

HOMOTOPY AND BOUND QUIVERS

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RÉSUMÉ. Le but de cette note est de faire un bref survol sur l'homotopie des carquois liés. S'il est vrai que la motivation principale pour ceci vient de la théorie des représentations des algèbres de dimension finie sur des corps algébriquement clos, le sujet est traité de façon indépendante. Même si plusieurs exemples viennent de ce domaine, la lecture de ces notes n'exige pas une grande connaissance de la théorie des représentations des algèbres mentionnées.

ABSTRACT. In this note we make a brief survey on homotopy of bound quivers. While the main motivation comes from representation theory of finite dimensional algebras over algebraically closed fields, the subject is treated rather independently. Even if many examples come from this area, almost no knowledge about representation theory is needed.

1. Introduction. The aim of these notes is to give a brief account on homotopy of bound quivers. The motivation for this comes mainly from representation theory of finite dimensional algebras over an algebraically closed field $\mathbb{k}$ (in the sequel this will be the meaning of "algebra"). Indeed, every such (basic) algebra can be expressed as a quotient of a path algebra $A \simeq \mathbb{k}Q_A/I$.

Originally, the simple connectedness of a representation finite algebra was defined in terms of the so-called separation condition of [11], and Galois coverings of the corresponding Auslander-Reiten quiver; see [13]. Nowadays, a triangular algebra is called *simply connected* if the fundamental group $\pi_1(Q, I)$ is trivial whenever $A \simeq \mathbb{k}Q/I$. On the one hand, simply connected algebras played an important role in the study of the representation theory of finite representation algebras. By means of covering techniques one can reduce the study of the representations of such an algebra to those over a suitably chosen simply connected algebra (may be of infinite dimension and without unit; see [26]). Moreover, even in the infinite representation case, one can still obtain a considerable amount of information by these means; see [21].

On the other hand, there are links between the fundamental groups $\pi_1(Q, I)$ of presentations of A and the first simplicial homology group of A , denoted by $\mathrm{SH}_1(A)$, as well as its first Hochschild cohomology group, $\mathrm{HH}^1(A)$; see [13, 4, 42]. These links are the best ones one can hope for incidence algebras $A = \mathbb{k}Q_\Sigma/I_\Sigma$ of finite posets $(\Sigma, \leq)$. In this case, the fundamental and the (co)homology groups (both simplicial

and Hochschild) coincide with the fundamental and the (co)homology groups of the simplicial complex of chains of Σ , respectively.

An important feature of the fundamental group, as well as the simplicial (co)homology groups, is that they are not invariants of the algebra: it may occur (see Section 7, for instance) that $\nu_1 : \mathbb{k}Q_A \rightarrow A$ and $\nu_2 : \mathbb{k}Q_A \rightarrow A$ are two presentations of A , so that $A \simeq \mathbb{k}Q_A/\text{Ker } \nu_1 \simeq \mathbb{k}Q_A/\text{Ker } \nu_2$, but $\text{SH}_1(A_{\nu_1}) \not\simeq \text{SH}_1(A_{\nu_2})$, and $\pi_1(Q_A, \text{Ker } \nu_1) \not\simeq \pi_1(Q_A, \text{Ker } \nu_2)$.

This leads to consider bound quivers instead of algebras when dealing with homotopy and simplicial (co)homology groups. Thus, in these notes we focus on bound quivers, specially on homotopical aspects, considering them as an object of study on their own. For this reason, references about algebras, modules or (co)homology theories are brief, and given for historical or motivating purposes.

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These notes are organized as follows:

1. Introduction.
2. Preliminaries.
3. Classifying spaces.
4. Fundamental groups.
5. Coverings.
6. Incidence algebras.
7. Dependence on the presentation.

2. Preliminaries.

2.1. Quivers, algebras, and modules. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are sets, and s, t are maps $s, t : Q_1 \rightarrow Q_0$. The elements of Q_0 and Q_1 are called the *vertices* and the *arrows* of Q , respectively. In addition, given an arrow $\alpha \in Q_1$, the vertex $s(\alpha)$ is its *source*, while $t(\alpha)$ is its *target*. A (non-trivial) path is a sequence of arrows $w = \alpha_1 \alpha_2 \cdots \alpha_r$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for every i such that $1 \leq i < r$. In addition, to every vertex x of Q we associate a *trivial* (or *stationary*) path ε_x . The source and the target of a path are defined in the natural way. Two paths w and w' sharing source and target are said to be *parallel*. If there exists a path from x to y we say that y is a *successor* of x , and that x is a *predecessor* of y . The set of successors of a vertex x is denoted by $x^{+\infty}$, whereas $x^{-\infty}$ stands for its predecessors. In case a vertex has no proper successors (or predecessors) we call it a *sink* (or a *source*, respectively).

Let Q be a quiver, and $\mathbb{k}$ a commutative field. The *path category* $\mathbb{k}Q$ is defined as follows: its object class is Q_0 , and, given $x, y \in Q_0$, the morphism set $\mathbb{k}Q(x, y)$ is the $\mathbb{k}$ -vector space having as basis the set of paths from x to y . The composition $w_1 \circ w_2$ of two morphisms $w_1 \in \mathbb{k}Q(y, z)$ and $w_2 \in \mathbb{k}Q(x, y)$ is given by the composition of paths: $w_1 \circ w_2 = w_2 w_1$. A *relation* from a vertex x to a vertex y is a linear combination $\rho = \sum_{i=1}^r \lambda_i w_i$ of paths from x to y . In case $r = 1$ we say that ρ is a *monomial relation*, whereas relations of the form $w_1 - w_2$ are called *commutativity relations*. Let F be the ideal of $\mathbb{k}Q$ generated by the arrows of Q , that is the ideal whose value $F(x, y)$ at a pair of vertices (x, y) is the $\mathbb{k}$ -vector space generated by the paths of positive length from x to y . A two sided ideal $I \trianglelefteq \mathbb{k}Q$ is said to be *admissible* if for every $x, y \in Q_0$ one has $I(x, y) \subseteq F^2(x, y)$ and for every $x \in Q_0$ there exists a

natural number $n_x \geq 2$ such that I contains every path of length greater than n_x having x as source or target. In this situation, the pair (Q, I) will be called a *bound quiver*. If in addition the quiver Q is *locally finite*, that is the sets $s^{-1}(x)$ and $t^{-1}(x)$ are finite for every vertex $x \in Q_0$, then the category $A = \mathbb{k}Q/I$ is said to be a *locally bounded $\mathbb{k}$ -category*; see [12]. In case the quiver Q is finite (which means that both Q_0 and Q_1 are finite sets), an ideal I is admissible if, and only if there exists an integer $m \geq 2$ such that $F^m \subseteq I \subseteq F^2$. In this case the category $A = \mathbb{k}Q/I$ is a finite dimensional algebra whose unit is $\sum_{x \in Q_0} e_x$, where $e_x = \varepsilon_x + I$ is the idempotent corresponding to the vertex x . We say that a full subcategory A' of A is *convex* if for any sequence of non-zero morphisms $x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_n$ in A , with x_0 and x_n belonging to A' , we have that x_i is in A' for every $i \in \{1, \dots, n-1\}$. In case A' is a full subcategory of A , we have a full embedding $\iota : \text{mod } A' \hookrightarrow \text{mod } A$. Note that in this situation, even if A and A' are algebras (that is if the corresponding quivers are finite) then A' is not a subalgebra of A . Indeed, $1_{A'} \neq 1_A$ in general.

Conversely, let A be a finite dimensional algebra over an algebraically closed field $\mathbb{k}$. It is well-known (see [26]) that if in addition A is basic, then there exists a unique quiver Q_A (which is connected if A is so), and a surjective morphism of algebras $\nu : \mathbb{k}Q_A \longrightarrow A$ such that $I = \text{Ker } \nu$ is an admissible ideal of $\mathbb{k}Q_A$. More precisely, the vertex set $\{1, 2, \dots, n\}$ of Q_A is in bijection with a complete set of primitive pairwise orthogonal idempotents $\{e_1, e_2, \dots, e_n\}$ of A . The set of arrows $i \longrightarrow j$ is in bijection with a basis $\{x_\alpha\}_\alpha$ of the $\mathbb{k}$ -vector space $e_i(\text{rad } A / \text{rad}^2 A)e_j$. In fact, the map $\nu : \mathbb{k}Q_A \longrightarrow A$, which is given by $\alpha \mapsto x_\alpha$, is nothing more than a choice of such bases, thus it is not unique; see Section 7. These surjective morphisms or, equivalently, the pairs $(Q_A, \text{Ker } \nu)$, are called *presentations* of A . In case Q_A has no oriented cycles, the algebra A is said to be a *triangular algebra*.

Let (Q, I) be a bound quiver, and $A = \mathbb{k}Q/I$. A *right A -module* M is a covariant functor $M : A \longrightarrow \text{Mod } \mathbb{k}$ (where $\text{Mod } \mathbb{k}$ stands for the category of $\mathbb{k}$ -vector spaces; see [12]). Such a module is *finitely generated* if it is the quotient of a finite direct sum of representable functors. A *$\mathbb{k}$ -linear representation* N of (Q, I) is the data of a $\mathbb{k}$ -vector space N_x for every vertex $x \in Q_0$, together with linear maps $N_w : N_x \longrightarrow N_y$ for every path $w : x \longrightarrow y$ in Q . In addition these maps must satisfy some compatibility conditions with respect to both, the composition of paths, and the ideal I ; that is $N_{\varepsilon_x} = \text{id}$, $N_{uv} = N_v \circ N_u$ whenever u and v are paths such that $s(v) = t(u)$, and, finally, $\sum_{i=1}^r \lambda_i N_{w_i} = 0$ whenever $\sum_{i=1}^r \lambda_i w_i \in I$. We naturally identify the $\mathbb{k}Q/I$ -modules with the $\mathbb{k}$ -linear representations of (Q, I) . The projective and injective indecomposable $\mathbb{k}Q/I$ -modules corresponding to a vertex x are represented by the functors $P_x(-) = A(x, -)$ and $I_x(-) = \text{Hom}_{\mathbb{k}}(A(-, x), \mathbb{k})$. In particular, $P_x(t)$ is the $\mathbb{k}$ -vector space generated by the set $\{w + I \mid w \text{ is a path } w : x \longrightarrow t\}$; see [8, 6] for further details.

2.2. Simplicial methods. The simplicial (co)homology groups of an algebra of finite representation type were introduced in [13] and played an essential role in the study of the representation theory of algebras; see [13, 37, 10]. For instance, the vanishing of the second cohomology group with coefficients in the multiplicative group $\mathbb{k}^*$ implies the existence of a multiplicative basis of A . Furthermore these groups are related to the Hochschild cohomology groups of A ; see [30, 39, 15].

Recall that any triangular $\mathbb{k}$ -algebra $A \simeq \mathbb{k}Q/I$ of finite representation type is *schurian*, that is $\dim_{\mathbb{k}} A(x, y) \leq 1$ for every pair of vertices x, y in Q . Following Bretscher and Gabriel [13], to a schurian triangular algebra $A \simeq \mathbb{k}Q/I$ we associate a simplicial complex $S_{\bullet}A$, which we call its *classifying space* (the original terminology was *simplicial frame* in [13]) in the following way:

- (1) The vertex set is Q_0 ;
- (2) The n -simplices are given by the $(n+1)$ -tuples $(x_0, x_1, \dots, x_n)$ of distinct vertices of Q such that the composition

$$A(x_{n-1}, x_n) \times A(x_{n-2}, x_{n-1}) \times \cdots \times A(x_0, x_1) \longrightarrow A(x_0, x_n)$$

is non-zero.

The simplicial complex $S_{\bullet}A$ gives rise to a differential complex $(C_{\bullet}A, d_{\bullet})$ in the natural way: $C_n A$ is the free abelian group with basis $S_n A$, and the differentials $d_n : C_n A \rightarrow C_{n-1} A$ are defined on the basis elements by

$$d_n(x_0, x_1, \dots, x_n) = \sum_{i=0}^n (-1)^i (x_0, \dots, \widehat{x}_i, \dots, x_n)$$

where, as usual, $(x_0, \dots, \widehat{x}_i, \dots, x_n)$ is the n -tuple obtained by deleting x_i from $(x_0, x_1, \dots, x_n)$. The *simplicial homology groups* of A are, by definition, the homology groups of $(C_{\bullet}A, d_{\bullet})$, and we denote them by $\mathrm{SH}_i(A)$. Naturally, given an abelian group G , the simplicial cohomology groups of A with coefficients in G are the cohomology groups of $\mathrm{Hom}_{\mathbb{Z}}(C_{\bullet}A, G)$, and we denote them by $\mathrm{SH}^i(A, G)$.

Example 2.2.1. Let $(\Sigma, \leq)$ be a finite partially ordered set (*poset* in the sequel). Without loss of generality, we can assume that $\Sigma = \{1, \dots, n\}$ and that the natural order on $\mathbb{Z}$ extends $\leq$. The *incidence algebra* of Σ with coefficients in a commutative field $\mathbb{k}$, denoted by $\mathbb{k}\Sigma$, is the subalgebra of $M_n(\mathbb{k})$ given by

$$\mathbb{k}\Sigma = \{(a_{i,j})_{(i,j)} \mid a_{i,j} = 0 \text{ if } i \not\leq j\}.$$

We can alternatively describe $\mathbb{k}\Sigma$ by a bound quiver in the following way. Let Q_{Σ} be the quiver having Σ as set of vertices, and whose arrows are given by the covering relations in Σ . More precisely, there is an arrow $\alpha_j^i : i \rightarrow j$ if, and only if, $i > j$ and there is no element $s \in \Sigma$ such that $i > s > j$. Finally, let I_{Σ} be the ideal of $\mathbb{k}Q_{\Sigma}$ generated by $\{w - w' \mid w \text{ and } w' \text{ are parallel paths in } Q_{\Sigma}\}$. With these notations, $\mathbb{k}\Sigma \simeq \mathbb{k}Q_{\Sigma}/I_{\Sigma}$. In this particular case, the i -simplices of $S_{\bullet}\mathbb{k}\Sigma$ are precisely the chains of length $i+1$ of Σ . Thus, the simplicial complex $S_{\bullet}\mathbb{k}\Sigma$ is the complex of chains of Σ ; see [20].

Besides the (co)homology groups, one can consider the homotopy groups of the space $S_{\bullet}A$. Among them, the fundamental group $\pi_1(S_{\bullet}A)$ has proven to be a useful tool in the representation theory of algebras. In the remaining part of these notes, we will be mainly interested in homotopical aspects, so we will not continue our discussion about simplicial (co)homology groups of algebras. However, we refer the reader to [10, 13, 37, 39] for information about these groups.

As a final remark, let us note that although we restricted ourselves to the construction of $S_{\bullet}A$ and the simplicial (co)homology groups of A in case the algebra A is schurian (that is, to the original context in which these groups arose), similar constructions can be made in much more general situations; see [39, 37, 15].

3. Classifying spaces.

3.1. The classifying space of a small category. The simplicial complex of non-empty chains of a poset is a particular case of a more general construction, that of the classifying space of a small category. Indeed, let $\mathcal{C}$ be such a category. The *classifying space* of $\mathcal{C}$, denoted by $\mathcal{BC}$, is the CW-complex whose 0-cells are the objects of $\mathcal{C}$. In addition, for $n \geq 1$, there is one n -cell corresponding to each diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

in $\mathcal{C}$ such that none of the f_i is an identity map. The corresponding n -cell is attached in the natural way to any cell of smaller dimension obtained by deleting one of the objects X_i in the diagram above, for $0 < i < n$, by replacing f_i and f_{i+1} by the composite $f_{i+1}f_i$, whenever this is not an identity map.

In the preceding example, the poset $(\Sigma, \leq)$ can be considered as a small category in a natural way: its object class is precisely the set Σ , and there is a unique morphism $\rho_y^x : x \rightarrow y$ if, and only if, $y \leq x$ and the composition is induced by the transitivity of $\leq$. Then the classifying space of (the small category) Σ is precisely the simplicial complex of the non-empty chains of $(\Sigma, \leq)$.

3.2. Homotopy relations in bound quivers. Let (Q, I) be a bound quiver, and x, y be two vertices of Q . Following [38] (see [33] also), a relation $\rho = \sum_{i=1}^r \lambda_i w_i \in I(x, y)$ (where $\lambda_i \in \mathbb{k}^*$ and the w_i 's are pairwise different paths of length at least two from x to y) is said to be a *minimal relation* if:

- (1) $r \geq 2$;
- (2) For every non-empty proper subset J of $\{1, \dots, r\}$ we have $\sum_{i \in J} \lambda_i w_i \notin I(x, y)$.

Given an arrow $\alpha : x \rightarrow y$ in Q , we denote by α^{-1} its formal inverse of source y and target x . A *walk* from a vertex x to a vertex y is a sequence $w = \alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \cdots \alpha_n^{\epsilon_n}$ of arrows and formal inverses of arrows such that the source of $\alpha_1^{\epsilon_1}$ is x , the target of $\alpha_n^{\epsilon_n}$ is y , and the source of $\alpha_i^{\epsilon_i}$ is the target of $\alpha_{i-1}^{\epsilon_{i-1}}$ for every i such that $1 < i \leq n$. If w and w' are walks in Q such that the source of w' is the target of w , one can consider the walk ww' . This yields a product on the set of walks, which, however, is not everywhere defined. We define the homotopy relation on the set of walks on Q as the smallest equivalence relation $\sim$ satisfying:

- (1) For each arrow $\alpha : x \rightarrow y$, one has $\alpha\alpha^{-1} \sim \varepsilon_x$ and $\alpha^{-1}\alpha \sim \varepsilon_y$;
- (2) For each minimal relation $\sum_{i=1}^m \lambda_i w_i$, one has $w_i \sim w_j$ for every i, j in $\{1, \dots, m\}$;
- (3) If u, v, w and w' are walks, and $u \sim v$, then $wuw' \sim vvw'$, whenever these compositions are defined.

We denote by $[w]$ the homotopy class of a walk w .

A closely related notion is that of natural homotopy of paths. Two parallel paths w and w' are said to be *naturally homotopic* (compare with the original definition in [3], p. 465) if either $w = w'$ or there exists a sequence of paths $w = w_0, w_1, \dots, w_s = w'$, such that for $i \in \{1, \dots, s\}$ there exist paths u_i, p_i, p'_i and v_i satisfying:

- (1) p_i and p'_i appear in a same minimal relation;
- (2) $w_i = u_i p_i v_i$ and $w_{i-1} = u_i p'_i v_i$.

In this case, we write $w \approx w'$, and $[w]$ will denote the natural homotopy class of w (of course, the natural homotopy is an equivalence relation).

Example 3.2.1. Consider the quiver Q :

$$\begin{array}{ccccc} & & \beta_1 & & \\ & & \curvearrowright & & \\ 1 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad \alpha \quad} & 3 \\ & & \beta_2 & & \end{array}$$

bound by the ideal $I = \langle \beta_1 \alpha - \beta_2 \alpha \rangle$. The arrows β_1 and β_2 are homotopic, but not naturally homotopic.

The following proposition gives an alternative description of the natural homotopy relation.

Proposition 3.2.2. *Let (Q, I) be a bound quiver. The natural homotopy relation is the smallest equivalence relation $\sim$ on the set of paths in Q verifying the following conditions :*

- (1) *For each minimal relation $\sum_{i=1}^m \lambda_i w_i$, one has $w_i \sim w_j$ for every i, j in $\{1, \dots, m\}$.*
- (2) *If u, v, w and w' are paths, and $u \sim v$ then $www' \sim vww'$, whenever these compositions are defined.*

Proof. Let $\sim$ be the smallest equivalence relation satisfying conditions (1) and (2) above. It is clear that these conditions are satisfied by $\approx$, so $w \sim w'$ implies $w \approx w'$. Conversely, assume that $w \approx w'$, with $w \neq w'$. Then, by definition, there exists a sequence of paths $w = w_0, w_1, \dots, w_s = w'$ and, for $i \in \{1, \dots, s\}$, there exist paths u_i, p_i, p'_i and v_i with p_i and p'_i appearing in the same minimal relation, and such that $w_i = u_i p_i v_i$ and $w_{i-1} = u_i p'_i v_i$. It follows from (1) that $p_i \sim p'_i$, for every i such that $1 \leq i \leq s$. It now follows from (2) that $u_i p_i v_i \sim u_i p'_i v_i$ and this yields $w \sim w'$. $\square$

Remark 3.2.3. Since the ideal I is assumed to be admissible, the natural homotopy class of an arrow α is reduced to $\{\alpha\}$. This does not hold for homotopy classes of arrows as Example 3.2.1 shows.

It is easily seen that if w and w' are two naturally homotopic paths, then they are homotopic, but the converse does not hold in general. However, as the following proposition shows, these relations may coincide.

Proposition 3.2.4. *Let $A = \mathbb{k}Q/I$ be a schurian algebra. Then the relations $\sim$ and $\approx$ coincide.*

Proof. (Sketch) The crucial point is that since A is schurian, the only minimal relations that one needs to completely determine homotopy relations are of the form $\lambda_1 w_1 + \lambda_2 w_2$, where w_1 and w_2 are non-zero paths sharing no arrows. $\square$

3.3. The classifying space of a bound quiver. We wish to build a CW-complex by successively attaching n -cells to a previously built $(n-1)$ -dimensional CW-complex. We begin by describing the sets $\mathcal{C}_n$ of n -dimensional cells:

- (1) $\mathcal{C}_0 = Q_0$;
- (2) $\mathcal{C}_1 = \{[\sigma] \mid \sigma : x \longrightarrow y \text{ is a path in } Q \text{ with } \sigma \notin I \text{ and } \sigma \neq \varepsilon_x, \varepsilon_y\}$;
- (3) For $n \geq 2$, $\mathcal{C}_n = \{([\sigma_1], \dots, [\sigma_n]) \mid \sigma_1 \sigma_2 \cdots \sigma_n \text{ is a path in } Q, \sigma_1 \cdots \sigma_n \notin I \text{ and } \sigma_i \text{ is not a stationary path}\}$.

Let $[\sigma] \in \mathcal{C}_1$, with $\sigma : x \longrightarrow y$. We attach the 1-cell $[\sigma]$ to the 0-cells corresponding to x and y in the natural way. Moreover, each n -dimensional cell $([\sigma_1], \dots, [\sigma_n])$ is attached in the obvious way to the $(n+1)$ -cells of dimension $n-1$ corresponding to $([\sigma_2], \dots, [\sigma_n]), \dots, ([\sigma_1], \dots, [\sigma_{i\sigma_{i+1}}], \dots, [\sigma_n]), \dots$ and $([\sigma_1], \dots, [\sigma_{n-1}])$. We refer the reader to Section 2.2 in [15] for a precise description of the construction.

Note that since I is an admissible ideal, there are only finitely many $k \in \mathbb{N}$ such that $\mathcal{C}_k \neq \emptyset$.

Definition 3.3.1. Let (Q, I) be a bound quiver. The CW-complex obtained by the preceding construction is the *classifying space* of (Q, I) , and is denoted by $\mathcal{B}(Q, I)$.

Remark 3.3.2. A slightly different approach consists in considering homotopy classes of paths, instead of natural homotopy classes to attach the cells. The complex obtained in this way will be denoted $\mathcal{B}^\sharp(Q, I)$, and called the *total classifying space* of (Q, I) . It may seem somehow artificial to consider *natural* homotopy classes instead of homotopy ones. However, in view of (co)homological computations, this is the right context. It is shown in [15] that if $A = \mathbb{k}Q/I$ has a semi-normed basis (see [39]) and $\mathrm{SH}^i(A, \mathbb{k}^+)$ are its simplicial cohomology groups, then $H^i(\mathcal{B}(Q, I)) \simeq \mathrm{SH}^i(A, \mathbb{k}^+)$ (see [15, (5.3)-(5.5)]).

Examples 3.3.3.

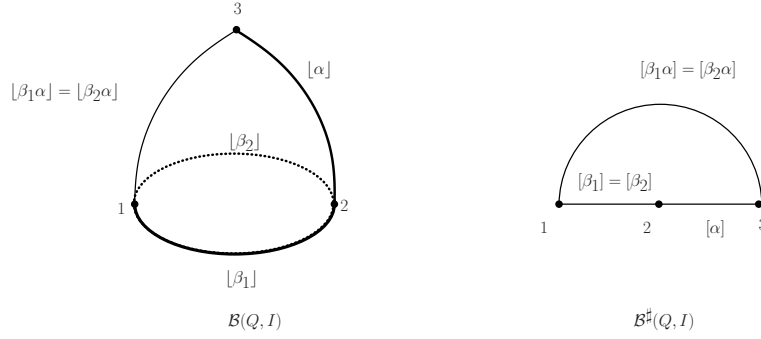
(1) Let $A = \mathbb{k}Q/I$ be a schurian algebra. As noted in the Proposition 3.2.4, in this case homotopy and natural homotopy coincide, thus $\mathcal{B}(Q, I) = \mathcal{B}^\sharp(Q, I)$. For $x, y \in Q_0$, there is a 1-cell joining them if and only if there is a non-zero path w from, say, x to y . Moreover, if there is another such path w' , then, since A is schurian, one has $w \approx w'$. Thus different paths give the same 1-cell. This 1-cell can be identified with the pair (x, y) . In a similar way, an n -cell corresponding to $([\sigma_1], \dots, [\sigma_n])$, with $\sigma_i \in A(x_{i-1}, x_i)$, is identified with the sequence of $n+1$ vertices $x_0, x_1, \dots, x_n$ in Q . This shows that for schurian algebras $A = \mathbb{k}Q/I$, the CW-complex $\mathcal{B}(Q, I)$ is precisely the simplicial complex $S_\bullet A$ of Section 2.2, above; see also [39].

(2) Let Q be a quiver, and I be the ideal of $\mathbb{k}Q$ generated by paths of length 2. In this case there are no minimal relations, so homotopy and natural homotopy are trivial relations and $\mathcal{B}(Q, I) = \mathcal{B}^\sharp(Q, I)$. Each arrow $\alpha : x \longrightarrow y$ of Q gives rise to a 1-cell. Moreover, since the only non-zero paths are the arrows, these are the only cells. For the same reason, there are no higher dimensional cells, so the space $\mathcal{B}(Q, I)$ is homeomorphic to $\overline{Q}$, the underlying graph of Q .

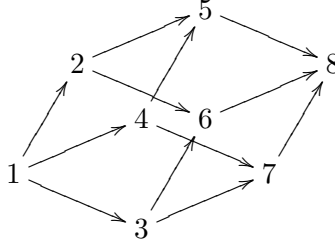
(3) The preceding example is a particular case of a more general result; see [15]. If $A = \mathbb{k}Q/I$ is a triangular algebra, with I generated by a set of paths, then there are no minimal relations, so the homotopy relations are trivial. A consequence of this is that in this case the graph $\overline{Q}$ is a strong deformation retract of $\mathcal{B}(Q, I)$.

(4) Consider the bound quiver of Example 3.2.1. On the one hand, the space $\mathcal{B}(Q, I)$ has two 2-cells, namely $([\beta_1], [\alpha])$ and $([\beta_2], [\alpha])$, sharing the 1-cells $[\alpha]$ and $[\beta_1\alpha] = [\beta_2\alpha]$ in their boundary. On the other hand, the space $\mathcal{B}^\sharp(Q, I)$ has only one 2-cell: $([\beta_1], [\alpha]) = ([\beta_2], [\alpha])$.

The spaces look as follows:



Consider the quiver



bound by the ideal generated by the commutativity relations and all the paths of length 3. The space $\mathcal{B}(Q, I)$ is homeomorphic to the 3-dimensional sphere S^2 . The commutativity relations tell how to "glue" the 2-dimensional cells. The existence of monomial relations implies that there are no 3-cells to "fill the hole".

Let (Q, I') be the same quiver bound by the commutativity relations (and only these). In particular, the natural homotopy relations induced by I and I' are the same. However, the space $\mathcal{B}(Q, I')$ is homeomorphic to B^3 , hence does not have the same homotopy type as $\mathcal{B}(Q, I)$. This shows that the monomial relations play an important role in the construction of $\mathcal{B}(Q, I)$, even though they are not taken into account to define the homotopy relations.

3.4. Some remarks. Before going into a deeper study of $\mathcal{B}(Q, I)$, we give a short list of general immediate remarks.

First of all, it is important to note that calling $\mathcal{B}(Q, I)$ a classifying space is an abuse of terminology. Indeed, $\mathcal{B}(Q, I)$ is not the classifying space of any (reasonable) category, in the sense of Section 3.1. Let $\mathcal{P}(Q)$ be the path category of Q (denoted $\mathbb{k}Q$ in Section 2.1). Its object class is precisely Q_0 , and the morphisms are the paths in Q . Moreover, let $\mathcal{P}(Q, I) = \mathcal{P}(Q)/\approx$ be the natural homotopy category of (Q, I) . One could think that $\mathcal{B}(Q, I)$ is the classifying space of $\mathcal{P}(Q, I)$. However, this is not the case. In general, $\mathcal{B}(Q, I)$ is a subcomplex of the classifying space of $\mathcal{P}(Q, I)$. The cells of the latter are in bijection with n -tuples of composable (classes of) morphisms $([\sigma_1], [\sigma_2], \dots, [\sigma_n])$ regardless whether their composition is zero or not. In fact, "zero" does not make any sense in the category $\mathcal{P}(Q, I)$, which is not linear.

In case the algebra $A = \mathbb{k}Q/I$ is triangular, then the boundary of any n -cell is homeomorphic to $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$, thus the complex $\mathcal{B}(Q, I)$ is a regular CW-complex. This holds because every path in Q passes at most once at any given vertex of Q .

As we noted in Remark 3.2.3, the natural homotopy class of an arrow α is reduced to $\{\alpha\}$. In this way we can identify the underlying graph of Q to a 1-dimensional subcomplex of $\mathcal{B}(Q, I)$. Note that this does not hold for $\mathcal{B}^\sharp(Q, I)$; see Example 3.2.1.

The (classical) construction of "classifying space" defines a functor from the category of small categories and functors to the category of CW-complexes and cellular maps. It is then natural to ask if the same holds for our construction. Recall that a *morphism of quivers* $f : Q \rightarrow Q'$ is given by a pair of maps $f_i : Q_i \rightarrow Q'_i$, for $i \in \{1, 2\}$, such that every arrow $\alpha : x \rightarrow y$ in Q induces an arrow $f_1(\alpha) : f_0(x) \rightarrow f_0(y)$ in Q' . Such a morphism induces in a natural way a $\mathbb{k}$ -linear map $f : \mathbb{k}Q \rightarrow \mathbb{k}Q'$. A *morphism of bound quivers* $f : (Q, I) \rightarrow (Q', I')$ is a morphism of quivers such that $f(I) \subseteq I'$. The following example shows that our construction is not functorial.

Example 3.4.1. Consider the Example (5) in 3.3.3. The identity map on Q induces a morphism of bound quivers $p : (Q, I') \rightarrow (Q, I)$. As we saw, $\mathcal{B}(Q, I') \simeq B^3$ which is contractible, and $\mathcal{B}(Q, I) \simeq S^2$, so there is no cellular map $p : \mathcal{B}(Q, I') \rightarrow \mathcal{B}(Q, I)$ besides the trivial one.

4. Fundamental groups.

4.1. The fundamental group of a bound quiver. Let (Q, I) be a bound quiver with Q connected, and fix a vertex $x_0 \in Q_0$. Let $\pi_1(Q, x_0)$ be the fundamental group of the underlying graph of Q at the vertex x_0 , which is isomorphic to the free group in $\chi(Q) = |Q_1| - |Q_0| + 1$ generators; see [46]. Moreover, let $N(Q, I, x_0)$ be the normal subgroup of $\pi_1(Q, x_0)$ generated by all the elements of the form $wuv^{-1}w^{-1}$ where w is a walk from x_0 to, say x , and u, v are two homotopic paths starting at x .

Definition 4.1.1. Let (Q, I) be a connected bound quiver. The *fundamental group* is defined to be $\pi_1(Q, I) = \pi_1(Q, x_0)/N(Q, I, x_0)$.

Since the quiver Q is assumed to be connected, this definition does not depend on the base point x_0 . However the definition of $\pi_1(Q, I)$ heavily depends on the minimal relations, which are given by the ideal I . Thus one can have $\mathbb{k}Q/I_1 \simeq \mathbb{k}Q/I_2$ but $\pi_1(Q, I_1) \not\simeq \pi_1(Q, I_2)$. We will discuss this point in Section 7.

At this point it is important to note that one could have chosen naturally homotopic paths in the definition of $N(Q, I, x_0)$ as well. Indeed, since in a group every element has an inverse, condition (1) in the definition of the homotopy relation (see Section 3.2) is automatically satisfied.

Following [7], a triangular algebra is called *simply connected* if for every bound quiver (Q, I) such that $A \simeq \mathbb{k}Q/I$ the fundamental group $\pi_1(Q, I)$ is trivial. It was shown in [48] that a triangular algebra A is simply connected if, and only if, it does not admit a proper Galois covering; see [38, 12, 26] or Section 5 for definition. This definition of simple connectedness extends to the infinite representation case those of [12] and of [11]: in [12], a representation finite algebra is called simply connected if its Auslander Reiten quiver, Γ_A is simply connected. Following [11], an A -module M is said to be *separated* if its restriction to each connected component of A is 0 or an indecomposable module. An algebra A is said to satisfy the *separation condition* if for every $x \in A_0$, the module $\text{rad } P_x$ is separated as a module over the full subcategory of A generated by the non-predecessors of x . It was established in [48] that if $A = \mathbb{k}Q/I$

is a triangular algebra which is separated, then it is simply connected. Moreover, if in addition A is of finite representation type, the converse also holds; see [11].

The definition of $\pi_1(Q, I)$ given above makes it difficult in practice to determine this group. It is thus interesting to obtain computational tools to do so. The following algorithm, due to E. Reynaud [44], reduces the computation of $\pi_1(Q, I)$ to the computation of the fundamental group of a quiver with relations (Q', I') having one vertex less than Q , but with I' not necessarily admissible. By successively repeating this process, one gets a convenient presentation of $\pi_1(Q, I)$.

Algorithm 4.1.2. [44] Let (Q, I) be a bound quiver having at least two vertices.

- (1) Choose an arrow $\alpha : x_1 \rightarrow x_2$ in Q .
- (2) Let Q' be the quiver obtained from Q by identifying the vertices x_1 and x_2 to a new vertex x . The arrows of Q' are those of Q except α . Note that an arrow parallel to α in Q becomes a loop in Q' .
- (3) Consider the map $p : \{\text{paths in } Q\} \rightarrow \{\text{paths in } Q'\}$ defined on the vertices and the arrows of Q (and extended in the obvious way) by

$$p(\varepsilon_y) = \begin{cases} \varepsilon_x & \text{if } y = x_1, \text{ or } x_2 \\ \varepsilon_y & \text{otherwise.} \end{cases} \quad \text{and} \quad p(\gamma) = \begin{cases} \varepsilon_x & \text{if } \gamma = \alpha \\ \gamma & \text{otherwise.} \end{cases}$$

This induces a linear map $p : \mathbb{k}Q \rightarrow \mathbb{k}Q'$ (which is not a morphism of algebras). Consider the ideal $I' = p(I)$, which may fail to be admissible. The groups $\pi_1(Q, I)$ and $\pi_1(Q', I')$ are isomorphic

- (4) If $|Q'_0| > 2$, we can start again. Otherwise the algorithm stops.

Example 4.1.3. Let Q be the quiver

$$\begin{array}{ccccc} & \alpha_1 & & \alpha_2 & \\ 3 & \xrightarrow{\quad} & 2 & \xrightarrow{\quad} & 1 \\ & \beta_1 & & \beta_2 & \end{array} \quad \text{bound by} \quad I = \langle \alpha_1 \alpha_2 - \beta_1 \beta_2, \alpha_1 \beta_2 - \beta_1 \alpha_2 \rangle.$$

For the first iteration choose the arrow α_1 , so we obtain the quiver

$$\begin{array}{ccc} & \alpha_2 & \\ \beta_1 \curvearrowright 2 & \xrightarrow{\quad} & 1 \\ & \beta_2 & \end{array} \quad \text{bound by} \quad I_1 = \langle \alpha_2 - \beta_1 \beta_2, \beta_1 \alpha_2 - \beta_2 \rangle.$$

For the second iteration choose the arrow α_2 , so we obtain the quiver

$$\beta_1 \curvearrowright 2 \curvearrowleft \beta_2 \quad \text{bound by} \quad I_2 = \langle \varepsilon_2 - \beta_1 \beta_2, \beta_1 - \beta_2 \rangle.$$

Since the obtained quiver has only one vertex, we have finished. The first relation in I_2 gives $[\beta_1]^{-1} = [\beta_2]$ while the second gives $[\beta_1] = [\beta_2]$ and this yields $\pi_1(Q, I) \simeq \mathbb{Z}_2$.

4.2. The fundamental group of $\mathcal{B}(Q, I)$. In this section we give a precise description of $\pi_1(\mathcal{B}(Q, I))$, and we see that it is isomorphic to $\pi_1(Q, I)$. This result generalizes similar ones obtained independently in case $A = \mathbb{k}Q/I$ is an incidence algebra in [43], or a schurian triangular algebra in [14].

We begin by briefly recalling a convenient way to compute the fundamental group of a connected CW-complex X . Fix a 0-cell x_0 . Then, by definition, the fundamental group of X is $\pi_1(X) = \{f : S^1 \rightarrow X \mid f \text{ continuous, } f(1, 0) = x_0\} / \sim$, where $\sim$ is the usual homotopy relation, relatively to x_0 , of maps of topological spaces; see [46]. In order to give a presentation of $\pi_1(X)$, fix a maximal tree M in X , that is a subcomplex

of dimension smaller than or equal to 1, which is acyclic and maximal for this property. Note that the maximality of M implies its connectedness. For every 2-cell e_λ^2 of X , let α_e be a path class which starts at some fixed 0-cell x in its boundary ∂e_λ^2 , and goes around it exactly once. Moreover, let β_x be the unique path class in X which goes from x to x_0 along M . Finally set $\gamma_e = \beta_x \alpha_e \beta_x^{-1}$ (compare with the definition of a *parade data* in [24]). Let G be the free group on the set of 1-cells of X , and N be the normal subgroup of G generated by:

- (1) The cells appearing in M ;
- (2) The elements γ_e constructed above.

With these notations, we get $\pi_1(X) \simeq G/N$; see [40, p. 213].

Remark 4.2.1. In case X is a graph, there are no 2-cells, so there are no elements γ_e . Moreover, in this situation M has exactly $|X_0| - 1$ one-dimensional cells, and we obtain that $\pi_1(X)$ is the free group in $|X_1| - |X_0| + 1$ generators; compare with Section 4.1.

We now turn our attention to the complex $\mathcal{B} = \mathcal{B}(Q, I)$. In this case, the 2-cells are given by pairs $e_\sigma^2 = ([\sigma_1], [\sigma_2])$ such that $\sigma_1 \sigma_2 \notin I$. Moreover, the boundary of such a cell is the union of the 1-cells: $[\sigma_1]$, $[\sigma_1 \sigma_2]$, and $[\sigma_2]$. This allows us to improve the presentation of $\pi_1(\mathcal{B})$.

Lemma 4.2.2. *Let (Q, I) be a bound quiver, and T be a maximal tree in Q . Then $\pi_1(\mathcal{B}) \simeq F/K$, where F is the free group with basis the set of arrows of Q , and K is the normal subgroup generated by the elements of the following two forms:*

- (1) α , for every arrow α in T ;
- (2) $(\alpha_1 \alpha_2 \cdots \alpha_r)(\beta_1 \beta_2 \cdots \beta_s)^{-1}$, whenever $\alpha_1 \alpha_2 \cdots \alpha_r$ and $\beta_1 \beta_2 \cdots \beta_s$ are two paths appearing in the same minimal relation.

Proof. (Sketch) Let T be a maximal tree in the quiver Q . In particular T is a set of arrows of Q . It follows from the construction of $\mathcal{B}(Q, I)$ that an arrow $\alpha : x \rightarrow y$ in T gives rise to a 1-cell $[\alpha]$ in $\mathcal{B}(Q, I)$. The set of 1-cells obtained from the arrows of T forms a maximal tree M in $\mathcal{B}(Q, I)$. Moreover, let G and N be as before. Define maps $\phi : F \rightarrow G/K$, and $\psi : G \rightarrow F/N$ in the natural way, and show that this gives group homomorphisms $\Phi : F/K \rightarrow G/N$ and $\Psi : G/N \rightarrow F/K$ which are mutually inverses. $\square$

This leads us to the following result.

Theorem 4.2.3. *Let (Q, I) be a bound quiver. Then the groups $\pi_1(\mathcal{B})$ and $\pi_1(Q, I)$ are isomorphic.*

Proof. This follows immediately from the description of $\pi_1(Q, I)$ done in Section 4.1, Remark 4.2.1 and the preceding lemma. $\square$

As noted before, the fundamental group is not an invariant of the algebra. However, in case $A \simeq \mathbb{k}Q/I$ is constricted (this includes all schurian triangular algebras, thus all triangular representation finite algebras) then $\pi_1(Q, I_1) \simeq \pi_1(Q, I_2)$ whenever $\mathbb{k}Q/I_1 \simeq \mathbb{k}Q/I_2$; see [9], or Section 7.

Corollary 4.2.4. *Let $A \simeq \mathbb{k}Q/I$ be a triangular representation finite algebra, and Γ_A be its Auslander-Reiten quiver. Then $\pi_1(\Gamma_A) \simeq \pi_1(\mathcal{B})$.*

Proof. This follows directly from [38, (3.9)(4.3)] and Theorem 4.2.3. $\square$

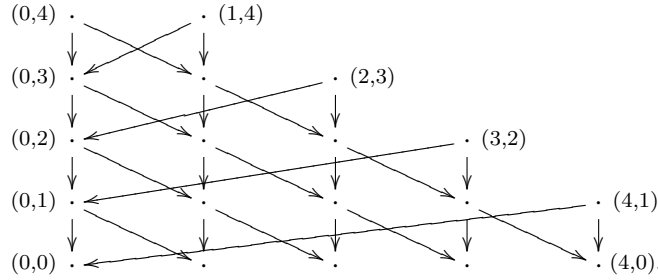
Another immediate remark is that the preceding theorem, together with Lemma 4.2.2 gives a presentation of $\pi_1(Q, I)$; see also [47].

Examples 4.2.5.

(1) Let m be an integer, with $m \geq 3$, and set $[m] = \{0, 1, \dots, m\}$. Following [29], consider the set

$$\Sigma_m^* = \{(i, j) \in [m] \times [m] \mid 0 \leq i \leq m+1-j \text{ if } j \neq 0, \text{ or } 0 \leq i \leq m \text{ if } j = 0\}$$

ordered by $(i, j) < (i, j+1)$, and $(i, j) < (i-1, j+1)$ where we identify $(-1, j+1)$ to $(m+1-j, j+1)$. We associate a quiver $Q_{(m)}$ to the poset Σ_m^* , in the way described in Example 2.2.1. For instance, the quiver $Q_{(4)}$ corresponding to Σ_4^* looks as follows:



Let I be the ideal of $\mathbb{k}Q_{(m)}$ generated by the differences of parallel paths, and write $m = 2q + r$ with $r \in \{0, 1\}$. We claim that $\pi_1(Q_{(m)}, I)$ is the free group in $q + r$ generators. Given $(i, j) \in \Sigma_m^*$, denote by α_j^i the arrow going from (i, j) to $(i, j-1)$. The other arrow starting at (i, j) is denoted by β_j^i . A maximal tree in Q is given by the arrows $\{\alpha_j^i\} \cup \{\beta_{m-i}^i\}$. From the minimal relation $\alpha_m^0 \beta_{m-1}^0 - \beta_m^0 \alpha_{m-1}^1$ we obtain $\beta_{m-1}^0 \in K$. Similarly, we obtain $\beta_j^i \in K$ for every arrow β_j^i with $(i, j) \neq (i, m-i+1)$. Now, since $m \geq 3$, we have a minimal relation $\beta_m^1 \beta_{m-1}^0 - \alpha_m^1 \alpha_{m-1}^1$, and this leads to $\beta_m^1 \in K$. Similarly, if $m \geq 4$ we have a minimal relation:

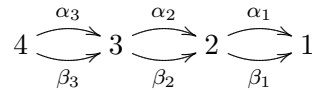
$$\beta_{m-1}^2 \beta_{m-2}^0 \beta_{m-3}^1 - \alpha_{m-1}^2 \alpha_{m-2}^2 \alpha_{m-3}^2,$$

hence $\beta_{m-1}^2 \in K$. More generally, in case $i \leq m/2$, we have $m - 2i + 1 \geq 1$, and this gives a minimal relation:

$$\beta_{m-i+1}^i \beta_{m-i}^0 \beta_{m-i-1}^1 \cdots \beta_{m-2i+1}^{i-1} - \alpha_{m-i+1}^i \alpha_{m-i}^i \cdots \alpha_{m-2i+1}^i,$$

from which we obtain $\beta_{m-i+1}^i \in K$ for $i \in \{1, \dots, q\}$. Since the remaining arrows β_j^i do not appear in any relation, they are free, so $\pi_1(Q_{(m)}, I)$ is the free group on these arrows. There are $m - q = 2q + r - q = q + r$ of them.

(2) Consider the quiver



bound by the ideal $I = \langle \alpha_3 \beta_2 - \beta_3 \alpha_2, \alpha_2 \beta_1 - \beta_2 \alpha_1, \beta_3 \beta_2 \beta_1 - \alpha_3 \alpha_2 \alpha_1 \rangle$. The arrows α_1, α_2 and α_3 form a maximal tree in Q . From the relation $\alpha_3 \beta_2 - \beta_3 \alpha_2$ we obtain $\beta_1 K = \beta_2 K$ and, similarly, $\beta_2 K = \beta_3 K$. Finally, the relation $\beta_3 \beta_2 \beta_1 - \alpha_3 \alpha_2 \alpha_1$ gives $(\beta_1 K)^3 = K$. Hence $\pi_1(Q, I) \simeq \mathbb{Z}_3$.

4.3. Some applications. The results in the previous section allow us to obtain some tools for computations involving fundamental groups of bound quivers. On the one hand, we can use the presentation given in Lemma 4.2.2, and on the other we can use results from algebraic topology.

Let $A = \mathbb{k}Q/I$ be a triangular algebra, x be source in Q , and $B = A/(Ae_xA)$. Letting $M = \text{rad}P_x$, which is a right B -module, we can write A in matrix form

$$A = \begin{bmatrix} B & 0 \\ {}_{\mathbb{k}}M_B & \mathbb{k} \end{bmatrix}$$

with the usual matrix operations (the multiplication being induced from the module structure of M). We say that A is a *one-point extension* of B , and write $A = B[M]$. Let (Q_A, I_{ν_A}) be a presentation of A , and (Q_B, I_{ν_B}) be the induced presentation of B . Let $Q_B^{(1)}, \dots, Q_B^{(c)}$ be the connected components of Q_B , and $I_{\nu}^{(j)} = I_{\nu_B} \cap kQ_B^{(j)}$, for $1 \leq j \leq c$. For each j , the embedding of $Q_B^{(j)}$ in Q_A induces a group homomorphism $\eta_j : \pi_1(Q_B^{(j)}, I_{\nu}^{(j)}) \longrightarrow \pi_1(Q_A, I_{\nu_A})$. We compute the cokernel of the induced homomorphism (compare with [1]):

$$\eta = (\eta_j)_j : \coprod_{j=1}^c \pi_1(Q_B^{(j)}, I_{\nu}^{(j)}) \longrightarrow \pi_1(Q_A, I_{\nu_A}).$$

As in [4], given a vertex x in a quiver Q , denote by $x^\rightarrow$ the set of arrows leaving x . The set $^\rightarrow x$ is defined analogously. Following [4, (2.1)], let $\sim$ be the least equivalence relation on $x^\rightarrow$ such that $\alpha \sim \beta$ whenever there exists a minimal relation $\sum_{i=1}^r \lambda_i w_i$ such that $w_1 = \alpha w'_1$ and $w_2 = \beta w'_2$. We denote by $[\alpha]_{\nu_A}$ the equivalence class of α , and by $t = t(\nu_A)$ the number of equivalence classes $[\alpha]_{\nu_A}$ in $x^\rightarrow$. Finally, we denote by $s = s(x)$ the number of indecomposable direct summands of M . It is shown in [4, (2.2)] that $c \leq t \leq s$.

Theorem 4.3.1. [1] *Let $A = B[M]$ be given an arbitrary presentation (Q_A, I_{ν_A}) . Then the cokernel of the morphism $\eta : \coprod_{j=1}^c \pi_1(Q_B^{(j)}, I_{\nu}^{(j)}) \longrightarrow \pi_1(Q_A, I_{\nu_A})$ induced by the inclusion of walks is the free group L_{t-c} in $t - c$ generators.*

Proof. (Sketch) Since $\delta \sim \gamma$ cannot happen whenever δ and γ are arrows lying in distinct components of Q_B , we assume that B is connected, that is $c = 1$. Set $t = t(\nu_A)$, and let $\{\alpha_1, \dots, \alpha_t\}$ be a complete set of representatives of the classes $[\alpha]_{\nu_A}$. For $i \in \{1, 2, \dots, t-1\}$ set $a_i = [\alpha_i]_{\nu_A}$.

Let T_B be a maximal tree in Q_B , and F_B, K_B as in Lemma 4.2.2. Denote by T_A the maximal tree in Q_A obtained from T_B by adding the arrow α_t . Moreover, let F_A and K_A be as in Lemma 4.2.2. We have $\pi_1(Q_B, I_{\nu_B}) \simeq F_B/K_B$ and $\pi_1(Q_A, I_{\nu_A}) \simeq F_A/K_A$ and the morphism $\eta : F_B/K_B \longrightarrow F_A/K_A$ is given by $\alpha K_B \mapsto \alpha K_A$. Let L be the free group with basis $\{a_i \mid 1 \leq i < t\}$. Define a group homomorphism $\tilde{\varphi} : F_A \longrightarrow L$ by

$$\tilde{\varphi}(\alpha) = \begin{cases} a_i & \text{if } \alpha \sim \alpha_i \text{ for some } i \in \{1, \dots, t-1\}, \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we verify that $\tilde{\varphi}$ induces a morphism $\varphi : F_A/K_A \longrightarrow L$ which is the cokernel of η . $\square$

The Van Kampen Theorem (see [46]) is perhaps one of the most useful tools to compute fundamental groups of topological spaces. Theorem 4.2.3 allows us to obtain this result in the context of bound quivers.

Given a bound quiver (Q, I) and a full convex subquiver Q^i , let I^i denote the ideal $I \cap \mathbb{k}Q^i$. With these notations, we have the following result.

Corollary 4.3.2. [15] *Let (Q, I) be a bound quiver, Q^1 and Q^2 be two full convex subquivers of Q such that every non-zero path of Q lies entirely in either Q^1 or Q^2 , and $Q^0 = Q^1 \cap Q^2$ is connected. Then $\pi_1(Q, I)$ is the pushout of the diagram*

$$\pi_1(Q^1, I^1) \longleftarrow \pi_1(Q^0, I^0) \longrightarrow \pi_1(Q^2, I^2)$$

where the arrows are the maps induced by the inclusions of walks.

It is worthwhile to note that a different approach to a Van Kampen Theorem has been considered in [43] for schurian algebras.

Example 4.3.3. Let $A = \mathbb{k}Q_A/I_A$ and $B = \mathbb{k}Q_B/I_B$ be two algebras given by quivers with relations. Following [36], let Q_C be a full convex and connected common subquiver of Q_A and Q_B such that $I_B \cap \mathbb{k}Q_C = I_A \cap \mathbb{k}Q_C$. Denote by I_C this ideal of $\mathbb{k}Q_C$, and let $e_C = \sum_{x \in (Q_C)_0} e_x$ be the identity of the algebra $C = \mathbb{k}Q_C/I_C$. Then $C \simeq e_C A e_C \simeq e_C B e_C$ is a common quotient of A and B . The pullback of the projections

$$A \xrightarrow{p_A} C \xleftarrow{p_B} B$$

is the algebra $R = \{(a, b) \in A \times B \mid e_C a e_C = e_C b e_C\}$. We have that $R \simeq \mathbb{k}Q_R/I_R$, where Q_R is the pushout of the inclusions (of quivers)

$$Q_A \xleftarrow{j_A} Q_C \xrightarrow{j_B} Q_B.$$

That is, Q_R is obtained from Q_A and Q_B by identifying the common subquiver Q_C (so that paths in Q_A or Q_B can be considered as paths in Q_R). The ideal I_R is then $I_R = I_A + I_B + \langle \mathcal{R} \rangle$, where $\mathcal{R}$ is the set of paths of Q_R linking $(Q_A)_0 \setminus (Q_B)_0$ and $(Q_B)_0 \setminus (Q_A)_0$. The hypothesis of Corollary 4.3.2 are satisfied, so the fundamental group $\pi_1(Q_R, I_R)$ is the pushout of

$$\pi_1(Q_A, I_A) \xleftarrow{\pi_1(j_A)} \pi_1(Q_C, I_C) \xrightarrow{\pi_1(j_B)} \pi_1(Q_B, I_B).$$

5. Coverings. As seen in the preceding section, the construction of $\mathcal{B}(Q, I)$ does not lead to a functor from the category of bound quivers to that of CW-complexes. This is somehow natural, since there is no reason why a bound quiver morphism should behave well with respect to the homotopy relations. However, if we consider covering morphisms of bound quivers, which do behave well from the homotopy point view, we will obtain covering maps of CW-complexes.

5.1. Coverings of bound quivers. We begin by recalling the following definition.

Definition 5.1.1. [41] A *Galois covering of a connected bound quiver* (Q, I) is a bound quiver morphism $p : (\hat{Q}, \hat{I}) \rightarrow (Q, I)$, where $(\hat{Q}, \hat{I})$ is connected, together with a group G of automorphisms of $(\hat{Q}, \hat{I})$ such that:

- (1) G acts freely on $\hat{Q}_0$;

- (2) $pg = p$ for every $g \in G$ and $p(x) = p(y)$ if, and only if, there exists $g \in G$ such that $y = g(x)$;
- (3) For every $x \in Q_0$ and $\hat{x} \in p^{-1}(x)$ the map p induces bijections $\hat{x} \rightarrow x$ and $\rightarrow \hat{x} \rightarrow x$;
- (4) I is the ideal generated by the elements of the form $p(\hat{\rho})$, with $\hat{\rho} \in \hat{I}$.

Among all Galois coverings of (Q, I) , there is one of particular interest: the *universal Galois covering* (see [38]) $p : (\tilde{Q}, \tilde{I}) \rightarrow (Q, I)$, given by the group $G = \pi_1(Q, I)$. More precisely, the universal cover $(\tilde{Q}, \tilde{I})$ at some fixed vertex $x_0 \in Q_0$ is defined in the following way: the vertices of $\tilde{Q}$ are the homotopy classes $[w]$ of walks of Q starting at x_0 , and there is an arrow from $[w]$ to $[w']$ whenever $[w'] = [w\alpha]$, with $\alpha \in Q_1$. This provides a map of quivers $p : \tilde{Q} \rightarrow Q$ defined by $p([w]) = x$, the target of w . Finally, the ideal $\tilde{I}$ is defined to be generated by the inverse images under p of the generators of I (compare this construction with that of the universal cover of a topological space in, for instance, [46]).

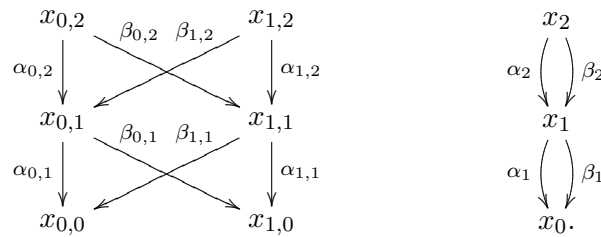
If there is another Galois covering $\hat{p} : (\hat{Q}, \hat{I}) \rightarrow (Q, I)$ given by a group H , then there exists a Galois covering $p^* : (\tilde{Q}, \tilde{I}) \rightarrow (\hat{Q}, \hat{I})$ such that $p = \hat{p}p^*$.

$$\begin{array}{ccc} (\tilde{Q}, \tilde{I}) & \xrightarrow{p^*} & (\hat{Q}, \hat{I}) \\ p \downarrow & \swarrow \hat{p} & \\ (Q, I) & & \end{array}$$

Moreover, in this case, there is a normal subgroup N of $\pi_1(Q, I)$ such that $\pi_1(\hat{Q}, \hat{I}) \simeq N$ and $\pi_1(Q, I)/N \simeq H$; see [38, 33, 13].

Example 5.1.2. Let $n \geq 2$ be an integer, and let $\hat{Q}_n$ be the quiver whose vertices are $x_{i,j}$, with $0 \leq i < n$ and $0 \leq j \leq n$. The arrows of $\hat{Q}_n$ are $\alpha_{i,j} : x_{i,j} \rightarrow x_{i,j-1}$ and $\beta_{i,j} : x_{i,j} \rightarrow x_{i+1,j-1}$ for $0 \leq i \leq n$ and $1 \leq j \leq n$, where indices are read modulo n . Moreover, let $\hat{I}_n = \langle w - w' \mid w \text{ and } w' \text{ are parallel paths} \rangle$, so that $\mathbb{k}\hat{Q}_n/\hat{I}_n$ is an incidence algebra. Consider the automorphism $g : (\hat{Q}_n, \hat{I}_n) \rightarrow (\hat{Q}_n, \hat{I}_n)$ defined on the vertices of $\hat{Q}_n$ by $g(x_{i,j}) = x_{i+1,j}$, and on the arrows by $g(\alpha_{i,j}) = \alpha_{i+1,j}$ and $g(\beta_{i,j}) = \beta_{i+1,j}$. Then g has order n . We can consider the quotient $(Q_n, I_n) = (\hat{Q}_n, \hat{I}_n)/\langle g \rangle$. The vertex set of the quiver Q_n is $\{x_0, x_1, \dots, x_n\}$, and the arrows are $\alpha_j, \beta_j : x_j \rightarrow x_{j-1}$, for $1 \leq j \leq n$. Moreover, $I_n = \langle \alpha_n \cdots \alpha_1 - \beta_n \cdots \beta_1, \alpha_i \beta_{i-1} - \beta_i \alpha_{i-1} \mid 1 < i \leq n \rangle$. An immediate computation shows that $\pi_1(\hat{Q}_n, \hat{I}_n) = 1$ (see also [28, 14]), so that $\pi_1(Q_n, I_n) \simeq \langle g \rangle \simeq \mathbb{Z}_n$.

For $n = 2$, the quivers $\hat{Q}_2$, and Q_2 look as follows:



A Galois covering of bound quivers $p : (\widehat{Q}, \widehat{I}) \rightarrow (Q, I)$ induces naturally a covering functor between $\mathbb{k}$ -categories $F : \widehat{A} = \mathbb{k}\widehat{Q}/\widehat{I} \rightarrow A = \mathbb{k}Q/I$, and following [12], we obtain two new functors:

- (1) The *pull-up functor* $F_\bullet : \text{mod } A \rightarrow \text{Mod } \widehat{A}$, given by $F_\bullet(M) = M \circ F$;
- (2) The *push-down functor* $F_\lambda : \text{Mod } \widehat{A} \rightarrow \text{mod } A$, the left adjoint to $F_\bullet$.

By means of the functors above and their nice properties (see [26, 12, 13, 18]), it is often possible to reduce the study of the category of finite dimensional representations of an algebra A to that of representations of a simply connected $\mathbb{k}$ -category.

5.2. Coverings of classifying spaces. Recall from [46], for instance, that if X is a topological space, then a covering space of X is a pair $(\hat{X}, p)$ where

- (1) $\hat{X}$ is an arcwise connected topological space;
- (2) $p : \hat{X} \rightarrow X$ is a continuous map;
- (3) Each $x \in X$ has an open neighborhood U_x such that $p^{-1}(U_x) = \bigcup_{i \in I} \hat{U}_i$, with $\hat{U}_i$ disjoint open sets, and $p|_{\hat{U}_i} : \hat{U}_i \rightarrow U_x$ an homeomorphism, for every $i \in I$.

In this situation, we obtain a morphism of groups $p_* = \pi_1(p) : \pi_1(\hat{X}) \rightarrow \pi_1(X)$ which is always a monomorphism. In case Imp_* is a normal subgroup of $\pi_1(X)$, the covering $(\hat{X}, p)$ is said to be *regular*. The set of all homeomorphisms $\phi : \hat{X} \rightarrow \hat{X}$ such that $p\phi = p$ is a group, called the *group of covering automorphisms* of $(\hat{X}, p)$, and is denoted by $\text{Cov}(\hat{X}/X)$. A covering $(\hat{X}, p)$ is regular if and only if $\text{Cov}(\hat{X}/X)$ acts transitively on $p^{-1}(x_0)$, the fiber over the base point; see [46], for instance.

Given a Galois covering $p : (\widehat{Q}, \widehat{I}) \rightarrow (Q, I)$, we wish to build a regular covering $\mathcal{B}p : \mathcal{B}(\widehat{Q}, \widehat{I}) \rightarrow \mathcal{B}(Q, I)$. We need the following lemma.

Lemma 5.2.1. *Let $p : (\widehat{Q}, \widehat{I}) \rightarrow (Q, I)$ be a Galois covering given by a group G . Then:*

- (1) *For every $x, y \in Q_0$, every minimal relation $\rho \in I(x, y)$ and every $\hat{x} \in p^{-1}(x)$, there exists $\hat{y} \in p^{-1}(y)$ and $\hat{\rho} \in \hat{I}(\hat{x}, \hat{y})$ such that $p(\hat{\rho}) = \rho$.*
- (2) *Let x be a vertex of Q , and w_1, w_2 be two paths with source x . Moreover, let $\hat{x} \in p^{-1}(x)$, and $\hat{w}_1, \hat{w}_2$ be two paths with source $\hat{x}$ such that $p(\hat{w}_i) = w_i$, for $i = 1, 2$. Then $w_1 \approx w_2$ if and only if $\hat{w}_1 \approx \hat{w}_2$.*
- (3) *For $x_0 \in Q_0$, and every $\hat{x}_0 \in p^{-1}(x_0)$, there is a bijective correspondence between the set of n -cells of $\mathcal{B} = \mathcal{B}(Q, I)$ having x_0 as boundary point, and the set of n -cells of $\hat{\mathcal{B}} = \mathcal{B}(\widehat{Q}, \widehat{I})$ having $\hat{x}_0$ as boundary point.*

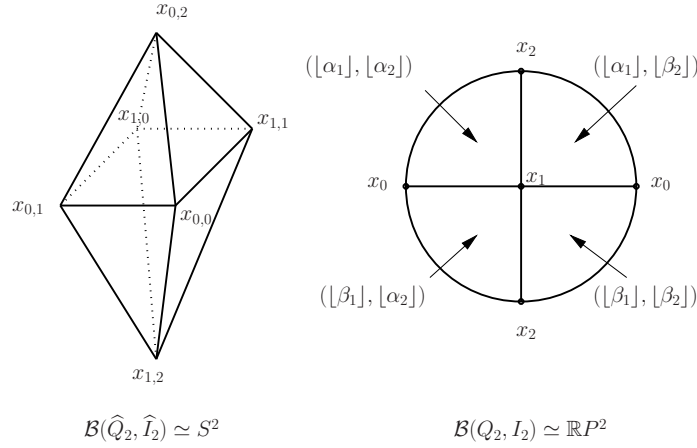
Proof. (Sketch) Since we assume Q to be connected, it is clear that p is surjective. Moreover, the second requirement in condition (2) of Definition 5.1.1 implies that G acts transitively on the fibers over the vertices of Q . This, together with the fact that G is a group of automorphisms, imply statement (1); see also [38, (1.4)]. Statements (2) and (3) follow directly. $\square$

Note that the same statement holds if one replaces natural homotopy by homotopy.

In the light of statement (3), we can define without any ambiguity $\mathcal{B}p : \hat{\mathcal{B}} \rightarrow \mathcal{B}$ as the map which maps homeomorphically an n -cell $([\sigma_1], \dots, [\sigma_n])$ onto the cell $([p\sigma_1], \dots, [p\sigma_n])$. This leads to the following theorem; see [15].

Theorem 5.2.2. [15] *Let $p : (\hat{Q}, \hat{I}) \rightarrow (Q, I)$ be a Galois covering given by a group G . Then $(\hat{B}, \mathcal{B}p)$ is a regular covering of $\mathcal{B}$ and $\text{Cov}(\hat{B}/\mathcal{B}) \simeq G$.*

Example 5.2.3. Consider the bound quivers $(\hat{Q}_2, \hat{I}_2)$ and (Q_2, I_2) of Example 5.1.2. The covering map $p : (\hat{Q}_2, \hat{I}_2) \rightarrow (Q_2, I_2)$ is given by $p(x_{i,j}) = x_j$. The classifying space $\hat{B} = \mathcal{B}(\hat{Q}_2, \hat{I}_2)$ is homeomorphic to S^2 . In addition, there are four 2-cells in $\mathcal{B} = \mathcal{B}(Q_2, I_2)$, namely $([\alpha_2], [\alpha_1])$, $([\alpha_2], [\beta_1])$, $([\beta_2], [\alpha_1])$ and $([\beta_2], [\beta_1])$. Keeping in mind that $\alpha_2\alpha_1 \approx \beta_2\beta_1$ and $\alpha_2\beta_1 \approx \beta_2\alpha_1$, it is easily seen that $\mathcal{B}$ is homeomorphic to the real projective plane $\mathbb{R}P^2$, and the map $\mathcal{B}p : \hat{B} \rightarrow \mathcal{B}$ identifies antipodal points.



5.3. Higher homotopy groups. Recall that given a topological space X , its higher homotopy groups are

$$\pi_n(X, x_0) = \{f : S^n \rightarrow X \mid f \text{ is continuous, } f(1, 0, \dots, 0) = x_0\} / \sim,$$

where, again, $\sim$ is the usual homotopy relation of maps relatively to x_0 . In view of the results of the preceding sections, it would be natural to define the higher homotopy groups of a bound quiver (Q, I) as the corresponding homotopy groups $\pi_n(\mathcal{B}(Q, I))$ of its classifying space.

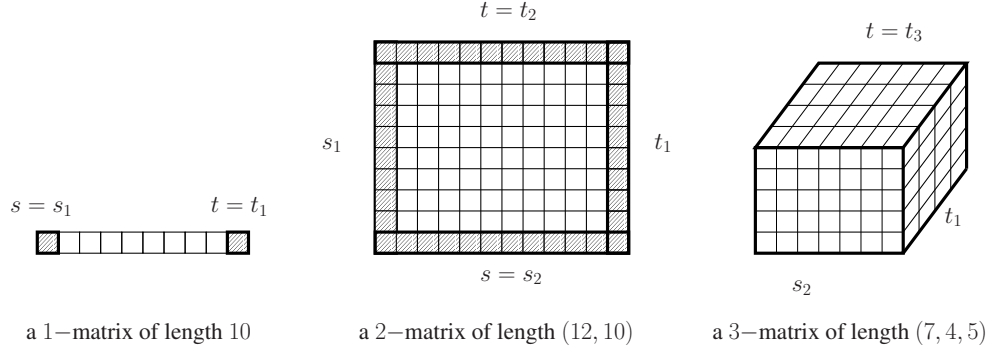
A different approach for the definition of $\pi_n(Q, I)$ has been considered by E. Reynaud in [45]. It consists of considering higher dimensional analogs of walks and the homotopy relation. In order to do so, consider a quiver Q having no multiple arrows. In this situation, a walk f of length $r - 1$ on Q can be considered as a map $f : \{1, \dots, r\} \rightarrow Q_0$ such that for every $i \in \{2, \dots, r - 1\}$ the vertices $f(i - 1)$ and $f(i)$ appear in a path of Q .

Now, let $n \in \mathbb{N}$, with $n \geq 2$, and $\{a_1, \dots, a_n\}$ be natural numbers. Denote by $[1, a_i]$ the set $\{1, 2, \dots, a_i\}$ and set $\mathbf{A} = [1, a_1] \times [1, a_2] \times \dots \times [1, a_n]$. An n -matrix of length $\{a_1, \dots, a_n\}$ is a map $f : \mathbf{A} \rightarrow Q_0$. An n -matrix is an n -walk matrix if for any $(i_1, \dots, i_n) \in \mathbf{A}$ there exists a path in Q containing the vertices $f(j_1, \dots, j_n)$ with $(j_1, \dots, j_n)$ satisfying $i_k \leq j_k \leq i_k + 1$ for any k . Denote by $\mathcal{M}_n(Q)$ the set of such matrices. The next step is to generalize the notions of source and target of an n -walk matrix $f \in \mathcal{M}_n(Q)$ of length $\{a_1, \dots, a_n\}$. For $i \in \{1, \dots, n\}$, let $s_i(f)$ and $t_i(f)$ be

the $(n-1)$ -matrices of length $(a_1, \dots, \widehat{a_i}, \dots, a_n)$ defined by:

$$\begin{cases} s_i(f)(y_1, \dots, y_{n-1}) &= f(y_1, \dots, y_{i-1}, 1, y_i, \dots, y_{n-1}), \\ t_i(f)(y_1, \dots, y_{n-1}) &= f(y_1, \dots, y_{i-1}, a_i, y_i, \dots, y_{n-1}). \end{cases}$$

The *source* $s(f)$ and the *target* $t(f)$ of an n -walk matrix f are the $(n-1)$ -matrices $s_n(f)$ and $t_n(f)$, respectively. Two n -matrices f and f' are said to be *equal alongside* if $s_i(f) = s_i(f')$, and $t_i(f) = t_i(f')$ for every $i \in \{1, \dots, n\}$.



In order to define the composition law on $\mathcal{M}_n(Q)$, let f and f' be two n -matrices of lengths $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$, such that $s(f') = t(f)$. In particular this implies $a_i = a'_i$ for $i \in \{1, \dots, n-1\}$. Let $f \times f'$ to be the n -matrix of length $(a_1, \dots, a_{n-1}, a_n + a'_n - 1)$ defined in such a way that $(f \times f')(x_1, \dots, x_{n-1}, t)$ takes the value

$$\begin{cases} f(x_1, \dots, x_{n-1}, t) & \text{if } 1 \leq t \leq a_n, \\ f'(x_1, \dots, x_{n-1}, t - a_n + 1) & \text{if } a_n + 1 \leq t \leq a_n + a'_n - 1. \end{cases}$$

Roughly speaking, $f \times f'$ is obtained by gluing f and f' along their common side. Naturally, if f and f' are n -walk matrices then $f \times f'$ is an n -walk matrix as well.

Finally, we can define inductively an homotopy relation $\sim$ on $\mathcal{M}_n(Q')$ as the equivalence relation generated by:

- (1) Two 0-walk matrices are homotopic if and only if they are equal.
- (2) Let f, f' be two n -walk matrices of length $(a_1, \dots, a_n)$. If there exists an $(n+1)$ -walk matrix F such that $s(F) = f$, $t(F) = f'$ and the matrices $F(-, t)$ are equal alongside for every $t \in \{1, \dots, a_n\}$ then $f \sim f'$.
- (3) Let f be an n -walk matrix of length $(a_1, \dots, a_n)$, $i \in \{1, \dots, n\}$ such that $a_i \geq 2$, and $k \in \{1, \dots, a_i\}$. Let $f_{i,k}$ be the n -matrix of length $(a_1, \dots, a_i - 1, \dots, a_n)$ defined by

$$f_{i,k}(x_1, \dots, x_n) = \begin{cases} f(x_1, \dots, x_i, \dots, x_n) & \text{if } 1 \leq x_i < k, \\ f(x_1, \dots, x_i + 1, \dots, x_n) & \text{if } k \leq x_i \leq a_i - 1. \end{cases}$$

If $f_{i,k}$ is an n -walk matrix such that:

- (a) $f(1 + \epsilon_1(a_1 - 1), \dots, 1 + \epsilon_n(a_n - 1)) = f_{i,k}(1 + \epsilon_1(a'_1 - 1), \dots, 1 + \epsilon_n(a'_n - 1))$ for $\epsilon_i \in \{0, 1\}$ (where $a'_i = a_i - 1$ and $a'_j = a_j$ for $j \neq i$);
 - (b) $s_j(f) \sim s_j(f_{i,k})$ and $t_j(f) \sim t_j(f_{i,k})$ in $\mathcal{M}_{n-1}(Q)$, for $j \in \{1, \dots, n\}$.
- Then $f \sim f_{i,k}$.

Intuitively, $f_{i,k}$ is obtained from f by deleting the k -th hyperplane in the i -th coordinate.

$$f = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}, \quad f_{1,1} = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}, \quad f_{2,2} = \begin{bmatrix} z_1 & z_3 & z_4 \\ y_1 & y_3 & y_4 \\ x_1 & x_3 & x_4 \end{bmatrix}.$$

Example 5.3.1. Let $a \leq x \leq b$ and $a \leq y \leq b$ be two parallel chains in a poset Σ . Then, it follows from the condition (1) that the corresponding 1-walk matrices are homotopic. Indeed, consider the 2-walk matrix of length $(3, 3)$ given by

$$F = \begin{bmatrix} a & a & a \\ x & a & y \\ b & b & b \end{bmatrix}.$$

This shows that if (Q, I) is a bound quiver arising from a poset, then the homotopy relation on 1-matrices extends the usual homotopy relation of paths.

The relation $\sim$ is compatible with the (partially defined) composition law $\times$ just defined. As for the fundamental group, fix a vertex $x \in Q_0$ and let $\mathcal{M}_n(Q, x)$ be the set of n -walk matrices f such that $s_i(f) = t_i(f)$ for every $i \in \{1, \dots, n\}$, and this common value is the $(n-1)$ -walk matrix of size $(1, \dots, 1)$ and with unique entry equal to x . On $\mathcal{M}_n(Q, x)$, the composition law is everywhere defined.

Now let (Q, I) be a bound quiver, with Q having no oriented cycles. Let $\mathcal{B} = \mathcal{B}(Q, I)$ be its classifying space, and $\mathcal{OB}$ be the set of cells of $\mathcal{B}$ ordered by inclusion. Let (Q', I') be the bound quiver associated to this poset. In particular, Q' has no multiple arrows, and we can carry the construction described above. This leads to the following definition:

Definition 5.3.2. [45] Let (Q, I) be a bound quiver with Q having no oriented cycles. Moreover, let (Q', I') be as in the preceding paragraph and $x \in Q'_0$. The n -th homotopy group of (Q, I) is defined to be $\pi_n(Q, I) = \mathcal{M}_n(Q', x)$.

This leads to the following theorem.

Theorem 5.3.3. [45] With the above notations, there is, for every $n \geq 1$, an isomorphism of groups $\pi_n(Q, I) \simeq \pi_n(\mathcal{B}(Q', I'))$.

Using the fact that $\mathcal{B}(Q', I')$ is the first barycentric subdivision of $\mathcal{B}(Q, I)$, we obtain immediately the equivalence of the two approaches.

Before giving an example, we recall that if $(\hat{X}, p)$ is a covering space of a topological space X , then the groups $\pi_n(\hat{X})$ and $\pi_n(X)$ are isomorphic for every $n \geq 2$; see [46].

Example 5.3.4. Consider the bound quiver $(\hat{Q}_2, \hat{I}_2)$ of Example 5.2.3. The universal cover of $\mathcal{B} = \mathcal{B}(\hat{Q}_2, \hat{I}_2)$ is the sphere S^2 so that $\pi_2(\mathcal{B}) \simeq \mathbb{Z}$.

Remark 5.3.5. At this point it is important to note that in general the higher homotopy groups are much harder to compute than either the (co)homology groups or the fundamental group. This is essentially due to the fact that neither the excision property (which allows to obtain the Mayer-Vietoris sequences) holds for homotopy, nor the Van Kampen Theorem holds for higher homotopy groups. Even for rather simple spaces, like spheres, the homotopy groups are unknown in general. Moreover, unlike homology

groups, if X is a i -dimensional CW-complex, it may happen that $\pi_n(X)$ is not trivial for $n \geq i + 1$. For instance, in the preceding example, $\pi_3(\mathcal{B}) \simeq \mathbb{Z}$, and $\pi_4(\mathcal{B}) \simeq \mathbb{Z}_2$; see [46].

6. Incidence algebras.

6.1. Some immediate results. As we noted in Section 3.1, in case Σ is a poset and (Q_Σ, I_Σ) is the associated bound quiver, then $\mathcal{B}(Q_\Sigma, I_\Sigma)$, which is a simplicial complex, is the classifying space of Σ considered as a category; see Example 3.3.3 (1). Besides, consider the incidence algebra $\mathbb{k}\Sigma$, and let $\mathrm{HH}^i(\mathbb{k}\Sigma) = \mathrm{Ext}_{\mathbb{k}\Sigma - \mathbb{k}\Sigma}^i(\mathbb{k}\Sigma, \mathbb{k}\Sigma)$ denote its Hochschild cohomology groups. It was proven in [30] by Gerstenhaber and Schack (see [39, 19, 15] also) that for every $i \geq 0$ there are isomorphisms of abelian groups $\mathrm{HH}^i(\mathbb{k}\Sigma) \simeq H^i(\mathcal{B}\Sigma, k^+)$. Moreover, these isomorphisms commute with cup-products. Thus, when dealing with incidence algebras, the links with algebraic topology are the best ones one can hope, and we can obtain some immediate results concerning the simple connectedness or the Hochschild cohomology of such algebras.

In [28], it was shown that if Σ is a poset having a greatest, or a least element, then the Hochschild cohomology groups of $\mathbb{k}\Sigma$ vanish in every positive degree. The following result, from which the second statement is in [14], is a slight generalization of this fact.

Theorem 6.1.1. *Let Σ be a finite poset. If there exists an element $x_0 \in \Sigma$ such that for every $y \in \Sigma$ one has either $x_0 \leq y$ or $y \leq x_0$, then*

- (1) $\pi_n(\mathbb{k}\Sigma) = 0$ for every $n \geq 1$;
- (2) $\mathrm{HH}^n(\mathbb{k}\Sigma) = 0$ for every $n \geq 1$.

Proof. Set $\Sigma' = \Sigma \setminus \{x_0\}$, so that $\mathcal{B}\Sigma$ is the cone over $\mathcal{B}\Sigma'$ with vertex x_0 . In particular it is a contractible topological space. $\square$

Let Σ be a finite poset. Recall from [23] that $x \in \Sigma$ is called *irreducible* if x has either a unique immediate successor, or a unique immediate predecessor. Naturally, it is called *doubly irreducible* if it has both, a unique immediate successor, and a unique immediate predecessor.

Proposition 6.1.2. *Let x be an irreducible element in a poset Σ , and set $\Sigma' = \Sigma \setminus \{x\}$. Then for every $n \geq 0$ we have isomorphisms:*

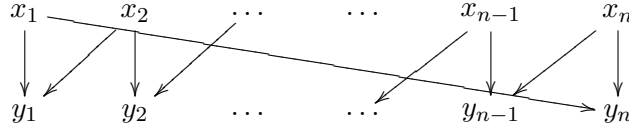
- (1) $\pi_n(\mathcal{B}\Sigma) \simeq \pi_n(\mathcal{B}\Sigma')$;
- (2) $\mathrm{HH}^n(\mathbb{k}\Sigma) \simeq \mathrm{HH}^n(\mathbb{k}\Sigma')$.

Proof. This follows from the fact that $\mathcal{B}\Sigma'$ is a strong deformation retract of $\mathcal{B}\Sigma$; see [34, 20]. $\square$

A nice consequence of this result is that one can simplify computations of homotopy and (co)homology groups of an incidence algebra by successively removing irreducible points. It follows that every incidence algebra has the same homotopy and (co)homology groups than incidence algebras whose underlying poset does not contain any irreducible element. In case the above process ends with a poset having a single element, we say that the poset is *dismantlable* and, of course, in this case $\mathcal{B}\Sigma$ is a contractible simplicial complex; see [23].

As noted in [5, 22, 28, 34], posets of a special kind, called *crowns*, play an important role when dealing with simple connectedness or Hochschild cohomology of incidence algebras.

Definition 6.1.3. Let $n \geq 2$ be an integer. The n -crown is the poset $C_n = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ ordered by $y_i < x_i$ and $y_{i-1} < x_i$ for all i in $\{1 \dots, n\}$, with $y_0 = y_n$; and $x_i \not\leq x_j, y_i \not\leq y_j$ for any $i \neq j$.



It is easy to see that $\pi_1(\mathcal{BC}_n)$ is isomorphic to $\mathbb{Z}$. Moreover, C_2 is the minimal poset whose incidence algebra is not simply connected.

Recall from [49] that a triangular algebra $A \simeq \mathbb{k}Q/I$ is said to be *strongly simply connected* if every full convex subcategory $A' \subseteq A$ is simply connected. Note that in the case of incidence algebras, "full convex subcategory" means incidence algebra of a convex subposet.

The following result shows the relation between the strong simple connectedness of an incidence algebra $\mathbb{k}\Sigma$ and the existence or not of crowns in the poset Σ .

Theorem 6.1.4. [22] *Let Σ be a finite poset. Its incidence algebra is strongly simply connected if and only if Σ does not contain a crown.*

Proof. On the one hand, it is clear that if Σ contains a crown then the algebra $\mathbb{k}\Sigma$ is not strongly simply connected. On the other hand, if Σ does not contain a crown, then the same holds for every convex subposet Σ' of Σ . It follows from Corollary 3.8 in [34] that the simplicial complex $\mathcal{B}\Sigma'$ is contractible, thus simply connected. $\square$

Remark 6.1.5. It follows from Proposition 2.1 in [27] that if Σ does not contain a crown, then $\mathrm{HH}^i(\mathbb{k}\Sigma') = 0$ for every convex subposet Σ' of Σ , for all $i \geq 1$.

The last result gives a sufficient (but not necessary) condition for the simple connectedness of an incidence algebra. A necessary condition has been given in [5].

Finally, let us mention that the notions and methods exposed in this section have successfully been generalized to schurian triangular algebras. We refer the reader to [5, 2].

6.2. Barycentric subdivisions. Let K be a simplicial complex and $\mathcal{O}K$ be the set of simplices of K ordered by inclusion. If K is of the form $\mathcal{B}\Sigma$, for some poset Σ , then the simplicial complex $\mathcal{B}\Sigma = \mathcal{B}\mathcal{O}K$ is the first barycentric subdivision of $\mathcal{B}\Sigma$. This proves the following result:

Proposition 6.2.1. *Let Σ be a finite poset. Then for every $n \geq 0$ there are isomorphisms:*

- (1) $\mathrm{HH}^n(\mathbb{k}\Sigma) \simeq \mathrm{HH}^n(\mathbb{k}\mathcal{O}\mathcal{B}\Sigma)$;
- (2) $\pi_n(\mathbb{k}\Sigma) \simeq \pi_n(\mathbb{k}\mathcal{O}\mathcal{B}\Sigma)$.

It is a well-known fact (see [46]) that every finitely presented group is the fundamental group of a finite simplicial complex. As a consequence, every finitely presented group arises as the fundamental group of an incidence algebra. This was first shown by Fischbacher and De la Peña [25].

In the light of the preceding result, and since in general the poset $\mathcal{OB}\Sigma$ is much bigger than Σ , it is an interesting question to determine whether a poset is of the form $\mathcal{OB}\Sigma$, and, if this is the case, to recover Σ . This is the aim of this section. In order to attain it, we will proceed in two steps. The first one is to determine when a poset is of the form $\mathcal{OK}$, for a simplicial complex K . The second step is to combine this characterization with Stanley's characterization [50] of the simplicial complexes of the form $\mathcal{B}\Sigma$. Some additional terminology will be needed.

Let Q be a quiver, x be a vertex of Q , and n be an integer. Denote by $Q^{-\infty}$ and Q^{∞} the sets of sources and sinks of Q respectively. The set of vertices y of Q such that there exists a path of length n from x to y is the set of n -th successors of x and will be denoted by x^{+n} : so the set of successors of x is $x^{+\infty} = \bigcup_{n \geq 0} x^{+n}$. In an analogous way we define the sets x^{-n} of n -th predecessors of x . The set $\mathbf{s}(x) = x^{+\infty} \cap Q^{\infty}$ is the set of sinks of x .

Let K be a simplicial complex, and $Q = Q_{\mathcal{OK}}$ be the quiver associated to the poset $\mathcal{OK}$. Given σ and σ' in Q , there exists an arrow $\sigma \rightarrow \sigma'$ if, and only if, $\sigma' \subset \sigma$ and $\dim \sigma' = \dim \sigma - 1$. It follows that, given a fixed vertex σ in Q , all the paths from σ to the sinks of Q have the same length which, moreover, is the dimension of σ .

Motivated by this, we call a quiver Q *ranked* if for every vertex x of Q , the length of a path from x to a sink of Q does not depend on the choices of the path and of the sink. This length is called the *rank* of x . For a given integer k , $N_k(Q)$ will denote the set of vertices of rank k in Q . In particular, $N_0(Q)$ is the set of sinks of Q . If K is a simplicial complex and σ is an m -simplex of K , then σ^{+k} is the set of $(m-k)$ -faces of K contained in σ . In particular, $\sigma^{+m} = \mathbf{s}(\sigma)$. Moreover, if $\mathcal{OK}_1 \simeq \mathcal{OK}_2$, then the complexes K_1 and K_2 are isomorphic, and one can easily see that $\mathcal{O}(K_1 \cup K_2) \simeq \mathcal{OK}_1 \cup \mathcal{OK}_2$.

These considerations lead us to the promised characterization of the posets of the form $\mathcal{OK}$, whose proof is omitted; see [17]. Before stating it, define $\mathcal{P}^*(X)$ to be the set of non-empty subsets of a set X .

Theorem 6.2.2. *Let Q be a finite quiver. The following conditions are equivalent:*

- (1) *There exists a simplicial complex K such that $Q = Q_{\mathcal{OK}}$;*
- (2)
 - (a) *Q is ranked;*
 - (b) *For any two different vertices of Q , $x, y \in N_k(Q)$, we have $x^{+k} \neq y^{+k}$;*
 - (c) *For every vertex $x \in N_k(Q)$ we have $|x^{+i}| = \binom{k+1}{i}$, for every $i \in \{0, 1, \dots, k\}$;*
- (3)
 - (a) *Q is ranked;*
 - (b) *The map $\mathbf{s} : Q_0 \rightarrow \mathcal{P}^*(Q^{\infty})$ is injective;*
 - (c) *For every vertex x of Q we have $\mathbf{s}(x^{+\infty}) = \mathcal{P}^*(\mathbf{s}(x))$.*

Moreover, if one of these conditions is satisfied, then $K = \bigcup_{x \in Q^{-\infty}} \mathcal{P}^*(\mathbf{s}(x))$.

The next step is to use Stanley's characterization of complexes of the form $\mathcal{B}\Sigma$, which asserts that a simplicial complex K is of the desired form if, and only if :

- (1) Its 1-skeleton (that is the graph formed by all its 1-simplices) is a comparability graph;
- (2) Every set of vertices of K which is not a simplex of K and is maximal for this property has at most 2 elements.

Example 6.2.3. Let K be the complete graph on 3 elements, with vertices $\{1, 2, 3\}$. Of course K is a comparability graph, but $\{1, 2, 3\}$ is not a simplex of K , so K is not of the form $\mathcal{B}\Sigma$.

Now, in [31] (see also [32]) a criterion allowing to recognize whether a given graph is a comparability graph or not is given. We do not go into details here, but simply mention that this is done by determining if an associated graph is bipartite or not. Adapting these results to our context we obtain the following result.

Corollary 6.2.4. *Let Q be a finite quiver. The following conditions are equivalent:*

- (1) *There exists a finite poset Σ such that $Q = Q_{\mathcal{O}\mathcal{B}\Sigma}$;*
- (2) *The quiver Q satisfies the conditions of Theorem 6.2.2 and*
 - (a) *The graph $\overline{Q}$ whose vertices are pairs (a, b) of different sinks of A such that $a^{-1} \cap b^{-1} \neq \emptyset$, and in which there exists an edge joining (a, b) and (c, d) if and only if $a = c$ or $a^{-1} \cap c^{-1} = \emptyset$ is bipartite;*
 - (b) *Every set V of sinks of Q such that $\bigcap_{v \in V} v^{-\infty} = \emptyset$ but $\bigcap_{v \in V'} v^{-\infty} \neq \emptyset$ for every proper subset V' of V , has 2 elements.*

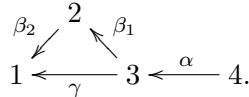
As a final remark, let us note that in the situation of the last result, the poset Σ is not unique. We refer the reader to [31, 32, 17] for further details.

7. Dependence on the presentation. As we mentioned in Section 4.1, the fundamental group is not an invariant for algebras of the form $A \simeq \mathbb{k}Q/I$. Moreover, we know that every finitely presented group is the fundamental group of some bound quiver (coming from a poset). With this in mind, we tackle the following problem: how distinct can be fundamental groups of presentations of algebras?

7.1. Preliminary considerations. We begin by giving two examples of algebras having different presentations leading to non-isomorphic fundamental groups.

Examples 7.1.1.

- (1) Let Q be the quiver



On the one hand, consider the ideal $I_1 = \langle \alpha\beta_1\beta_2 \rangle$ and let $A \simeq \mathbb{k}Q/I_1$. Since the ideal I_1 is monomial, we have $\pi_1(Q, I_1) \simeq \mathbb{Z}$. On the other hand, consider the morphism of algebras $\nu_2 : \mathbb{k}Q \rightarrow A$ defined by $\nu_2(\gamma) = (\gamma + \beta_1\beta_2) + I_1$, $\nu_2(\alpha) = \alpha + I_1$, $\nu_2(\beta_1) = \beta_1 + I_1$, and $\nu_2(\beta_2) = \beta_2 + I_1$. A straightforward computation shows that ν_2 is a presentation, and, moreover, that $I_2 = \text{Ker } \nu_2 = \langle \alpha(\gamma - \beta_1\beta_2) \rangle$. This yields to a trivial group $\pi_1(Q, I_2)$.

- (2) Consider the quiver $4 \begin{array}{c} \xrightarrow{\alpha_3} \\ \xleftarrow{\beta_3} \end{array} 3 \begin{array}{c} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} 1$ bound by the ideal

$$I = \langle \alpha_3\beta_2 - \beta_3\alpha_2, \alpha_2\beta_1 - \beta_2\alpha_1, \beta_3\beta_2\beta_1 - \alpha_3\alpha_2\alpha_1 \rangle.$$

As we saw in Examples 4.2.5 (2) (see also Example 5.1.2), we have $\pi_1(Q, I) \simeq \mathbb{Z}_3$. On the other hand, consider the morphism of algebras $\nu : \mathbb{k}Q \rightarrow \mathbb{k}Q/I$ defined by $\nu(\alpha_3) = (\alpha_3 - \beta_3) + I$, and $\nu(\gamma) = \gamma + I$ for every arrow $\gamma \neq \alpha_3$. Let $J = \text{Ker } \nu$. We leave to the reader the verification that $\pi_1(Q, J)$ is trivial; see [16].

In the two preceding examples, the existence of a pair (α, p) , where α is an arrow, and p a non-zero path parallel to α allowed us to define the changes of presentations that lead to changes of fundamental groups. Following [2], such a pair is called a *bypass*. Recall from [9] that an algebra $A \simeq \mathbb{k}Q/I$ is said to be *constricted* if for every arrow $\alpha : x \rightarrow y$ we have $\dim_{\mathbb{k}} e_x A e_y = 1$, that is, if there is no bypass in (Q, I) . This leads us to the following result.

Theorem 7.1.2. [9] *Let $A = \mathbb{k}Q/I$ be a constricted algebra. Given (Q, I_1) , (Q, I_2) , two presentations of A , one has $\pi_1(Q, I_1) \simeq \pi_1(Q, I_2)$.*

In order to carry the construction of the free product of two fundamental groups of bound quivers, and in analogy with algebraic topology where pointed spaces are considered, we define a *pointed bound quiver* (Q, I, x) to be a bound quiver (Q, I) together with a distinguished vertex x . Given two pointed bound quivers $Q' = (Q', I', x')$ and $Q'' = (Q'', I'', x'')$, we can assume, without loss of generality, that $Q'_0 \cap Q''_0 = Q'_1 \cap Q''_1 = \emptyset$. We define the quiver $Q = Q' \amalg Q''$ in the following way: Q_0 is $Q'_0 \cup Q''_0$ in which we identify x' and x'' to a single new vertex x , and $Q_1 = Q'_1 \cup Q''_1$. Then, Q' and Q'' are identified to two full convex subquivers of Q , so walks on Q' or Q'' can be considered as walks on Q . Thus, I' and I'' generate two-sided ideals of $\mathbb{k}Q$ which we denote again by I' and I'' . We define I to be the ideal $I' + I''$ of $\mathbb{k}Q$. It follows from this definition that the minimal relations of I are precisely the minimal relations of I' together with the minimal relations of I'' , both I' and I'' being viewed as ideals of $\mathbb{k}Q$. Moreover, we can consider an element $[w] \in \pi_1(Q', I', x')$ as an element $[w] \in \pi_1(Q, I, x)$. Conversely, any (reduced) walk w in Q has a decomposition $w = w'_1 w''_1 w'_2 w''_2 \cdots w'_n w''_n$ where w'_i and w''_i are walks in Q' and Q'' respectively. Moreover, this decomposition is unique, up to reduced walk, and compatible with the homotopy relations involved. This leads us to the following proposition.

Proposition 7.1.3. *With the notations above we have:*

- (1) (Q, I, x) is the coproduct of (Q', I', x) and (Q'', I'', x) in the category of pointed bound quivers;
- (2) $\pi_1(Q, I, x) \simeq \pi_1(Q', I', x') \amalg \pi_1(Q'', I'', x'')$.

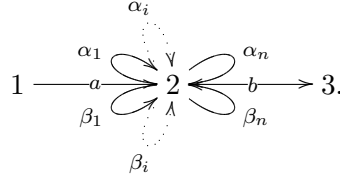
Proof. The first statement follows from a direct computation, while the second follows immediately from the above discussion. $\square$

7.2. Dependence on the presentation. In Examples 7.1.1, we encountered examples of triangular algebras having different presentations which led to cyclic or trivial fundamental groups. The following proposition generalizes this to arbitrary finitely presented groups but considering non-triangular algebras.

Proposition 7.2.1. [16] *Let G be a finitely presented group. Then there exists an algebra $A \simeq \mathbb{k}Q_G/I_G \simeq \mathbb{k}Q_G/I'_G$ such that $\pi_1(Q_G, I_G) \simeq G$ and $\pi_1(Q_G, I'_G) \simeq 1$.*

Proof. Let $G = \langle \alpha_i | w_j, 1 \leq i \leq n, 1 \leq j \leq m \rangle$ be a finitely presented group. More precisely, G is the factor group of the free group having basis $\{\alpha_i\}_{i=1}^n$ by a normal subgroup generated by $\{w_j\}_{j=1}^m$. Without loss of generality, we assume that the words w_j are reduced, non-empty, pairwise different, and, moreover, that each w_j contains at least one letter α_i 's with positive power (otherwise we can replace w_j by

w_j^{-1}). Consider the quiver Q_G :



Identifying the arrow β_i with the letter α_i^{-1} , we obtain a correspondence between the words in $\{\alpha_i, \alpha_i^{-1}\}_{i=1}^n$ and (some of) the paths of Q_G . With this identification in mind, define the ideal

$$I_G = \langle a\alpha_i\beta_i b - ab, aw_j b - ab, F^N \mid 1 \leq i \leq n, 1 \leq j \leq m \rangle$$

where $N = \max\{l(w_j) + 3, 6 \mid 1 \leq j \leq m\}$. An immediate computation shows that the generators of I_G which are not monomial relations are in fact minimal relations.

Therefore, $a\alpha_i\beta_i b \sim ab$, and $aw_j b \sim ab$. From this, we get $\alpha_i\beta_i \sim e_2$, and $w_j \sim e_2$, for all i, j . This shows that $\pi_1(Q_G, I_G) \simeq G$.

For the second statement, consider the morphism of algebras $\nu : \mathbb{k}Q_G \rightarrow \mathbb{k}Q_G/I_G$ defined on the arrows of Q_G by

$$\nu(\gamma) = \begin{cases} \alpha_i + \beta_i + \beta_i^2 + I_G & \text{if } \gamma = \alpha_i, \\ \gamma + I_G & \text{otherwise.} \end{cases}$$

Let $I'_G = \text{Ker } \nu$. We leave to the reader to verify that this defines a new presentation of A and, moreover, that $\rho = a\alpha_i\beta_i b - a\beta_i^2 b - a\beta_i^3 b - ab$ is a minimal relation. From this, it is easily seen that $\pi_1(Q_G, I'_G)$ is trivial. $\square$

Before stating the main theorem of this section, we need the following lemma.

Lemma 7.2.2. [16] *For $i \in \{1, 2\}$, let $A_i \simeq kQ_{A_i}/I_i \simeq kQ_{A_i}/I'_i$ be algebras with two different presentations. Denote by $\pi_1(Q_{A_i}, I_i) \simeq G_i$, and $\pi_1(Q_{A_i}, I'_i) \simeq G'_i$. Then there exists an algebra A having two presentations $A \simeq kQ/I \simeq kQ/I'$ such that $\pi_1(Q, I) \simeq G_1 \amalg G_2$ and $\pi_1(Q, I') \simeq G'_1 \amalg G'_2$.*

Proof. For $i \in \{1, 2\}$, let $\nu_i : kQ_{A_i} \rightarrow A_i$, and $\nu'_i : kQ_{A_i} \rightarrow A_i$ be the presentations of the algebra A_i such that $I_i = \text{Ker } \nu_i$, and $I'_i = \text{Ker } \nu'_i$. Consider the pointed bound quiver $(Q, I, x) = (Q_{A_1}, I_1, x_1) \amalg (Q_{A_2}, I_2, x_2)$ and let $A = kQ/I$. It follows from Proposition 7.1.3 that $\pi_1(Q, I, x) \simeq G_1 \amalg G_2$. Now, consider the presentation $\nu' : kQ \rightarrow kQ/I \simeq A$ given by $\nu'(\alpha) = \nu'_i(\alpha)$, where α is an arrow of Q_{A_i} . Then, we have $I' = \text{Ker } \nu' = I'_1 + I'_2$, and, again, from Proposition 7.1.3, we obtain $\pi_1(Q, I', x) \simeq G'_1 \amalg G'_2$. $\square$

We are now able to state the main result of this section.

Theorem 7.2.3. [16] *Let $G_1, \dots, G_n$ be finitely presented groups. Then, there exists a finite dimensional algebra A having presentations $A \simeq kQ_A/I_i$, for $i \in \{1, \dots, n\}$, such that $\pi_1(Q_A, I_i) \simeq G_i$.*

Proof. (Sketch) Using Proposition 7.2.1, we can build algebras A_i having presentations $A_i \simeq kQ_i/J_i \simeq kQ_i/J'_i$ with $\pi_1(Q_i, J_i) \simeq G_i$, and $\pi_1(Q_i, J'_i) \simeq 1$. For

$i \in \{1, \dots, n\}$ consider the bound quiver

$$(Q_A, I_i) = \left(\prod_{l=1}^{i-1} (Q_l, J'_l) \right) \amalg (Q_i, J_i) \amalg \left(\prod_{l=i+1}^n (Q_l, J'_l) \right).$$

It follows from proposition 7.1.3, statement (2), that $\pi_1(Q_A, I_i) \simeq G_i$. Moreover, using the argument of the proof of the above lemma, one gets $kQ_A/I_i \simeq kQ_A/I_j$, for all $i, j \in \{1, \dots, n\}$. $\square$

Remark 7.2.4. As a final remark, let us note that the homotopy relation in a bound quiver does make sense even if we do not ask the ideal I to be admissible, nor the quiver Q to be finite. These requirements lead to finite dimensional algebras (with unit).

If one wishes to consider the family of all finitely generated groups, similar constructions to those which were made in Proposition 7.2.1 can be done. Indeed given any finitely generated group G , as one can construct a quiver Q_G and a two-sided ideal $I_G \trianglelefteq kQ_G$ such that $\pi_1(Q_G, I_G) \simeq G$. In this case, the quiver would still be finite, but the ideal I will not be admissible, so this would lead to infinite dimensional algebras which can be seen as k -categories.

As further generalization, one may wish to consider arbitrary groups. Again, this can be performed, as in Proposition 7.2.1. In this case, the quiver will still have three vertices, but will have an infinite number of arrows, and, again, the ideal would not be admissible.

Also, in the light of Theorem 7.1.2, one may ask how big is the family of groups that rise as fundamental groups of presentation of a fixed algebra A ?

Finally, let us note that a "universal" presentation of some triangular algebras has been constructed in [35]. A *double bypass* is a quadruple (α, u, β, v) such that (α, u) and (β, v) are bypasses such that the arrow β appears in the path u . Let A be a triangular $\mathbb{k}$ -algebra having no double bypasses, with $\text{char } \mathbb{k} = 0$, then there exists a presentation $A \simeq \mathbb{k}Q_A/I$ such that if $A \simeq \mathbb{k}Q_A/I'$ is another presentation then there exists a surjective morphism of groups $p : \pi_1(Q_A, I) \rightarrow \pi_1(Q_A, I')$.

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Résumé substantiel en français. Le but de ces notes est de fournir un bref survol au sujet de l'homotopie des carquois liés. La motivation pour ceci vient principalement de la théorie des représentations des algèbres de dimension finie sur des corps algébriquement clos $\mathbb{k}$ (par la suite, mot "algèbre" sera utilisé dans ce sens). En effet, toute algèbre (sobre) de cette classe peut être exprimée comme le quotient d'une algèbre de chemins par un idéal admissible $A \simeq \mathbb{k}Q_A/I$.

À l'origine, la connexité simple d'une algèbre de représentation finie a été définie en termes de la "condition de séparation", au sens de [11], ou encore des revêtements de Galois du carquois d'Auslander-Reiten associé à A , voir [13]. À présent, une algèbre

triangulaire est dite *simplement connexe* si, lorsque $A \simeq \mathbb{k}Q_A/I$, le groupe fondamental $\pi_1(Q, I)$ est trivial. Les algèbres simplement connexes ont joué un rôle fondamental dans la théorie des représentations des algèbres de dimension finie. Moyennant des techniques utilisant les revêtements, l'étude de la catégorie des représentations d'une telle algèbre peut être ramenée à l'étude des représentations d'une algèbre simplement connexe (souvent de dimension infinie) convenablement choisie; voir [26]. En fait, même dans le cas de représentation infinie, ces techniques permettent d'obtenir une grande quantité d'informations; voir [21].

D'autre part, il y a des liens entre les groupes fondamentaux $\pi_1(Q, I)$ des présentations d'une algèbre A , le premier groupe de cohomologie simpliciale de A , que l'on note $\mathrm{SH}_1(A)$, ainsi que son premier groupe de cohomologie de Hochschild, noté $\mathrm{HH}^1(A)$; voir [13, 4, 42]. Ces liens sont les meilleurs possibles dans le cas des algèbres d'incidence $A = \mathbb{k}Q_\Sigma/I_\Sigma$ d'ensembles ordonnés $(\Sigma, \leq)$. En effet, dans ce cas, le groupe fondamental, ainsi que les groupes de cohomologie (simpliciale et de Hochschild) coïncident avec le groupe fondamental et le premier groupe de cohomologie du complexe simplicial des chaînes non vides de Σ .

Une caractéristique importante du groupe fondamental ainsi que des groupes de cohomologie simpliciale est qu'ils ne sont pas des invariants de l'algèbre: il peut arriver (voir Section 7) que $\nu_1 : \mathbb{k}Q_A \rightarrow A$ et $\nu_2 : \mathbb{k}Q_A \rightarrow A$ soient deux présentations différentes de la même algèbre, de sorte que $A \simeq \mathbb{k}Q_A/\mathrm{Ker} \nu_1 \simeq \mathbb{k}Q_A/\mathrm{Ker} \nu_2$, mais $\mathrm{SH}_1(A_{\nu_1}) \not\simeq \mathrm{SH}_1(A_{\nu_2})$, et $\pi_1(Q_A, \mathrm{Ker} \nu_1) \not\simeq \pi_1(Q_A, \mathrm{Ker} \nu_2)$.

Ceci mène à considérer les carquois liés, plutôt que les algèbres, lorsqu'il s'agit d'étudier l'homotopie, ou la (co)homologie simpliciale. Ainsi, dans ces notes, nous nous concentrons sur les carquois liés, plus particulièrement sur les aspects homotopiques les considérant comme un sujet d'étude à part entière. Les références aux algèbres, aux modules ou aux théories de cohomologie qui leur sont associées sont données de façon très brève et surtout à des fins de motivation ou pour des raisons historiques.

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