i \leq n-1 \text{ and } 1 \leq k \leq t\} \cup \{\alpha_{i,i+1}^1 : 0 \leq i \leq n-2\}.$$

Thus, K is generated by the arrows in T and the walks arising from the parallel paths in Q . Now, using induction, we prove that for all $h \in \mathbb{N}$ such that $1 \leq h \leq n-1$, the arrows $\alpha_{i,i+h}^k$ and $\alpha_{i+h,i}^k$ are in K , for $1 \leq k \leq t$ and $0 \leq i \leq n-1-h$.

Suppose $h = 1$. For $0 \leq i \leq n-2$, since $\alpha_{i+1,i+1}^1 \alpha_{i,i+1}^2 = \alpha_{i,i+1}^1 \alpha_{i,i}^2$, and $\alpha_{i+1,i+1}^1, \alpha_{i,i+1}^1$ and $\alpha_{i,i}^2$ are in K , then $\alpha_{i,i+1}^2$ must also be in K . Recursively $\alpha_{i,i+1}^k$ is also in K , for $1 \leq k \leq t$.

In addition, for all i, k such that $0 \leq i \leq n-2$ and $2 \leq k \leq t$, we have $\alpha_{i+1,i+1}^{k-1} \alpha_{i+1,i+1}^k = \alpha_{i,i+1}^{k-1} \alpha_{i+1,i}^k$. Then $\alpha_{i+1,i}^k \in K$ since $\alpha_{i+1,i+1}^{k-1}, \alpha_{i+1,i+1}^k, \alpha_{i,i+1}^{k-1}$ is in K .

Assume $2 \leq h \leq n-1$, and that the result holds for any h' such that $1 \leq h' < h \leq n-1$. We have that to consider two cases:

- (i) If $t = 2$, then $n > t$. Since $\alpha_{i+h,i+h}^1 \alpha_{i,i+h}^2 = \alpha_{i+h-1,i+h}^1 \alpha_{i,i+h-1}^2$, and $\alpha_{i+h,i+h}^1, \alpha_{i+h-1,i+h}^1$ and $\alpha_{i,i+h-1}^2$ (using the inductive hypothesis) are in K , we have that $\alpha_{i,i+h}^2$ is also in K . In an analogous way, we can see that $\alpha_{i+h,i}^2 \in K$.
- (ii) Let $t \geq 3$. Using the inductive hypothesis and the fact that

$$\alpha_{i+h,i}^{t-2} \alpha_{i+h-1,i+h}^{t-1} \alpha_{i,i+h+1}^t = \alpha_{i,i}^{t-2} \alpha_{i,i}^{t-1} \alpha_{i,i}^t,$$

we infer that $\alpha_{i,i+h+1}^t \in K$. Since $\alpha_{i+h+1,i+h+1}^{t-1} \alpha_{i,i+h+1}^t = \alpha_{i,i+h+1}^{t-1} \alpha_{i,i}^t$, and $\alpha_{i+h+1,i+h+1}^{t-1}, \alpha_{i,i+h+1}^t, \alpha_{i,i}^t$ are in K , we see that $\alpha_{i,i+h+1}^{t-1}$ is in K . In an analogous way, we decrease t to obtain that $\alpha_{i,i+h+1}^k$ is in K , for $1 \leq k \leq t$.

Notice that dual arguments hold for $\alpha_{i+h+1,i}^k$.

Hence, all the arrows of the quiver are in K and this implies that $\pi_1({}^n_t U)$ is trivial. $\square$

Remark 4.2. If $n = 1$, then ${}^1_1 U$ is contractible and so $\pi_1({}^1_1 U)$ is trivial.

In the next lemma we compute the fundamental group of the algebras A_m^* using Theorem 2.3 and Corollary 2.9 which allow us to reduce the number of points in the quiver.

Theorem 4.3. *Let A_m^* , with $m \geq 3$, be the algebra defined in Section 3. Let $m = 2q+r$, with $q \in \mathbb{N}$ and $0 \leq r \leq 1$. Then $\pi_1(A_m^*) = F_{q+r}$.*

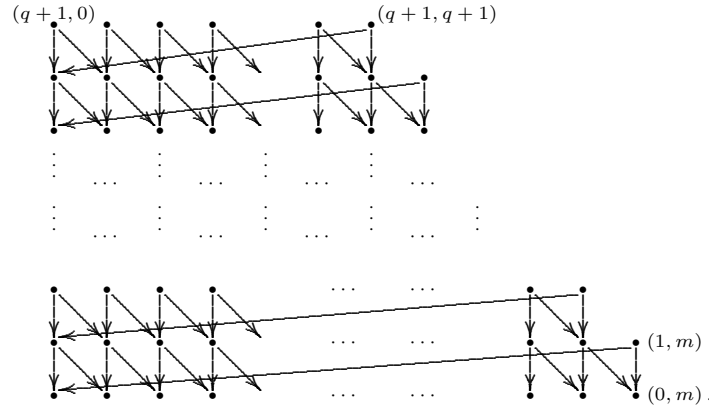
Proof. In order to compute the fundamental group of A_m^* , we remove, applying Theorem 2.3, some vertices until we obtain a full subcategory of A_m^* whose ordered quiver has no relations and its fundamental group is isomorphic to $\pi_1(A_m^*)$. Hence, the statement will follow directly from Corollary 2.9.

In fact, suppose first that m is odd, that is, $m = 2q + 1$. It is easy to see that the sources $(m, 0)$ and $(m, 1)$ are not tops of any weak crown since $\text{supp rad } P_{(m,0)} = \text{supp rad } P_{(m,1)}$ contains no crowns. Moreover, the module $\text{rad } P_{(m,0)} = \text{rad } P_{(m,1)}$ has a connected support, and therefore it is indecomposable by Lemma 2.5. Thus, it follows from Theorem 2.3 that we can eliminate them.

In a similar way one can delete the $q - 1$ crowns which are on the top. That is, we can remove the vertices (i, j) , with $q + 2 \leq i \leq m - 1$ and $0 \leq j \leq m + 1 - i$, in the following order:

$$\left\{ \begin{array}{l} (m-1, 0), (m-1, 1), (m-1, 2); \\ (m-3, 0), (m-3, 1), \dots, (m-3, 4); \\ \dots \\ (q+3, 0), (q+3, 1), \dots, (q+3, q-1); \\ (q+2, 0), (q+2, 1), \dots, (q+2, q). \end{array} \right.$$

Thus, $\pi_1(A_{2q+1}^*)$ is isomorphic to the fundamental group of the algebra associated to the following ordered quiver:



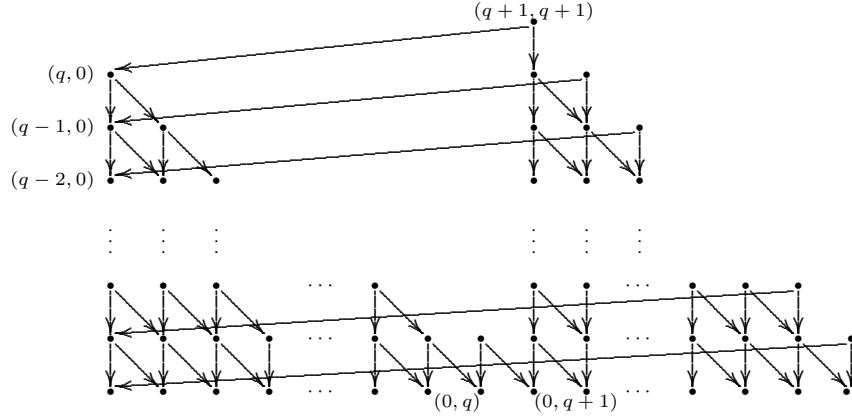
Notice that now the vertices $(q+1, j)$ can also be eliminated, for $0 \leq j \leq q$. However, we cannot delete the vertex $(q+1, q+1)$ because the radical of $P_{(q+1, q+1)}$ is clearly decomposable.

With a similar procedure, we can delete the vertices (i, j) , with $2 \leq i \leq q$ and $q+1-i \leq j \leq q$, in the following order:

$$\left\{ \begin{array}{l} (q, 1), (q, 2), \dots, (q, q); \\ (q-1, 2), (q-1, 3), \dots, (q-1, q); \\ \dots \\ (3, q-2), (3, q-1), (3, q); \\ (2, q-1), (2, q). \end{array} \right.$$

Doing this, we get that the fundamental group of the algebra A_{2q+1}^* coincides with the

fundamental group of the algebra whose associated ordered quiver is the following:

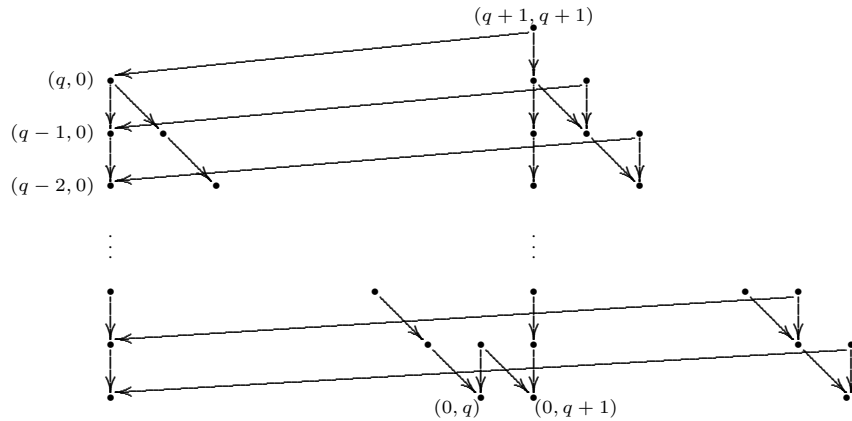


We are now going to use the dual of Theorem 2.3 to remove the parallelism relations. We first consider the vertices $(0, j)$, with $1 \leq j \leq 2q$, $j \neq q$, and $j \neq q+1$. It is clear all these vertices are sinks. Moreover, notice that $\text{supp soc } I_{(0,j)}$ is connected and has no crowns. Then, by Lemma 2.5 and Remark 2.4, the hypotheses of Theorem 2.3 hold and we can eliminate these vertices.

In the same way, we can remove the vertices (i, j) , in the following order:

$$\left\{ \begin{array}{l} (1, 1), (1, 2), \dots, (1, q-2) \text{ and } (1, q+2), (1, q+3), \dots, (1, 2q-1); \\ (2, 1), (2, 2), \dots, (2, q-3) \text{ and } (2, q+2), \dots, (2, 2q-2); \\ \dots \\ (q-3, 1), (q-3, 2) \text{ and } (q-3, q+2), (q-3, q+3); \\ (q-2, 1) \text{ and } (q-2, q+2). \end{array} \right.$$

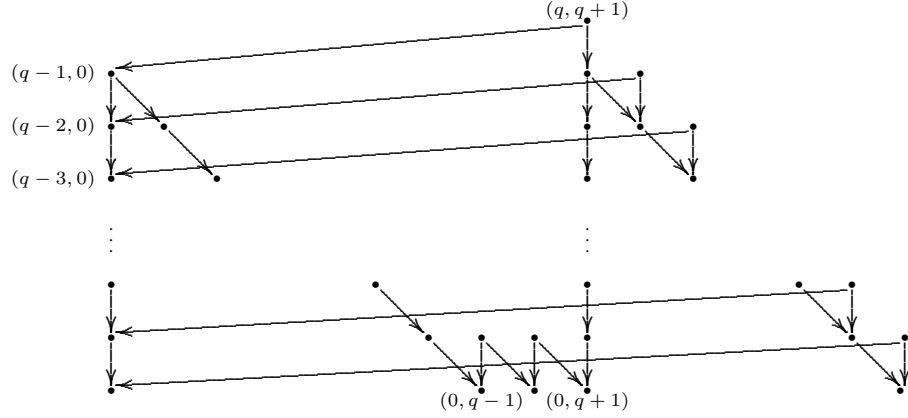
Therefore, we get that $\pi_1(A_{2q+1}^*)$ is isomorphic to the algebra whose associated quiver is the following:



Since this algebra is hereditary we conclude, from Corollary 2.9, that $\pi_1(A_{2q+1}^*)$ is the free group generated by $|Q_1| - |Q_0| + 1 = (6q+4) - (5q+4) + 1 = q+1$ elements.

We can apply the same idea when m is even, that is $m = 2q$. As we have done in the first step of the proof, it is possible to eliminate the first q crowns on the top and then

the vertices appearing in the parallelism relations. Thus, we get that the fundamental group of A_{2q}^* is isomorphic to the fundamental group of the hereditary algebra whose associated quiver is



Then the statement follows directly from Corollary 2.9 since this algebra has $6q + 6$ arrows and $5q + 6$ vertices. $\square$

In the remaining part of this paper, we compute the fundamental group of the incidence algebras whose associated reduced poset are superpositions of crowns of the same width plus a bigger or a smaller crown which lies completely over the stack of crowns. Such algebras have been denoted by $A_{n,m}^{t,t'}$ and $A_{n,m}^r$ respectively.

Theorem 4.4. *Let $A_{n,m}^{t,t'}$, with $n \geq 3$, be the incidence algebra defined in Section 3. Then*

$$\pi_1(A_{n,m}^{t,t'}) = \begin{cases} \mathbb{Z} & \text{if } m \geq n - 1, \\ F_2 & \text{if } 1 \leq m \leq n - 2. \end{cases}$$

Proof. Let $P_{n,m}^{t,t'}$ be the poset associated to the algebra $A_{n,m}^{t,t'}$, that is $P_{n,m}^{t,t'}$ is the superposition of m crowns of width n plus a crown of width $n + t + t'$ where t, t' are not simultaneously zero. We denote by

$$\left\{ \begin{array}{l} \alpha_i^j \text{ the arrow from } (i, j) \text{ to } (i - 1, j), \text{ for } 1 \leq i \leq m \text{ and } 0 \leq j \leq n - 1; \\ \beta_i^j \text{ the arrow from } (i, j) \text{ to } (i - 1, j - 1), \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1; \\ \beta_i^0 \text{ the arrow from } (i, 0) \text{ to } (i - 1, n - 1), \text{ for } 1 \leq i \leq m; \\ \alpha^j \text{ the arrow from } (m + 1, j) \text{ to } (m, j), \text{ for } -t \leq j \leq n - 1 + t'; \\ \gamma^j \text{ the arrow from } (m + 1, j) \text{ to } (m, j - 1), \text{ for } -t + 1 \leq j \leq n - 1 + t'; \\ \gamma \text{ the arrow from } (m + 1, -t) \text{ to } (m, n - 1 + t'). \end{array} \right.$$

First, we consider the case $m \geq n - 1$. We compute $\pi_1(A_{n,m}^{t,t'})$ using the presentation of the fundamental group of an incidence algebra given in Theorem 2.7.

Consider the maximal tree T in $A_{n,m}^{t,t'}$ given by all the arrows α_i^j , α^j , β_1^j , and γ^j , for $-t + 1 \leq j \leq 0$ and $n \leq j \leq n - 1 + t'$. Let F be the free group with the set of arrows as basis, and let K be the normal subgroup of F generated by all the arrows in T and all the walks arising from the commutativity relations in $A_{n,m}^{t,t'}$.

It is easy to prove using an inductive argument that all the arrows β_i^j belong to K since $\alpha_i^j \beta_{i+1}^{j+1} = \beta_i^{j+1} \alpha_{i+1}^{j+1}$, and β_1^j, α_i^j are in K . Moreover, since $m \geq n-1 \geq 2$, we have

$$\beta_1^0 \beta_2^1 \cdots \beta_{m-1}^{n-3} \beta_m^{n-2} \gamma^{n-1} = \alpha_1^{n-1} \alpha_2^{n-1} \cdots \alpha_{m-1}^{n-1} \alpha_m^{n-1} \alpha^{n-1}$$

and therefore β_1^0 belongs to K . Then, using the fact that $\alpha_{i-1}^{n-1} \beta_i^0 = \beta_{i-1}^0 \alpha_i^0$ for all i , and recalling that the arrows α_i^j and β_i^0 are in K , we can show, by induction on (i, j) , that all the arrows β_i^0 are in K . Hence, the only arrow that is not in K is γ and therefore $F/K \simeq \mathbb{Z}$.

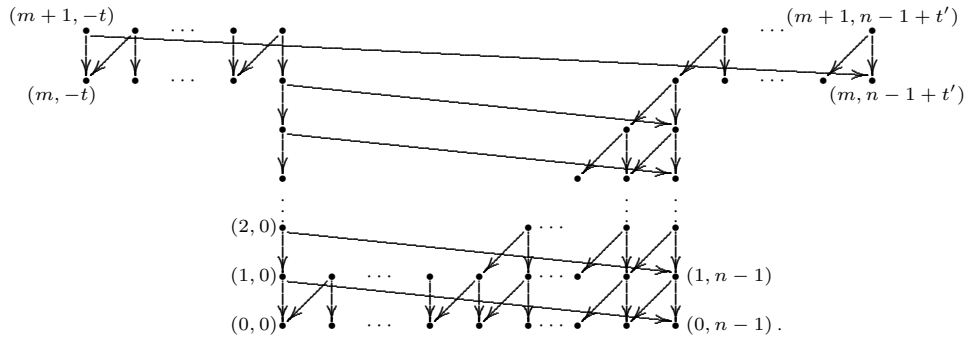
Now, consider the case where $1 \leq m < n-1$. In order to prove that $\pi_1(A_{n,m}^{t,t'}) \simeq F_2$, we remove, by applying Theorem 2.3 in a convenient way, some vertices $(i, j) \in (Q_{n,m}^{t,t'})_0$ until we get a full subcategory of $A_{n,m}^{t,t'}$ whose ordered quiver has no relations and its fundamental group is isomorphic to $\pi_1(A_{n,m}^{t,t'})$. Thus, Corollary 2.9 implies the desired result.

In fact, since $m \leq n-2$, we can see that the sources $(m+1, j)$, with $1 \leq j \leq n-1$, are not tops of any weak crown. Moreover $\text{supp rad } P_{(m+1,j)}$ is connected for all $1 \leq j \leq n-1$, and therefore, by Lemma 2.5, the hypotheses of Theorem 2.3 hold and we can delete all these vertices.

Next, we iterate this procedure to eliminate the vertices (i, j) , for $2 \leq i \leq m$ and $1 \leq j \leq n-m-2+i$, in the following order:

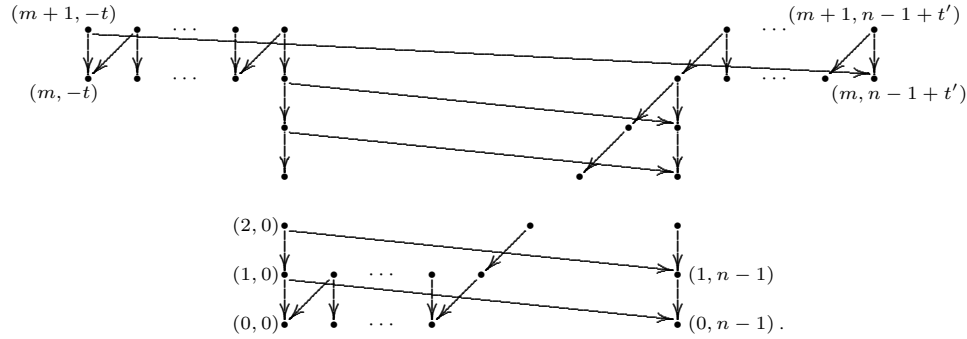
$$\left\{ \begin{array}{l} (m, 1), (m, 2), \dots, (m, n-3), (m, n-2); \\ (m-1, 1), (m-1, 2), \dots, (m, n-3); \\ \dots \\ (3, 1), \dots, (3, n-m), (3, n-m+1); \\ (2, 1), \dots, (2, n-m). \end{array} \right.$$

Thus we obtain that $\pi_1(A_{n,m}^{t,t'})$ is isomorphic to the fundamental group of the incidence algebra whose ordered quiver is the following:



Now, we apply the dual of Theorem 2.3 in order to remove the vertices (i, j) , for $0 \leq i \leq m-1$ and $n-m+i \leq j \leq n-2$. These vertices are no sustain of any weak crown and their supports $\text{supp soc } I_{(i,j)}$ are connected. Thus, $\pi_1(A_{n,m}^{t,t'}) \simeq \pi_1(H)$, where H is

the hereditary algebra corresponding to the following quiver:



The remaining arrows are the following: α_i^0 , for $1 \leq i \leq m$; α_m^{n-1} and α_1^j , for $1 \leq j \leq n - m - 1$; β_1^j , for $1 \leq j \leq n - m$; β_i^j , for $2 \leq i \leq m$ and $j = i + m - n - 1$; α^j , for $-t \leq j \leq 0$ or $n \leq j \leq t' + n - 1$; γ^j , for $n \leq j \leq t' + n - 1$; and γ and β_m^0 .

There are $2(n + t + t' + 1)$ arrows and $2(t + t' + n) + 1$ vertices. Therefore our claim follows from Corollary 2.9. $\square$

Remark 4.5. If $n = 2$, then $\pi_1(A_{2,m}^{t,t'}) \simeq \mathbb{Z}$ since $m \geq 1$.

In the final part of this section we compute the fundamental group of the incidence algebra $A_{n,m}^r$ whose associated poset $P_{n,m}^r$ is the superposition of m crowns of the same width n plus a smaller crown of width r , with $r < n$.

Theorem 4.6. For $n \geq 3$, we have

$$\pi_1(A_{n,m}^r) = \begin{cases} 1 & \text{if } m \geq n, \\ \mathbb{Z} & \text{if } r - 1 \leq m < n, \\ F_2 & \text{if } m < r - 1. \end{cases}$$

Proof. As before, let $A_{n,m}^r$ be the incidence algebra defined in Section 3. Moreover, let

$$\begin{cases} \alpha_i^j \text{ be the arrow from } (i, j) \text{ to } (i - 1, j), \text{ for } 1 \leq i \leq m \text{ and } 0 \leq j \leq n - 1; \\ \beta_i^j \text{ be the arrow from } (i, j) \text{ to } (i - 1, j - 1), \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n - 1; \\ \beta_i^0 \text{ be the arrow from } (i, 0) \text{ to } (i - 1, n - 1), \text{ for } 1 \leq i \leq m. \end{cases}$$

Since the smaller crown lies completely over the stack of crowns, there exists s , with $0 \leq s \leq n - 2$, such that $s + r \leq n$. Hence, we denote by

$$\begin{cases} \alpha^j \text{ the arrow from } (m + 1, j) \text{ to } (m, j), \text{ for } s \leq j \leq s + r - 1; \\ \gamma^j \text{ the arrow from } (m + 1, j) \text{ to } (m, j - 1), \text{ for } s + 1 \leq j \leq s + r - 1; \\ \gamma \text{ the arrow from } (m + 1, s) \text{ to } (m, s + (r - 1)). \end{cases}$$

In order to prove our statement, we will consider three cases:

- (a) $m \geq n$;
- (b) $r - 1 \leq m < n$;
- (c) $m < r - 1$.

In the cases (a) and (b) we shall apply Theorem 2.7 and in the third case we shall use Theorem 2.3 and Corollary 2.9.

(a) Consider the maximal tree in $A_{n,m}^r$ given by all the arrows α_i^j , α^j and β_1^j . Using the same arguments as in Theorem 4.4 with $m \geq n - 1$, it can be proven that since all

the arrows α_i^j , α^j and β_1^j are in the maximal tree, the arrows β_i^j , with $2 \leq i \leq m$ and $1 \leq j \leq n-1$, and γ^j , with $s+1 \leq j \leq s+r-1$, are also in K . Then, using the fact that for all i , $\alpha_{i-1}^{n-1} \beta_i^0 = \beta_{i-1}^0 \alpha_i^0$, it is easy to see that all the arrows β_i^0 , for $2 \leq i \leq n$, are also in K . Moreover, the arrow γ belongs to K since

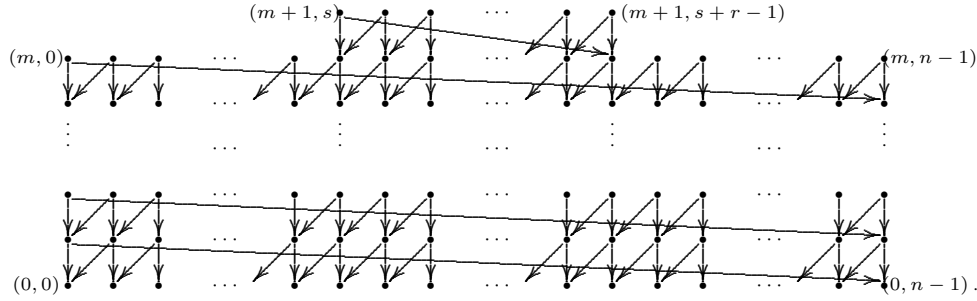
$$\beta_{m-r+1}^{s+1} \cdots \beta_{m-1}^{s+r-2} \beta_m^{s+r-1} \gamma = \alpha_{m-r+1}^s \cdots \alpha_{m-1}^s \alpha_m^s \alpha^s.$$

Therefore the fundamental group of $A_{n,m}^r$ is the trivial group when $m \geq n$.

(b) Keeping in mind the maximal tree considered in case (a), it is easy to see that all the arrows α_i^j , with $1 \leq i \leq m$ and $0 \leq j \leq n-1$; α^j , with $s \leq j \leq s+r-1$; β_i^j , with $1 \leq i \leq m$ and $1 \leq j \leq n-1$; γ^j , with $s+1 \leq j \leq s+r-1$; and γ are in K . But now the arrow β_1^0 does not belong to K since $m < n$. Hence, $\pi_1(A_{n,m}^r)$ is generated by the class $\beta_1^0 \cdot K$ since $\alpha_{i-1}^{n-1} \beta_i^0 = \beta_{i-1}^0 \alpha_i^0$ for all i .

(c) As in Theorem 4.4 with $m < n$, the idea is to remove some convenient vertices until we get a full subcategory H of $A_{n,m}^r$, with H a hereditary algebra, such that $\pi_1(H) \simeq \pi_1(A_{n,m}^r)$.

In fact, $A_{n,m}^r$ is the incidence algebra whose associated quiver is the following:

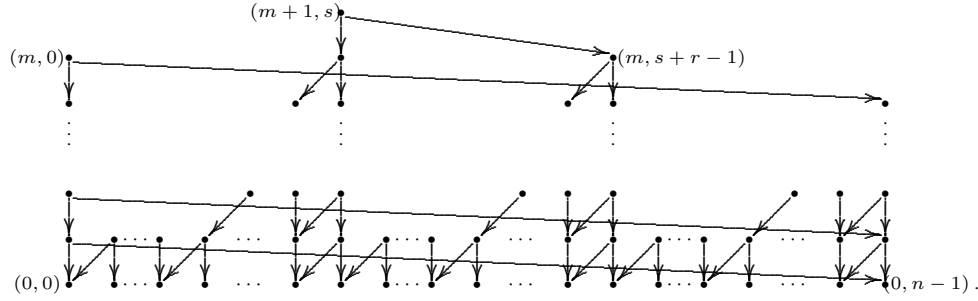


We eliminate first the vertices (i, j) in the following order:

$$\left\{ \begin{array}{l} (m+1, j), \text{ for } s+1 \leq j \leq s+r-1; \\ (m, j), \text{ for } 1 \leq j \leq s-1, s+1 \leq j \leq s+r-2, \text{ and } s+r \leq j \leq n-1; \\ (m-1, j), \text{ for } 1 \leq j \leq s-2, s+1 \leq j \leq s+r-3, \text{ and } s+r \leq j \leq n-1; \\ \dots \\ (2, j), \text{ for } 1 \leq j \leq s-m+1, s+1 \leq j \leq s+r-m, \text{ and } s+r \leq j \leq n-1. \end{array} \right.$$

Notice that in each step, the vertex (i, j) does not top any weak crown, and moreover $\text{supp rad } P_{(i,j)}$ is connected. Therefore, by Lemma 2.5, it is indecomposable. Hence, the hypotheses of Theorem 2.3 hold and $\pi_1(A_{n,m}^r)$ is isomorphic to the fundamental group of the incidence algebra described in Theorem 2.3.

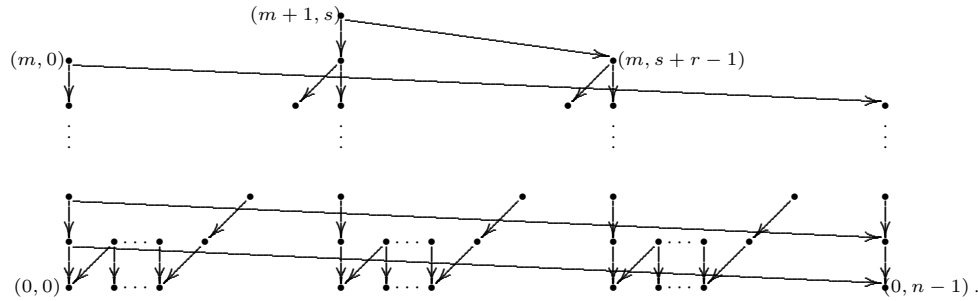
Here it is the incidence algebra having the following ordered quiver:



Finally, we eliminate the vertices (i, j) in the following order:

$$\begin{cases} (0, j), \text{ for } 1 \leq j \leq s-1, s+1 \leq j \leq s+r-2; \\ (1, j), \text{ for } 2 \leq j \leq s-1, s+2 \leq j \leq s+r-2; \\ \dots \\ (m-2, j), \text{ for } m-1 \leq j \leq s-1, s+m-1 \leq j \leq s+r-2. \end{cases}$$

Hence, the fundamental group of $A_{n,m}^r$ is isomorphic to the fundamental group of the incidence algebra H with ordered quiver



The arrows in this quiver are the following: α_i^s , for $1 \leq i \leq m$; α_i^{s+r-1} , for $1 \leq i \leq m$; α^s and β_i^j , for $1 \leq i \leq m$ and $j = s - m + i$; β_i^j , for $1 \leq i \leq m$ and $j = s + r - 1 - m + i$; β_1^0 , α_1^0 and α_1^j , for $s + r \leq j \leq n - 1$; β_1^j , for $s + r \leq j \leq n - 1$; α_1^j , for $s + 1 \leq j \leq s + r - m - 1$; β_1^j , for $s + 1 \leq j \leq s + r - m - 1$ and γ .

There are $2(m + n - s + 1)$ arrows and $2(m + n - s) + 1$ vertices. Therefore our claim follows from Corollary 2.9. $\square$

Acknowledgements. The first author has a fellowship from CONICET and the second author has a fellowship from ANPCyT. This work has been partially financed by PICT 03-08280 ANPCyT and UBACYT X169.

Résumé substantiel en français. Soient $(\Sigma, \leq)$ un ensemble partiellement ordonné fini et k un corps algébriquement clos. On définit l'algèbre d'incidence $A(\Sigma)$ de Σ comme étant la sous-algèbre de l'algèbre de matrices $M_n(k)$ où n est le nombre d'éléments de Σ . Ses éléments sont de la forme (x_{ij}) , avec $x_{ij} = 0$ si $i \leq j$.

Il a été démontré que pour toute algèbre A de dimension finie sur k , il existe un unique carquois Q et au moins un idéal admissible I de kQ , l'algèbre de chemins de Q , tels que A soit isomorphe à kQ/I .

Considérons un sommet x_0 de Q_0 et la plus petite relation d'équivalence $\sim$ sur l'ensemble des marches de Q vérifiant :

- (1) Si $\alpha : x \rightarrow y$ est une flèche de Q , alors $\alpha\alpha^{-1} \sim e_y$ et $\alpha^{-1}\alpha \sim e_x$;
- (2) Si $\sum_{i=1}^m \lambda_i w_i$ est une relation minimale, alors $w_i \sim w_j$ pour tout $1 \leq i, j \leq m$;
- (3) Si $u \sim v$, alors $uwv' \sim wv'w'$ lorsque ces compositions existent.

Alors, l'ensemble quotient de l'ensemble des marches fermées en x_0 par la relation d'équivalence précédente est un groupe ne dépendant pas de x_0 lorsque Q est connexe. Ce groupe est noté $\pi_1(Q, I)$.

D'autre part, étant donné un ensemble partiellement ordonné Σ et le complexe simplicial associé $|\Sigma|$, il existe un isomorphisme entre le groupe fondamental algébrique de Σ et le groupe fondamental topologique de $|\Sigma|$ (voir [8]).

Il y a aussi une relation importante entre le groupe fondamental algébrique et le premier groupe de cohomologie de Hochschild de ce type d'algèbres. Plus précisément,

$$\text{Hom}(\pi_1(Q, I), k^+) \simeq HH^1(A).$$

Cet article est organisé comme suit : dans la première section, nous rappelons quelques propriétés et définitions élémentaires. Les méthodes qui sont utilisées pour calculer le groupe fondamental sont décrites dans la section 2. Enfin, nous utilisons ces résultats pour simplifier les calculs des groupes fondamentaux des algèbres définies à la section 3. Nous démontrons les résultats suivants :

Théorème 4.1. *Pour tout $n \in \mathbb{N}$, $n \geq 2$, nous avons*

$$\pi_1({}^n_t U) = \begin{cases} F_{(n-1)^2} & \text{si } t = 1, \\ 1 & \text{si } t > 1, \end{cases}$$

où ${}^n_t U$ est l'algèbre définie à la section 3 et F_r est le groupe libre sur r générateurs.

Théorème 4.3. *Soit A_m^* , avec $m \geq 3$, l'algèbre définie à la section 3. Soit $m = 2q + r$, avec $q \in \mathbb{N}$ et $0 \leq r \leq 1$. Alors $\pi_1(A_m^*) = F_{q+r}$.*

Théorème 4.4. *Soit $A_{n,m}^{t,t'}$, avec $n \geq 3$, l'algèbre définie à la section 3. Alors*

$$\pi_1(A_{n,m}^{t,t'}) = \begin{cases} \mathbb{Z} & \text{si } m \geq n - 1, \\ F_2 & \text{si } 1 \leq m \leq n - 2. \end{cases}$$

Théorème 4.6. *Soit $A_{n,m}^r$ l'algèbre définie à la section 3. Alors pour $n \geq 3$,*

$$\pi_1(A_{n,m}^r) = \begin{cases} 1 & \text{si } m \geq n, \\ \mathbb{Z} & \text{si } r - 1 \leq m < n, \\ F_2 & \text{si } m < r - 1. \end{cases}$$

REFERENCES

1. I. Assem and J. A. de la Peña, *The fundamental groups of a triangular algebra*, Comm. Algebra **24** (1996), 187–208.
2. I. Assem, M. I. Platzeck, M. J. Redondo and S. Trepode, *Simply connected incidence algebras*, Discrete Math. **269** (2003), 333–355.

3. M. J. Bardzell and E. Marcos, $H^1(\Lambda)$ and presentations of finite dimensional algebras, Representations of algebras (São Paulo, 1999), 31–38, Lecture Notes in Pure and Appl. Math. **224**, Dekker, New York, 2002.
4. J. C. Bustamante, On the fundamental group of a schurian algebra, Comm. Algebra **30** (2002), 5307–5329.
5. M. A. Gatica and A. Rey, Computing the Hochschild cohomology groups of incidence algebras, Comm. Algebra **34** (2006), 2039–2056.
6. E. L. Green, Graphs with relations, coverings and group-graded algebras, Trans. Amer. Math. Soc. **279** (1983), 297–310.
7. R. Martínez-Villa and J. A. de la Peña, The universal cover of a quiver with relations, J. Pure Appl. Algebra **30** (1983), 277–292.
8. E. Reynaud, Le groupe fondamental algébrique. Ph.D. Thesis. Université de Montpellier, 2002.

M. A. GATICA
INSTITUTO DE MATEMÁTICA BAHÍA BLANCA
UNIVERSIDAD NACIONAL DEL SUR
BAHÍA BLANCA
ARGENTINA
agatica@criba.edu.ar

A. A. REY
DEPARTAMENTO DE MATEMÁTICA,
FACULTAD DE CIENCIAS EXACTAS Y NATURALES,
UNIVERSIDAD DE BUENOS AIRES,
BUENOS AIRES, ARGENTINA
aarey@bigua.dm.uba.ar