

STRATIFICATIONS OF ALGEBRAS WITH RADICAL SQUARE ZERO

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RÉSUMÉ. Nous étudions les stratifications des algèbres de radical carré nul. Nous montrons qu'une telle algèbre est standardement stratifiée pour un certain ordre si et seulement si elle est quasi-triangulaire. Nous montrons également que dans ce cas, il existe un ordre tel que $F(\Delta) = Proj$ et un ordre tel que $F(\Delta) = P^{<\infty}$.

ABSTRACT. We study the stratifications of algebras with radical square zero. We show that an algebra with radical square zero is standardly stratified in some order if and only if it is quasi-triangular. We also show that for such algebras there is an order such that $F(\Delta) = Proj$ and also an order such $F(\Delta) = P^{<\infty}$.

1. Introduction. In this paper all algebras are associative, finite dimensional, basic and indecomposable K -algebras, where K is an algebraically closed field. A theorem of Gabriel states that all such algebras are, up to isomorphism, of the form $\Lambda = \frac{KQ}{I}$, where Q is a finite quiver and I an admissible ideal.

Let $v_1, \dots, v_n$ be the vertices of Q in a fixed order, $S_1, \dots, S_n$ be the corresponding order of simple modules, and P_i the projective cover of S_i . The *standard module* Δ_i is defined as the maximal quotient of P_i with composition factors S_j , with $j \leq i$; see [5].

Given an order on the set of the vertices $\{v_1, \dots, v_n\}$, take $\Delta = \{\Delta_1, \dots, \Delta_n\}$ and consider $F(\Delta)$, the full subcategory of $\text{mod } \Lambda$ whose objects are the modules $M \in \text{mod } \Lambda$ having a filtration with factors in Δ , namely there is a filtration $0 = M_0 \subset M_1 \subset \dots \subset M_t = M$, with $\frac{M_i}{M_{i-1}} \in \Delta$ for each i . The modules in $F(\Delta)$ will be called the *good modules*. Because of the connection with the study of some subcategories in the category of modules over a Lie Algebra, these modules are also called *Verma modules* by some authors. The algebra Λ is called *standardly stratified* if $\Lambda \in F(\Delta)$, or equivalently if every projective module is a good module.

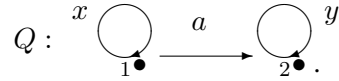
In [2], we showed that an algebra is standardly stratified in all orders if and only if it is an algebra for which all idempotent ideals are projective. In that case, we also proved that there is always an order where the good modules are exactly the projective modules, and an order where the good modules are the modules of finite projective dimension. It is known that if an algebra is standardly stratified then these are the possible extreme cases, since the category of good modules, in any order, always contains the projective modules and is always contained in the category of modules of finite projective dimension; see [6].

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The algebras with radical square zero which are standardly stratified are very special. The result presented in this paper implies that if an algebra with radical square zero is standardly stratified in some order, then it is *quasi-triangular*, that is every oriented cycle on its quiver is a sequence of loops. As a consequence, we get that if a radical square zero algebra is such that all its idempotent ideals are projective, then any path of length two in the quiver starts with a loop, and the vertex attached to this loop is a source in the quiver obtained after dropping the loop. We also get that if an algebra with radical square zero is standardly stratified in some order then there is an order where the good modules are the projective modules, and an order where the good modules are modules of finite projective dimension. Another consequence of our work is that an algebra with radical square zero with no simple projective modules has finitistic dimension zero, that is all non-projective modules have infinite projective dimension.

2. Algebras with radical square zero and stratification. Throughout this section, all algebras are of the form KQ/J^2 where Q is a finite connected quiver and J is the ideal generated by arrows.

Example 2.1. Consider the radical square zero algebra whose quiver Q is the following:



In this example, the algebra is standardly stratified in the order $(2, 1)$, but it is not in the order $(1, 2)$. Observe that in this case the only modules with finite projective dimension are the projective modules. Also, with the order $(2, 1)$, the good modules are the projective modules.

The following lemma is a particular case of Corollary 1.4 in [3], in the case where the oriented cycle is a product of loops. We include its proof for convenience.

Lemma 2.2. Let $\Lambda = \frac{KQ}{J^2}$, x be a vertex of the quiver Q and S_x be the corresponding simple module. If there is an oriented cycle that contains x , then $\text{pd } S_x = \infty$.

Proof. Since $\Lambda = \frac{KQ}{J^2}$ is a Koszul algebra, each simple S_0 has the following minimal projective resolution:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow S_0 \longrightarrow 0,$$

where $P_n = \Lambda \otimes_{\Lambda_0} \langle \text{paths of length } n \text{ ending at } v_0 \text{ in } KQ \rangle$ and Λ_0 is the semisimple subalgebra of Λ generated by the idempotents associated to the vertices of Q ; see [1]. Therefore, if $\text{pd } S_0 < \infty$, there exists t such that $P_t = 0$, and hence there is no path of length greater than or equal to t ending in v_0 . In particular, there is no oriented cycle containing v_0 . $\square$

We recall that $\Lambda = KQ/I$ is quasi-triangular if and only if any oriented cycle on Q is a product of loops.

Lemma 2.3. Let $\Lambda = \frac{KQ}{J^2}$, and assume that Q has an oriented cycle which is not a product of loops. Let $S_1, \dots, S_m$ be an order on the simple modules. Then there exists a vertex a_1 such that $\text{pd } \Delta(a_1) = \infty$.

Proof. Let $S_1, \dots, S_m$ be the given order and $a_1, \dots, a_n$, for $n > 1$, be the vertices of an oriented cycle. Without loss of generality we can assume that $a_1 < a_2$ and that there exists an arrow from a_1 to a_2 . Since Λ has radical square zero, all modules have Loewy length at most two. We then have the exact sequence

$$0 \longrightarrow \coprod_{\substack{a_1 \xrightarrow{\quad} a \\ a_1 < a}} S_a \longrightarrow P_{a_1} \longrightarrow \Delta_{a_1} \longrightarrow 0.$$

From this exact sequence, it follows that $\text{pd } \Delta_{a_1} < \infty$ if and only if $\text{pd } S_a < \infty$ for all $a_1 < a$ such that there is an arrow from a_1 to a . In particular, if $\text{pd } \Delta_{a_1} < \infty$, then $\text{pd } S_{a_2} < \infty$ and the result follows from the previous lemma. $\square$

We can now prove the following result. For an algebra Λ , we let, here and in the sequel, Proj denote the subcategory of projective modules. We also denote by $P^{<\infty}$ the subcategory of modules of finite projective dimension.

Proposition 2.4. *Let $\Lambda = \frac{KQ}{J^2}$. The following conditions are equivalent:*

- (1) *There is an order where $F(\Delta) = \text{Proj}$;*
- (2) *The algebra is quasi-triangular;*
- (3) *Λ is standardly stratified in some order.*

Proof. The equivalence between (1) and (2) is true in general, without the hypothesis of radical square zero; see for instance [4, 2]. It is also clear that (1) implies (3). It remains to show that (3) implies (2). As stated in the introduction, if an algebra is standardly stratified in some order, then the category of good modules is contained $P^{<\infty}$. In particular, all standard modules Δ_i would have finite projective dimension [6], but this contradicts Lemma 2.3. $\square$

Example 2.5. Let $\Lambda = \frac{KQ}{J^2}$, where Q is the quiver $\bullet_1 \rightrightarrows \bullet_2 \rightarrow \bullet_3$, and J is the ideal generated by the arrows. Our result implies that Λ is not standardly stratified in any order.

Proposition 2.4 shows that if a radical square zero algebra is standardly stratified in some order, then it is quasi-triangular and there is at least one order where the good modules are the projective modules. For such algebras we always fix an order with this property, that we shall call an *order where the good modules are the projective modules*.

Since, for an algebra which is standardly stratified, all good modules have finite projective dimension, the following question is now natural.

Question: *Is there some order such that $F(\Delta) = P^{<\infty}$?*

This is a desirable fact, among other reasons, because it implies that the category of modules of finite projective dimension is a functorially finite resolving category. We answer this question now.

In the first place observe that if there exists some order on the vertices such that $F(\Delta) = P^{<\infty}$, it is clear that the algebra is standardly stratified in this order. Then, by the previous proposition, Q has no oriented cycle which is not a product of loops. This provides a necessary condition, which we will show to be also sufficient.

We assume from now on that our quivers admit no oriented cycles which are not a product of loops. Observe that in this case Λ has finite global dimension if and only if Q has no loops by Lemma 2.2, and this happens if and only if Λ is a triangular algebra.

We continue our analysis dividing our classes of algebras in three cases:

Case 1: The algebra has finite global dimension. The quiver of Λ has no loops if and only if $\text{gl.dim. } \Lambda < \infty$. In this case, if we take the reverse of an order for which the good modules are the projective modules, we see that all $\Delta(i)$ are simple and $F(\Delta) = P^{<\infty} = \text{mod } \Lambda$.

We consider now only cases where the algebra has infinite global dimension. Under our hypothesis, this is the same as assuming that there is at least one loop in the quiver.

Case 2: No projective module is simple and the quiver has at least one loop. Assume now that there are no projective simple modules. Observe that this happens if and only if there is no sink in the quiver. Using that $\text{rad}^2 \Lambda = 0$ and considering an order such that the good modules are projective, we see that the following property holds: in each Δ_i there is a path which starts at i , is not in J^2 , and such that the product with every non-trivial path is in the ideal J^2 . We now invoke Theorem 2.5 of [4], and we get that $F(\Delta) = \text{Proj} = P^{<\infty}$.

We stop our analysis for a moment, in order to state the following consequence of our argument.

Corollary 2.6. *Let $\Lambda = KQ/J^2$ be such that there are no simple projective modules, and any cycle in Q is a product of loops. Then the projective modules are the only Λ -modules of finite projective dimension.*

The only case which remains to be considered is the following:

Case 3: There is at least one simple projective module and the quiver has at least one loop. We show that also in this case there exists an order such that $F(\Delta) = P^{<\infty}$. This is done by induction on the number of vertices. Note that our hypothesis implies that the quiver has at least two vertices. Assume that Q has two vertices. Then it is easy to see that under our hypothesis Λ is such that all its idempotent ideals are projective, and therefore Λ has an order such that $F(\Delta) = P^{<\infty}$; see [2].

Let now $\Lambda = \frac{KQ}{J^2}$ and v be a sink on the quiver. Consider $\Lambda' = \frac{\Lambda}{\langle v \rangle}$, namely $\Lambda' = \frac{KQ'}{(J')^2}$, where Q' is the quiver obtained from Q by deleting the vertex v and J' is the ideal of KQ' generated by the arrows in Q' .

We have a functor which associates to each $M \in \text{mod } \Lambda$ the module $\frac{M}{\langle v \rangle M} \in \Lambda'$. We now prove that

$$\text{pd}_\Lambda M < \infty \iff \text{pd}_{\Lambda'} \frac{M}{\langle v \rangle M} < \infty.$$

First, assume that $\text{pd}_\Lambda M < \infty$. We show that $\text{pd}_{\Lambda'} \frac{M}{\langle v \rangle M} < \infty$. Note that there is an exact sequence

$$0 \rightarrow \langle v \rangle M \rightarrow M \rightarrow \frac{M}{\langle v \rangle M} \rightarrow 0.$$

Also, $\langle v \rangle M \simeq S_v^{\dim \langle v \rangle M}$ which is projective because v is a sink. Since we have $\frac{M}{\langle v \rangle M} \simeq \frac{\Lambda}{\langle v \rangle} \otimes_\Lambda M$, we get a natural isomorphism of functors of Λ' -modules

$$\text{Ext}_{\Lambda'}^i \left(\frac{\Lambda}{\langle v \rangle} \otimes_\Lambda M, - \right) \simeq \text{Ext}_\Lambda^i (M, -).$$

We will see now that if $M \in \text{mod } \Lambda'$, then $\text{pd}_{\Lambda'} M < \infty$ implies $\text{pd}_\Lambda M < \infty$. We prove this fact by induction on $\text{pd}_{\Lambda'} M$. Assume first that M is indecomposable

projective, that is $M = \Lambda'e_w$, with w a vertex of Q . We have two possibilities: w is a neighbor of v or not. If w is not a neighbor of v , then $\Lambda'e_w = \Lambda e_w$, and thus $\text{pd}_\Lambda M = 0$. If w is a neighbor of v , we have the sequence

$$0 \rightarrow S_v^t \rightarrow \Lambda e_w \rightarrow \Lambda'e_w \rightarrow 0.$$

Since S_v^t is a Λ -projective module, it follows that $\text{pd}_\Lambda \Lambda'e_w = 1$. Let us suppose now that $\text{pd}_{\Lambda'} M = n < \infty$. We have the exact sequence

$$0 \rightarrow \Omega_{\Lambda'}(M) \rightarrow P_{\Lambda'}(M) \rightarrow M \rightarrow 0.$$

It follows from $\text{pd}_{\Lambda'} \Omega_{\Lambda'}(M) < n$, and the induction hypothesis, that $\text{pd}_\Lambda \Omega_{\Lambda'}(M)$ is finite. Also, since $P_{\Lambda'}(M)$ is a Λ' -projective module, we have that $\text{pd}_\Lambda P_{\Lambda'}(M)$ is finite, and therefore $\text{pd}_\Lambda M$ is finite.

We collect the above observations in the following lemma.

Lemma 2.7. *The following conditions are equivalent:*

- (1) $\text{pd}_\Lambda M$ is finite;
- (2) $\text{pd}_\Lambda \frac{M}{\langle v \rangle M}$ is finite;
- (3) $\text{pd}_{\Lambda'} \frac{M}{\langle v \rangle M}$ is finite.

Thus $P^{<\infty}(\Lambda') \subset P^{<\infty}(\Lambda)$ and v is a sink. Therefore P_v is simple and $\Delta_v = P_v = S_v$.

Assume now that there is an order on the simple Λ' -modules $S_1, \dots, S_t$ such that $F(\Delta) = P^{<\infty}(\Lambda')$. We extend this order to an order on the simple modules in $\text{mod } \Lambda$, declaring the simple module associated to the vertex v as the last element, that is the order is $S_1, \dots, S_t, S_v$.

One sees that for this order we have $\Delta_i(\Lambda) \simeq \Delta_i(\Lambda')$ for $i = 1, \dots, t$. Let now M be a Λ -module such that $\text{pd}_\Lambda(M) < \infty$ and consider the canonical exact sequence

$$0 \rightarrow \langle v \rangle M \rightarrow M \rightarrow \frac{M}{\langle v \rangle M} \rightarrow 0.$$

Clearly,

$$\langle v \rangle M = S_v^{\dim \langle v \rangle M} = P_v^{\dim \langle v \rangle M} = \Delta_v^{\dim \langle v \rangle M} \in F_\Lambda(\Delta).$$

Moreover $\frac{M}{\langle v \rangle M}$ has finite projective dimension as a Λ' -module. Consequently $\frac{M}{\langle v \rangle M} \in F_{\Lambda'}(\Delta)$. Finally, using that $\Delta_i(\Lambda) \simeq \Delta_i(\Lambda')$ for $i = 1, \dots, t$, we get that $\frac{M}{\langle v \rangle M} \in F_\Lambda(\Delta)$. As a consequence, we get $M \in F_\Lambda(\Delta)$.

We collect our observations in the following proposition:

Proposition 2.8. *Let $\Lambda = \frac{KQ}{J^2}$. The following statements are equivalent:*

- (1) *There is an order such that $F(\Delta) = P^{<\infty}$;*
- (2) *Every cycle in Q is a sequence of loops;*
- (3) *Λ is quasi-triangular.*

If we put together the statements of Proposition 2.4 and Proposition 2.8, we have the following theorem.

Theorem 2.9. *Let $\Lambda = \frac{KQ}{J^2}$. The following conditions are equivalent:*

- (1) *There is an order where $F(\Delta) = \text{Proj}$;*
- (2) *There is an order where $F(\Delta) = P^{<\infty}$;*

- (3) Q has no cycle which is not a sequence of loops;
- (4) Λ is standardly stratified in some order.

Résumé substantiel en français. Dans cet article, nous présentons quelques résultats sur les K -algèbres associatives, sobres, connexes, indécomposables et de dimension finie sur un corps algébriquement clos K . En vertu d'un théorème fondamental dû à Gabriel, ces algèbres sont, à isomorphisme près, de la forme $\Lambda = KQ/I$, où Q est un carquois fini et I est un idéal admissible de KQ . Pour cette raison, toutes les algèbres considérées dans cet article sont supposées être de cette forme. Nous décrivons maintenant les résultats obtenus.

L'objectif de cet article est de classifier toutes les algèbres de radical carré nul admettant un ordre sur l'ensemble de leurs modules simples et pour lequel l'algèbre est standardement stratifiée. Rappelons la définition de telles algèbres:

Définitions. Soient $v_1, \dots, v_n$ les sommets de Q dans un ordre fixé, $S_1, \dots, S_n$ l'ordre correspondant sur les modules simples, et P_i la couverture projective de S_i . Le *module standard* Δ_i est, par définition, le quotient maximal de P_i admettant les modules simples S_j , avec $j \leq i$, comme uniques facteurs de composition.

Soit $\Delta = \{\Delta_1, \dots, \Delta_n\}$, et soit $F(\Delta)$ la sous-catégorie pleine de $\text{mod}\Lambda$ ayant comme objets les modules M possédant une filtration ayant ses facteurs de composition dans Δ , c'est-à-dire une filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_t = M$$

telle que $M_i/M_{i-1} \in \Delta$ pour tout i . Les modules appartenant à $F(\Delta)$ sont appelés les *bons modules*.

L'algèbre Λ est dite *standardement stratifiée* (pour l'ordre fixé) si $\Lambda \in F(\Delta)$.

Dans [2], nous montrons qu'une algèbre est standardement stratifiée pour tous les ordres possibles si et seulement si tous ses idéaux idempotents sont projectifs. Dans ce cas, nous montrons également qu'il existe toujours un ordre pour lequel les bons modules coïncident avec les modules projectifs, ainsi qu'un ordre pour lequel les bons modules sont les modules de dimension projective finie. Il est connu que ces deux cas sont extrémaux puisque la catégorie des bons modules relativement à un ordre donné contient toujours l'ensemble des modules projectifs, et est toujours formée de modules de dimension projective finie; voir [6].

Les algèbres de radical carré nul standardement stratifiées sont très particulières. En effet, les résultats contenus dans cet article impliquent notamment que si une algèbre de radical carré nul est standardement stratifiée pour un ordre quelconque, alors elle est quasi-triangulaire, c'est-à-dire que tout cycle orienté dans son carquois est une composition de boucles.

Dans l'Exemple 2.1, nous montrons qu'une algèbre de radical carré nul peut parfois n'admettre qu'un unique ordre pour lequel elle est standardement stratifiée. Pour sa part, le Lemme 2.2 montre que pour une algèbre de radical carré nul, s'il existe une boucle rattachée à un certain sommet du carquois, alors le module simple correspondant à ce sommet est de dimension projective infinie.

Nous avons également la proposition suivante. Dans ce qui suit, Proj représente la sous-catégorie formée des modules projectifs.

Proposition 2.4. *Soit Λ une algèbre de radical carré nul. Les conditions suivantes sont équivalentes:*

- (1) *Il existe un ordre pour lequel $F(\Delta) = \text{Proj}$;*
- (2) *L'algèbre Λ est quasi-triangulaire;*
- (3) *Il existe un ordre pour lequel Λ est standardement stratifiée.*

Nous déduisons de cette proposition le résultat suivant.

Corollaire 2.6. *Soit Λ une algèbre de radical carré nul, et soit Q son carquois associé. Supposons qu'il n'existe aucun module simple projectif et que tout cycle orienté de Q est une composition de boucles. Alors un module est projectif si et seulement s'il est de dimension projective finie.*

Notre dernier résultat est le suivant, dans lequel on note $P^{<\infty}$ la sous-catégorie formée des modules de dimension projective finie.

Théorème 2.9. *Soit Λ une algèbre de radical carré nul, et soit Q son carquois associé. Les conditions suivantes sont équivalentes:*

- (1) *Il existe un ordre pour lequel $F(\Delta) = \text{Proj}$;*
- (2) *Il existe un ordre pour lequel $F(\Delta) = P^{<\infty}$;*
- (3) *Tous les cycles orientés de Q , s'il y en a, sont des compositions de boucles;*
- (4) *Il existe un ordre pour lequel Λ est standardement stratifiée.*

Une conséquence immédiate des deux derniers énoncés est que si Λ est une algèbre de radical carré nul n'admettant aucun module simple projectif, alors la dimension finitiste de Λ est zéro, puisque dans ce cas tout module non-projectif est de dimension projective infinie.

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