

COSEMISIMPLE COALGEBRAS

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RÉSUMÉ. Le but de cet article est de présenter la cosemisimplicité des $\mathbb{K}$ -cogèbres (où $\mathbb{K}$ est un corps) de façon intrinsèque, sans tenir compte des propriétés correspondantes pour les algèbres. Nous étudions aussi les catégories de $\circ$ -cogèbres et de modules cointaires et nous exprimons la définition de CQG (groupe quantique compact) en termes de $\circ$ -cosemisimplicité.

ABSTRACT. We give an intrinsic approach to cosemisimplicity in coalgebras without invoking the corresponding properties of algebras. We study the categories of $\circ$ -coalgebras and unitary comodules, and explain the definition of CQG in terms of $\circ$ -cosemisimplicity.

1. Introduction. This article has two purposes, the first is to give an intrinsic approach to cosemisimplicity and the second is to clarify the definition of compact quantum group.

The standard method for study cosemisimplicity in coalgebras is to reduce the problem to semisimplicity in algebras by properties of finiteness and duality. This is the method used, for example, in [11], [1] and [9]. Even more recent books like [7] and [10] propose this approach. Here we give direct proofs of these results without invoking the corresponding properties of algebras.

The definition of compact quantum group (CQG) was first given by S. L. Woronowicz [13], and it was later redefined in a purely algebraic way independently by N. Andruskiewitsch [4], and M. Dijkhuizen and T. Koornwinder [5]; see also [6]. In [3] the definition of CQG was given by means of $\circ$ -coalgebras. In this article we study the categories of $\circ$ -coalgebras and unitary comodules, and explain the definition of CQG in terms of $\circ$ -cosemisimplicity. For us, the approach to CQG from $\circ$ -coalgebras in [3] is more natural than the approach from $\ast$ -Hopf algebras in [5] and [4], because we do not need the antipode in the definition of unitary comodule. We can indeed prove many results of compact quantum groups for compact quantum semigroups; see for instance Propositions 6.5 and 6.6.

The eminent part of the definition of a CQG is concentrated in its coalgebra structure, so we include a section on $\circ$ -coalgebras in which we study their main properties.

In this paper, we work over a fixed field $\mathbb{K}$ and use a simplified Sweedler's notation for the comultiplication and coaction: $\Delta(x) = x_{(0)} \otimes x_{(1)}$ and $\delta(v) = v_{(0)} \otimes v_{(1)}$. All

comodules over a coalgebra are right comodules unless specified explicitly. We consider a coalgebra as a comodule over itself via the comultiplication, this is the *regular corepresentation*. A coalgebra is called *simple* if it is not zero and it has no proper subcoalgebras, and called *cosemisimple* if it is a direct sum of simple subcoalgebras. A comodule is called *irreducible* if it is not zero and it has no proper subcomodules, and called *completely reducible* if it is a direct sum of simple subcomodules. Let V be a completely reducible comodule and $\{M_i \mid i \in I\}$ a set of representatives of the isomorphism classes of irreducible C -comodules. For each $i \in I$, let V_i be the sum of all subcomodules of V that are isomorphic to M_i . We call V_i the *isotypical component of type M_i* of V . It is well-known that $V = \bigoplus_{i \in I} V_i$. If V and W are C -comodules, we denote by $\text{Hom}^C(V, W)$ the space of C -comodule maps from V to W , and by $\text{End}^C(V)$ the space $\text{Hom}^C(V, V)$.

2. Coalgebras and comodules. In this section we recall some well-known properties of coalgebras and comodules.

Definition 2.1. Let V be a finite dimensional vector space, $\{e_1, \dots, e_m\}$ be a basis of V , and $\{e^1, \dots, e^m\}$ be its dual basis in V^* . The *coendomorphism coalgebra* of V is $C_V = V^* \otimes V$ with comultiplication and counit defined by

$$\Delta(\alpha \otimes v) = \sum_{i=1}^m (\alpha \otimes e_i) \otimes (e^i \otimes v) \quad \text{and} \quad \langle \varepsilon, \alpha \otimes v \rangle = \langle \alpha, v \rangle.$$

The *matrix coalgebra* $M_m(\mathbb{K})^c$ is $M_m(\mathbb{K})$ as vector space with coalgebra structure given by $\Delta(e_{ij}) = \sum_{k=1}^m e_{ik} \otimes e_{kj}$ and $\langle \varepsilon, e_{ij} \rangle = \delta_{ij}$, where $\{e_{ij} \mid 1 \leq i, j \leq m\}$ is the canonical basis of $M_m(\mathbb{K})$.

It is well-known that C_V is a simple coalgebra isomorphic to $M_m(\mathbb{K})^c$. The vector space V , endowed with the coaction $\delta(v) = \sum_{i=1}^m e_i \otimes (e^i \otimes v)$, is an irreducible right C_V -comodule. Each $R_i = e^i \otimes V \subset C_V$, with $i = 1, \dots, m$, is an irreducible right coideal isomorphic to V (as C_V -comodules) such that $C_V = \bigoplus_{i=1}^m R_i$.

The following proposition relates comodules and coalgebra maps.

Proposition 2.2. [12] *Let V be a finite dimensional vector space and C be a coalgebra. For any coalgebra map $\phi : C_V \rightarrow C$, we define a right C -coaction $\delta : V \rightarrow V \otimes C$ on V by*

$$\delta(v) = \sum_{i=1}^m e_i \otimes \phi(e^i \otimes v)$$

for all $v \in V$. Conversely, for any right coaction $\delta : V \rightarrow V \otimes C$ we define a coalgebra map $\phi : C_V \rightarrow C$ by

$$\phi(\alpha \otimes v) = (\alpha \otimes \text{id})\delta(v)$$

for all $\alpha \in V^*$ and $v \in V$.

If V is a finite dimensional C -comodule, the corresponding map $\phi : C_V \rightarrow C$ is called the *associated coalgebra map*.

Remark 2.3. We recall that the coalgebra C is a cogenerator in the category of C -comodules, that is every C -comodule is isomorphic to a subcomodule of a direct sum of copies of C .

3. Cosemisimplicity.

Let C be a coalgebra.

Theorem 3.1. [10] *The following assertions are equivalent:*

- (1) C is cosemisimple;
- (2) C is completely reducible via the regular corepresentation;
- (3) Every C -comodule is completely reducible;
- (4) C is a sum of simple subcoalgebras;
- (5) If D is any subcoalgebra of C , then there exists a subcoalgebra E of C such that $C = D \oplus E$;
- (6) Every subcoalgebra of C is cosemisimple;
- (7) Every finite dimensional subcoalgebra of C is cosemisimple.

Let C be a coalgebra and V be a C -comodule. We consider the space of *matrix coefficients* of V defined by

$$\text{Coeff}(V) := \mathbb{K} \{ \langle \alpha, v_{(0)} \rangle v_{(1)} \mid \alpha \in V^*, v \in V \} \subset C.$$

We have that $\text{Coeff}(V)$ is the smallest subcoalgebra of C such that V is a $\text{Coeff}(V)$ -comodule. If V is finite dimensional, then $\text{Coeff}(V) = \text{Im}(\phi)$, where $\phi : C_V \rightarrow C$ is the associated coalgebra map.

Proposition 3.2. *Let V be a C -comodule.*

- (1) *If $V \neq 0$, then $\text{Coeff}(V) \neq 0$.*
- (2) *If U is a subcomodule of V , then $\text{Coeff}(U) \subset \text{Coeff}(V)$.*
- (3) *For each $i \in I$, let V_i be a subcomodule of V . Then $\text{Coeff}(\sum_{i \in I} V_i) = \sum_{i \in I} \text{Coeff}(V_i)$.*

Proof. (1) If $\text{Coeff}(V) = 0$, then $\delta(v) = 0$ for all $v \in V$. Hence $V = 0$.

(2) Let $u \in U$ and $\delta(u) = \sum_{i=1}^n u_i \otimes c_i$, where $u_1, \dots, u_n$ are linearly independent. For each $i = 1, \dots, n$ there exists $\alpha_i \in V^*$ such that $\alpha_i(u_j) = \delta_{ij}$ (the Kronecker delta). Hence $c_i = \sum_{j=1}^n \alpha_i(u_j) c_j \in \text{Coeff}(V)$, for all $i = 1, \dots, n$.

(3) The inclusion $\sum_{i \in I} \text{Coeff}(V_i) \subset \text{Coeff}(\sum_{i \in I} V_i)$ follows immediately from (2). For each $i \in I$, we get $\delta(V_i) \subset V_i \otimes \text{Coeff}(V_i)$, hence for each finite subset $L \subset I$ we get $\delta(\sum_{i \in L} V_i) \subset \sum_{i \in L} V_i \otimes \text{Coeff}(V_i) \subset (\sum_{i \in L} V_i) \otimes \sum_{i \in L} \text{Coeff}(V_i)$. This implies $\delta(\sum_{i \in I} V_i) \subset (\sum_{i \in I} V_i) \otimes \sum_{i \in I} \text{Coeff}(V_i)$. Hence the minimality of Coeff implies $\text{Coeff}(\sum_{i \in I} V_i) \subset \sum_{i \in I} \text{Coeff}(V_i)$. $\square$

The next proposition gives a necessary condition for two comodules to be isomorphic.

Proposition 3.3. *Let V and W be C -comodules. If there exists a surjective C -comodule map $\phi : V \rightarrow W$, then $\text{Coeff}(W) \subset \text{Coeff}(V)$. In particular, if V and W are isomorphic as C -comodules, then $\text{Coeff}(V) = \text{Coeff}(W)$.*

Proof. Let $w \in W$, $\beta \in W^*$ and $v \in V$ be such that $\phi(v) = w$. It follows that $\langle \beta, w_{(0)} \rangle w_{(1)} = \langle \beta, \phi(v_{(0)}) \rangle v_{(1)} = \langle {}^t\phi(\beta), v_{(0)} \rangle v_{(1)}$, where ${}^t\phi : W^* \rightarrow V^*$ is the transpose of ϕ . Hence $\text{Coeff}(W) \subset \text{Coeff}(V)$. $\square$

Proposition 3.4. *Let C be a non-zero coalgebra. The following assertions are equivalent:*

- (1) C is simple;
- (2) For every non-zero C -comodule V , we get $\text{Coeff}(V) = C$;

- (3) *There exists an irreducible C -comodule V such that $\text{Coeff}(V) = C$;*
- (4) *C is finite dimensional and completely reducible via the regular corepresentation with only one isotypical component.*

Proof. First, (1) implies (2) since if V is a non-zero C -comodule, then $\text{Coeff}(V)$ is a non-zero subcoalgebra of the simple coalgebra C . Hence $\text{Coeff}(V) = C$.

Also, (2) implies (3) because every non-zero comodule has an irreducible subcomodule and there exists non-zero comodules (C itself is a non-zero comodule).

We now show that (3) implies (4). Indeed, let $C_V = \bigoplus_{i=1}^m R_i$ be as before. If we define a C -comodule structure on R_i such that $V \simeq R_i$ is a C -comodule isomorphism, then the associated coalgebra map $\phi : C_V \rightarrow C$ becomes a surjective comodule map. So $C \simeq C_V / \text{Ker } \phi \simeq V^r$ for some $r \leq m$ by the complete reducibility of C_V . This shows (4).

Finally, assume that (4) holds, and let $C = \sum_{i=1}^n T_i$, where each T_i is a right coideal isomorphic to some irreducible C -comodule V . We have

$$C = \sum_{i=1}^n T_i \subset \sum_{i=1}^n \text{Coeff}(T_i) = \text{Coeff}(V) \subset C,$$

hence $C = \text{Coeff}(V)$. Let D be a non-zero subcoalgebra of C . Then D is a right coideal, so by the complete reducibility of C we get that $D = \sum_{j=1}^m S_j$, where each S_j is a right C -coideal isomorphic to V as a C -comodule. Since D is a subcoalgebra of C , we get $C = \text{Coeff}(V) = \text{Coeff}(S_j) \subset D$, for all j . Hence $D = C$, and (1) holds as well. $\square$

Corollary 3.5. *There exists a bijection between the isomorphism classes of irreducible C -comodules and simple subcoalgebras of C , taking a class $[V]$ to $\text{Coeff}(V)$.*

Proof. The map $[V] \mapsto \text{Coeff}(V)$ is well-defined by the above propositions. The injectivity follows because if V is an irreducible comodule, then $\text{Coeff}(V)$ is a simple coalgebra and V is isomorphic to any irreducible right coideal of $\text{Coeff}(V)$. For the surjectivity, if D is a simple subcoalgebra of C and $R \subset D$ is an irreducible right coideal, then $D = \text{Coeff}(R)$. $\square$

The following shows the relation between the decomposition of a cosemisimple coalgebra as sum of simple subcoalgebras and sum of irreducible right coideals.

Proposition 3.6. *Let C be a cosemisimple coalgebra. If $C = \bigoplus_{i \in I} C_i$ is the decomposition of C in isotypical components, then $\{C_i \mid i \in I\}$ is the set of all simple subcoalgebras of C .*

Proof. Let $C = \bigoplus_{i \in I} C_i$ be the decomposition of C as sum of simple subcoalgebras. We have $C_i = \bigoplus_{j=1}^{m_i} R_j^i$, where the R_j^i 's are isomorphic irreducible right coideals. Each R_j^i is a non-zero right C_i -coideal, hence $\text{Coeff}(R_j^i) = C_i$ for all j , and $C = \bigoplus_{i \in I} C_i$ is the decomposition of C in isotypical components. Also, Corollary 3.5 implies that every simple subcoalgebra of C is C_i , for some i . $\square$

Corollary 3.7. *If $\{V_l \mid l \in I\}$ is a family of mutually non-isomorphic irreducible C -comodules, then $\sum_{l \in I} \text{Coeff}(V_l)$ is a direct sum.*

Our next objective is to show the relation between simple coalgebras and finite dimensional division algebras.

Proposition 3.8. *Let V be a completely reducible finite dimensional C -comodule. Then $\text{End}^C(V) \simeq \prod_{i=1}^k M_{n_i}(A_i)$ as algebras, where each A_i is a finite dimensional division algebra.*

Proof. Let $V = \bigoplus_{i=1}^k V_i^{n_i}$, where the V_i 's are mutually non-isomorphic irreducible subcomodules. Then by Schur's lemma for comodules we get

$$\text{End}^C(V) \simeq \prod_{i,j=1}^k \text{Hom}^C(V_i^{n_i}, V_j^{n_j}) \simeq \prod_{i=1}^k \text{End}^C(V_i^{n_i}) \simeq \prod_{i=1}^k M_{n_i}(A_i)$$

where $A_i = \text{End}^C(V_i)$. $\square$

Proposition 3.9. *There is an isomorphism of algebras $\text{End}^C(C) \simeq (C^{\text{cop}})^* = (C^*)^{\text{op}}$. In particular, if C is finite dimensional, then $C \simeq ((\text{End}^C(C))^{\text{op}})^*$ as coalgebras.*

Proof. The isomorphism $\text{End}^C(C) \simeq (C^{\text{cop}})^*$ is given by $\text{End}^C(C) \rightarrow (C^{\text{cop}})^*, \phi \mapsto \varepsilon \circ \phi$, with inverse $(C^{\text{cop}})^* \rightarrow \text{End}^C(C), \alpha \mapsto (\alpha \otimes \text{Id}) \circ \Delta$. $\square$

Remark 3.10. If A is an algebra, then $M_n(A)^{\text{op}} \simeq M_n(A^{\text{op}})$ as algebras via the transpose map.

Proposition 3.11. *Let C be a coalgebra. Then C is simple if and only if C is isomorphic to $M_n(A)^*$, where A is a finite dimensional division algebra.*

Proof. Necessity: Let $C = \bigoplus_{i=1}^m R_i$, where each R_i is a right coideal isomorphic to some irreducible C -comodule V . Then

$$C \simeq (\text{End}^C(C)^{\text{op}})^* \simeq (M_n(\text{End}^C(V))^{\text{op}})^* \simeq M_n(A)^*,$$

with $A = \text{End}^C(V)^{\text{op}}$.

Sufficiency: This is because there is a one-to-one correspondence between subcoalgebras of C and bilateral ideals of $M_n(A)$, and the fact that $M_n(A)$ is a simple algebra since A is a division algebra. $\square$

Remark 3.12. If $\mathbb{K}$ is algebraically closed, then, since $\text{End}^C(V) \simeq \mathbb{K}$ for every irreducible C -comodule V , every simple coalgebra C is isomorphic to a matrix coalgebra $M_m(\mathbb{K})^c$. In this situation, there exists a basis $\{t_{ij} \mid i, j = 1, \dots, m\}$ of C such that

$$\Delta(t_{ij}) = \sum_{k=1}^m t_{ik} \otimes t_{kj} \quad \text{and} \quad \langle \varepsilon, t_{ij} \rangle = \delta_{ij}.$$

Proposition 3.13. *Let $\mathbb{K}$ is an algebraically closed field, V be an irreducible C -comodule, $\{v_1, \dots, v_m\}$ be a basis of V , and $\delta(v_i) = \sum_{j=1}^m v_j \otimes t_{ji}$. Then the set $\{t_{ij} \mid i, j = 1, \dots, m\}$ is a basis of $\text{Coeff}(V)$.*

Proof. The C -comodule V is irreducible; this is equivalent to saying that V is irreducible as a $\text{Coeff}(V)$ -comodule. The subcoalgebra $\text{Coeff}(V)$ is simple and, since $\mathbb{K}$ algebraically closed, it is isomorphic to a matrix coalgebra $M_n(\mathbb{K})$. We have $M_n(\mathbb{K}) = \bigoplus_{i=1}^n R_i$ where each R_i is an irreducible right coideal of $M_n(\mathbb{K})$ isomorphic to $\mathbb{K}^n$. Since every irreducible C -comodule is isomorphic to an irreducible right coideal of C , we get $n = m$ and $\dim \text{Coeff}(V) = m^2$. Hence the set of generators $\{t_{ij} \mid i, j = 1, \dots, m\}$ is a basis. $\square$

4. $\circ$ -coalgebras. From now on we consider $\mathbb{K} = \mathbb{C}$ the field of complex numbers.

A $\circ$ -coalgebra is a pair $(C, \circ)$ such that C is a coalgebra and $\circ : C \rightarrow C$ is an anticomultiplicative conjugate-linear involution, that is a map verifying:

- (1) $x^{\circ\circ} = x$, for all $x \in C$;
- (2) $(\lambda x + y)^{\circ} = \bar{\lambda}x^{\circ} + y^{\circ}$, for all $x, y \in C$ and $\lambda \in \mathbb{C}$;
- (3) $(x^{\circ})_{(1)} \otimes (x^{\circ})_{(2)} = (x_{(2)})^{\circ} \otimes (x_{(1)})^{\circ}$, for all $x \in C$.

From these conditions we get $\langle \varepsilon, x^{\circ} \rangle = \overline{\langle \varepsilon, x \rangle}$, for all $x \in C$.

Remark 4.1. The concept of $\circ$ -coalgebra is the dual concept of a $*$ -algebra (see Remark 4.5) and was introduced in [3] with the purpose of obtaining new examples of compact quantum groups.

We say that a subcoalgebra D of a $\circ$ -coalgebra C is a $\circ$ -subcoalgebra if $x^{\circ} \in D$ for all $x \in D$. In this situation, D becomes a $\circ$ -coalgebra. If C is a $\circ$ -coalgebra and $D \subset C$ is a subcoalgebra then it is easy to see that $D^{\circ} = \{d^{\circ} \mid d \in D\}$ is a subcoalgebra and $D + D^{\circ}$ is a $\circ$ -subcoalgebra.

A $\circ$ -coalgebra map between $\circ$ -coalgebras C and $\tilde{C}$ is a coalgebra map $\phi : C \rightarrow \tilde{C}$ such that $\phi(x^{\circ}) = \phi(x)^{\circ}$ for all $x \in C$. We call $\circ$ -Coalg the category of all $\circ$ -coalgebras and $\circ$ -coalgebra maps.

Remark 4.2. Let C be a $\circ$ -coalgebra. It is easily seen that the intersection and sum of any family of $\circ$ -subcoalgebras of C is again a $\circ$ -subcoalgebra of C .

If x is an element of C , then x belongs to some finite dimensional subcoalgebra D . Hence x belongs to the finite dimensional $\circ$ -subcoalgebra $D + D^{\circ}$. This implies that the $\circ$ -subcoalgebra of C generated by a single element is finite dimensional.

We say that a $\circ$ -coalgebra C is a *simple $\circ$ -coalgebra* if it is a simple object in $\circ$ -Coalg, that is $C \neq 0$ and C has no proper $\circ$ -subcoalgebras. In the same way, we define a *simple $\circ$ -subcoalgebra*. We observe that a simple $\circ$ -coalgebra may not be simple as a coalgebra. Indeed we have the following.

Proposition 4.3. *Let C be a $\circ$ -coalgebra.*

- (1) *If C is a simple $\circ$ -coalgebra, then C is a simple coalgebra or there exists a simple subcoalgebra $D \subset C$ such that $C = D \oplus D^{\circ}$.*
- (2) *If $D \subset C$ is a simple subcoalgebra, then $D = D^{\circ}$ or $D \cap D^{\circ} = 0$, and $D \oplus D^{\circ}$ is a simple $\circ$ -subcoalgebra of C .*

Hence the simple $\circ$ -subcoalgebras of C have the form $D + D^{\circ}$ where D is a simple subcoalgebra of C .

Proof. (1) This follows because if $0 \neq D \subset C$ is a simple subcoalgebra then we have $C = D + D^{\circ}$.

(2) Suppose that $D \cap D^{\circ} = 0$. Let $E \subset D \oplus D^{\circ}$ be a $\circ$ -subcoalgebra and assume that $E \cap D = E \cap D^{\circ} = 0$. Note that E is stable under the left $(D \oplus D^{\circ})^*$ -module action given by

$$f \cdot e = e_{(1)} \langle f, e_{(2)} \rangle,$$

for $f \in (D \oplus D^{\circ})^*$ and $e \in E$. Denote by π_D and $\pi_{D^{\circ}}$ the canonical projections of $D \oplus D^{\circ}$ onto D and D° respectively. Then

$$\varepsilon \pi_D \cdot E = \varepsilon \cdot \pi_D(E) \subset E \cap D = 0 \quad \text{and} \quad \varepsilon \pi_{D^{\circ}} \cdot E \subset E \cap D^{\circ} = 0.$$

Hence $E = \varepsilon \cdot E = \varepsilon \pi_D \cdot E + \varepsilon \pi_{D^\circ} \cdot E = 0$. Thus any non-zero $\circ$ -subcoalgebra E of $D \oplus D^\circ$ must contain D or D° and hence equals $D \oplus D^\circ$. $\square$

Proposition 4.4.

- (1) Every simple $\circ$ -coalgebra is finite dimensional.
- (2) Every $\circ$ -coalgebra is a sum of finite dimensional $\circ$ -subcoalgebras.
- (3) Every non-zero $\circ$ -coalgebra contains a simple $\circ$ -subcoalgebra.
- (4) For a given $\circ$ -coalgebra C , the following assertions are equivalent:
 - (a) C is a sum of simple $\circ$ -subcoalgebras;
 - (b) C is the direct sum of its simple $\circ$ -subcoalgebras;
 - (c) If D is any $\circ$ -subcoalgebra of C , then there exists a $\circ$ -subcoalgebra E of C such that $C = D \oplus E$;
 - (d) If D is any $\circ$ -subcoalgebra of C and E is any $\circ$ -subcoalgebra of D , then there exists a $\circ$ -subcoalgebra F of D such that $D = E \oplus F$;
 - (e) C is cosemisimple.

Proof. (1), (2) and (3) follow from Remark 4.2. Concerning (4), the equivalence between (a) and (e) follows from Proposition 4.3. Moreover, we have the following implications:

(a) implies (b). Let $C = \sum_{i \in I} C_i$, where each C_i is a simple $\circ$ -subcoalgebra, and let $\mathcal{S} = \{K \subset I \mid \sum_{i \in K} C_i \text{ is a direct sum}\}$. By Zorn's lemma there exists a maximal subset J of I such that $\sum_{j \in J} C_j$ is a direct sum. Let $l \in I \setminus J$. Then $C_l \cap \sum_{j \in J} C_j$ is a $\circ$ -subcoalgebra of the simple $\circ$ -subcoalgebra C_l , and $C_l \cap \sum_{j \in J} C_j \neq 0$ because of the maximality of $\sum_{j \in J} C_j$. Then $C_l \subset \sum_{j \in J} C_j$ for all $l \in I \setminus J$ and $\sum_{j \in J} C_j = C$.

We now show that (b) implies (c). Let $0 \neq D$ be a $\circ$ -subcoalgebra of C and let $C = \bigoplus_{i \in I} C_i$, where each C_i is a simple $\circ$ -subcoalgebra. We consider $\mathcal{B} = \{K \subset I \mid D + \sum_{i \in K} C_i \text{ is a direct sum}\}$. By Zorn's lemma there exists a maximal subset J of I such that $D + \sum_{j \in J} C_j$ is a direct sum. As before we get $C = D + \sum_{j \in J} C_j$. Then we can take $E = \sum_{j \in J} C_j$.

Also, (c) implies (d). Indeed, assume that (c) holds. Then, since E becomes a $\circ$ -subcoalgebra of C , there exists a $\circ$ -subcoalgebra $\tilde{F}$ of C such that $E \oplus \tilde{F} = C$. Let $F = \tilde{F} \cap D$. Then F is a $\circ$ -subcoalgebra of D and $D = E \oplus F$, and (d) holds as well.

Finally, we show that (d) implies (a). Since C is a sum of finite dimensional $\circ$ -subcoalgebras, it is enough to prove that if C is finite dimensional and verifies (d) then C is a sum of simple $\circ$ -subcoalgebras.

If C is simple there is nothing to prove. Otherwise, there exists a $\circ$ -subcoalgebra $0 \neq C_1 \subsetneq C$. Hence there exists a $\circ$ -subcoalgebra F_1 of C such that $C = C_1 \oplus F_1$ where $\dim C_1 < \dim C$ and $\dim F_1 < \dim C$. The result follows by induction on the dimension. $\square$

Remark 4.5. If C is a $\circ$ -coalgebra, then the dual algebra C^* is a $*$ -algebra with respect to the involution $*$ given by

$$\langle \alpha^*, x \rangle = \overline{\langle \alpha, x^\circ \rangle},$$

with $\alpha \in C^*$, $x \in C$. This construction gives an isomorphism between the category of finite dimensional $\circ$ -coalgebras and the category of finite dimensional $*$ -algebras.

In case $C = M_m(\mathbb{C})^c$, we can identify C^* with $M_m(\mathbb{C})$ (with its usual algebra structure) via the bilinear form $\langle x, y \rangle = \text{tr}(x {}^t y)$, where ${}^t y$ the transpose of y . Then

$M_m(\mathbb{C})$ is a $*$ -algebra by defining $(x_{ij})^* = (\overline{x_{ji}})$, and $M_m(\mathbb{C})^c$ becomes a $\circ$ -coalgebra for the $\circ$ -structure $(x_{ij})^\circ = (x_{ij})^* = (\overline{x_{ji}})$.

Proposition 4.6. *Let ν be a $*$ -structure in $M_m(\mathbb{C})$, and $*$ be the usual $*$ -structure of $M_m(\mathbb{C})$, that is $(x_{ij})^* = (\overline{x_{ji}})$. Then there exists an invertible element u in $M_m(\mathbb{C})$ such that $u^\nu = u^* = u$ and $x^\nu = u x^* u^{-1}$ for all x in $M_m(\mathbb{C})$. Also,*

- (1) *All the eigenvalues of u have the same sign if and only if there exists an isomorphism of $*$ -algebras between $(M_m(\mathbb{C}), \nu)$ and $(M_m(\mathbb{C}), *)$.*
- (2) *If u has eigenvalues of distinct sign, then there does not exist any norm $\| \cdot \|$ in $M_m(\mathbb{C})$ such that $(M_m(\mathbb{C}), \nu, \| \cdot \|)$ is a C^* -algebra.*

Proof. Every algebra automorphism of $M_m(\mathbb{C})$ is inner (since the map $x \rightarrow (x^\nu)^*$ is an algebra automorphism), and then there exists an invertible element u_0 such that $x^\nu = (u_0^*)^{-1} x^* u_0^*$ for all x . Using $x^{\nu\nu} = x$ we get $u_0 = \lambda u_0^*$ for some λ in $\mathbb{C}$ with $|\lambda| = 1$. If we take $u = (\sqrt{\lambda} u_0^*)^{-1}$, then we have the first assertion. We now prove (1) and (2).

(1) Suppose that ϕ is an algebra automorphism of $M_m(\mathbb{C})$; it is of the form $\phi(x) = w x w^{-1}$ for some w . We have $\phi(x^*) = \phi(x)^\nu$ for all x if and only if there exists η in $\mathbb{R} \setminus 0$ such that $w w^* = \eta u$. Hence if there exists an isomorphism of $*$ -algebras between $(M_m(\mathbb{C}), \nu)$ and $(M_m(\mathbb{C}), *)$, then all the eigenvalues of u have the same sign. Conversely, if all the eigenvalues of u have the same sign, then it is easily seen that there exists w such that $w w^* = \pm u$.

(2) If $\| \cdot \|$ is a norm such that $(M_m(\mathbb{C}), \nu, \| \cdot \|)$ is a C^* -algebra, then for all a in $M_m(\mathbb{C})$ we have $\text{Spec}(a^\nu a) = \{\text{eigenvalues of } a^\nu a\} \subset [0, +\infty)$. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of u and suppose that $\lambda_1 \lambda_2 < 0$. There exists v such that $v^* v = \text{id}$ and $v u v^* = d$, where $d = \text{diag}(\lambda_1, \dots, \lambda_m)$. Let $a = (a_{ij})$ be defined by $a_{12} = 1$ and $a_{ij} = 0$ elsewhere. We have $d a^* d^{-1} a = v b^\nu b v^{-1}$, where $b = v^* a v$. Then $\text{Spec}(b^\nu b) = \text{Spec}(d a^* d^{-1} a) = \{0, \lambda_1^{-1} \lambda_2\} \not\subset [0, +\infty)$. $\square$

If C is a $\circ$ -coalgebra, a *unitary comodule* is a pair $(V, (\mid))$ where V is a C -comodule and $(\mid)$ is an inner product in V such that

$$(u_{(0)} | v) u_{(1)} = (u | v_{(0)}) (v_{(1)})^\circ$$

for all $u, v \in V$. It is clear that a subcomodule of a unitary comodule is also a unitary comodule.

Example 4.7. In $C = M_m(\mathbb{C})^c$, we define a $\circ$ -structure by $e_{ij}^\circ = e_{ji}$. Then C is a simple $\circ$ -coalgebra and $\mathbb{C}^m$ is an irreducible unitary comodule with the usual inner product.

Example 4.8. Let $C = C_1 \oplus C_2$, with $C_2^{\text{cop}} = C_1 = M_m(\mathbb{C})^c$. We consider $\{e_{ij}^l\}_{ij}$ the canonical basis of C_l , with $l = 1, 2$, and define a $\circ$ -structure in C by $(e_{ij}^1)^\circ = e_{ij}^2$. Then C is a simple $\circ$ -coalgebra but not a simple coalgebra. Let $\{e_1, \dots, e_m\}$ be the canonical basis of $\mathbb{C}^m$. We take $V_1 = \mathbb{C}^m$ with the C -comodule structure $\delta(e_i) = \sum_{j=1}^m e_j \otimes t_{ji}^1$, and $V_2 = \mathbb{C}^m$ with the C -comodule structure $\delta(e_i) = \sum_{j=1}^m e_j \otimes t_{ij}^2$. Then V_1 and V_2 are irreducible C -comodules and $\text{Coeff}(V_l) = C_l$, for $l = 1, 2$. We have $C \simeq V_1^m \oplus V_2^m$ as C -comodules. We observe that since $C_l^\circ = C_2$ and $C_2^\circ = C_1$, then V_l cannot be a unitary C -comodule for any inner product in V_l . Hence, since C is cosemisimple, there are no unitary C -comodules.

Remark 4.9. Let V be a finite dimensional vector space with a fixed inner product. The inner product in V induces a conjugated linear bijection $\varphi : V \rightarrow V^*$. If $\{v_1, \dots, v_m\}$ is an orthonormal basis in V , then $\varphi(v_i) = v^i$. We obtain a structure of $\circ$ -coalgebra on C_V by defining $(\alpha \otimes v)^\circ = \varphi(v) \otimes \varphi^{-1}(\alpha)$. With this structure, V becomes a unitary C_V -comodule. We observe that if we identify V with $\mathbb{C}^m$ via $v_i \leftrightarrow e_i$, and C_V with $M_m(\mathbb{C})^c$ via $v^i \otimes v_j \leftrightarrow e_{ij}$, then C_V is the $\circ$ -coalgebra and V is the unitary comodule of Example 4.7.

With the $\circ$ -structure in C_V given in the above remark, we have the following proposition.

Proposition 4.10. *Let C be a $\circ$ -coalgebra and V be a finite dimensional C -comodule with a fixed inner product $(\cdot | \cdot)$. Then $(V, (\cdot | \cdot))$ is a unitary comodule if and only if the associated coalgebra map $\phi : C_V \rightarrow C$ is a $\circ$ -coalgebra map.*

Proof. If the coactions are given by $\delta_C : V \rightarrow V \otimes C$, $\delta_C(v) = v_{(0)} \otimes v_{(1)}$, and $\delta_{C_V} : V \rightarrow V \otimes C_V$, $\delta_{C_V}(v) = v_{[0]} \otimes v_{[1]}$, then we have $v_{(0)} \otimes v_{(1)} = v_{[0]} \otimes \phi(v_{[1]})$, for all $v \in V$. Assume that ϕ is a $\circ$ -coalgebra map. Then

$$\begin{aligned} (u_{(0)} | v) u_{(1)} &= (u_{[0]} | v) \phi(u_{[1]}) = \phi((u_{[0]} | v) u_{[1]}) = \phi((u | v_{[0]})(v_{[1]})^\circ) \\ &= (u | v_{[0]}) (\phi(v_{[1]})^\circ) = (u | v_{[0]}) (\phi(v_{[1]}))^\circ = (u | v_{(0)}) (v_{(1)})^\circ. \end{aligned}$$

Conversely, if $(V, (\cdot | \cdot))$ is a unitary comodule, $\{v_1, \dots, v_m\}$ is an orthonormal basis in V and $\delta_C(v_i) = \sum_{j=1}^m v_j \otimes t_{ji}$, then $t_{ij}^\circ = t_{ji}$, for all i, j . We have $\phi(v^i \otimes v_j) = t_{ij}$ and $(v^i \otimes v_j)^\circ = v^j \otimes v_i$. Hence ϕ is a $\circ$ -coalgebra map. $\square$

Remark 4.11. If V is a finite dimensional C -comodule and $B = \{v_1, \dots, v_m\}$ is a basis of V , with $\delta(v_i) = \sum_{j=1}^m v_j \otimes t_{ji}$, then it is easily seen that there exists an inner product $(\cdot | \cdot)$ in V , such that V is a unitary comodule and B is an orthonormal basis of V if and only if $t_{ij}^\circ = t_{ji}$ for all i, j .

Corollary 4.12. *If V is a unitary comodule, then $\text{Coeff}(V)$ is a $\circ$ -subcoalgebra of C . Hence, if V is a unitary irreducible comodule, then $\text{Coeff}(V) \subset C$ is a simple subcoalgebra and a $\circ$ -subcoalgebra.*

Proof. Write $V = \sum_{i \in I} V_i$, where the V_i 's are finite dimensional unitary comodules. Then the above proposition implies that $\text{Coeff}(V) = \sum_{i \in I} \text{Coeff}(V_i)$ is a $\circ$ -subcoalgebra. The last assertion follows from Proposition 3.4. $\square$

From Proposition 3.13 and Remark 4.11 we get:

Proposition 4.13. *Let C be a $\circ$ -coalgebra and D be a subspace of C . We consider the following assertions:*

- (1) *There exists a basis $\{t_{ij} \mid i, j = 1, \dots, m\}$ of D such that*

$$\Delta(t_{ij}) = \sum_{k=1}^m t_{ik} \otimes t_{kj}, \quad \langle \varepsilon, t_{ij} \rangle = \delta_{ij} \quad \text{and} \quad t_{ij}^\circ = t_{ji};$$

- (2) *$D = \text{Coeff}(V)$, for some irreducible unitary C -comodule V ;*
 (3) *D is a simple subcoalgebra and a $\circ$ -subcoalgebra.*

Then the assertions (1) and (2) are equivalent, and these imply (3).

Example 4.14. We consider $u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and define a new $*$ -structure in $M_2(\mathbb{C})$ by $x^\nu = ux^*u^{-1}$. Proposition 4.6 implies that $(M_2(\mathbb{C}), \nu)$ is not isomorphic to $(M_2(\mathbb{C}), *)$ as $*$ -algebras. Then $(M_2(\mathbb{C})^c, \diamond)$ is not isomorphic to $(M_2(\mathbb{C})^c, \circ)$ as $\circ$ -coalgebras, where $\circ$ and $\diamond$ the $\circ$ -structures corresponding to $*$ and ν respectively. Hence we get that $(M_2(\mathbb{C})^c, \diamond)$ is a simple coalgebra and a $\circ$ -coalgebra that cannot verify assertion (1) in Proposition 4.13.

A *unitary comodule map* between two unitary comodules V and W is a comodule map $\phi : V \rightarrow W$, that is an isometry: $(\phi(u)|\phi(v)) = (u|v)$ for all $u, v \in V$. So a unitary comodule map is always injective. We denote by $\mathcal{M}^{(C, \circ)}$ the category of unitary comodules over the $\circ$ -coalgebra C .

Let V and W be unitary comodules. An *isomorphism* of unitary comodules is a surjective unitary comodule map $\phi : V \rightarrow W$.

Suppose there exists a $\circ$ -coalgebra map $\phi : C \rightarrow D$ between two $\circ$ -coalgebras. If $(V, (|))$ is a unitary C -comodule with C -coaction δ_C , then $(V, (|))$ is a unitary D -comodule with D -coaction given by $\delta_D = (\text{Id} \otimes \phi)\delta_C$. It is easy to see that if $\alpha : V \rightarrow W$ is a unitary C -comodule map, then α is also a unitary D -comodule map when we regard V and W as D -comodules as above. Hence we have a functor between $\mathcal{M}^{(C, \circ)}$ and $\mathcal{M}^{(D, \circ)}$.

For the rest of this section C will be a $\circ$ -coalgebra.

Proposition 4.15. *Let V be a unitary comodule and W be a subcomodule of V . Then $W^\perp$ is a subcomodule of V .*

Proof. Let $u \in W^\perp$ and $\delta(u) = \sum_{i=1}^n u_i \otimes \tilde{u}_i$, where $\{\tilde{u}_1, \dots, \tilde{u}_n\}$ is a linearly independent set. Let $w \in W$ and $\delta(w) = \sum_{j=1}^m w_j \otimes \tilde{w}_j \in W \otimes C$. Then we have

$$0 = \sum_{j=1}^m (u|w_j)(\tilde{w}_j)^\circ = \sum_{i=1}^n (u_i|w)\tilde{u}_i.$$

Hence $(u_i|w) = 0$ for all $w \in W$, with $i = 1, \dots, n$, and then $\delta(W^\perp) \subset W^\perp \otimes C$. $\square$

Since every comodule is a sum of finite dimensional subcomodules, by the above proposition we immediately obtain:

Corollary 4.16. *If V is a unitary comodule, then V is completely reducible.*

Proposition 4.17. *Let $(V_i, (|)_i)$, with $i \in I$, be an arbitrary family of unitary comodules. Then $\bigoplus_{i \in I} V_i$ is a unitary comodule and the inclusions $V_i \hookrightarrow \bigoplus_{i \in I} V_i$ are unitary comodule maps.*

Proof. It suffices to define an inner product $(|)$ in $\bigoplus_{i \in I} V_i$ in such a way that $V_i \perp V_j$, if $i \neq j$, and $(|)$ coincides with $(|)_i$ in V_i . $\square$

The following proposition is easy to prove.

Proposition 4.18. *Let V and W be comodules, $\phi : V \rightarrow W$ be an injective comodule map and suppose that W is a unitary comodule. We consider in V the unique inner product that makes ϕ an isometry. Then V , with this inner product, is a unitary comodule and ϕ is a unitary comodule map. Hence, if ϕ is an isomorphism of comodules, then it becomes an isomorphism of unitary comodules.*

Proposition 4.19. *Let V and W be irreducible unitary comodules. If V and W are isomorphic as comodules, then they are isomorphic as unitary comodules.*

Proof. Let $\phi : V \rightarrow W$ be an isomorphism of comodules, $\{e_1, \dots, e_m\}$ be an orthonormal basis of V , and $\delta(e_i) = \sum_{j=1}^m e_j \otimes t_{ji}$. Then, since V is a unitary comodule, we get $t_{ij}^\circ = t_{ji}$ for all i, j . Because ϕ is an isomorphism we get that $\{\phi(e_1), \dots, \phi(e_m)\}$ is a basis of W and $\delta(\phi(e_i)) = \sum_{j=1}^m \phi(e_j) \otimes t_{ji}$.

The comodule V is irreducible, so $\{t_{ij} \mid i, j = 1, \dots, m\} \subset C$ is a linearly independent set. Since W is a unitary comodule we get, for all i, j , that

$$(\phi(e_i)_{(0)} | \phi(e_j))_W \phi(e_i)_{(1)} = (\phi(e_i) | \phi(e_j)_{(0)})_W (\phi(e_j)_{(1)})^\circ$$

if and only if

$$\sum_{k=1}^m (\phi(e_k) | \phi(e_j))_W t_{ki} = \sum_{h=1}^m (\phi(e_i) | \phi(e_h))_W t_{jh}.$$

Then, $(\phi(e_i) | \phi(e_j))_W = 0$, for all $i \neq j$, and $(\phi(e_i) | \phi(e_i))_W = (\phi(e_j) | \phi(e_j))_W$, for all i, j . So there exists $a \in \mathbb{R}^+$ such that $(\phi(e_i) | \phi(e_j))_W = a\delta_{ij}$ for all i, j . Then the map $\psi : V \rightarrow W$, defined by $\psi(v) = \frac{1}{\sqrt{a}}\phi(v)$, is an isomorphism of unitary comodules. $\square$

From the proof of the above proposition we immediately obtain the following.

Corollary 4.20. *Let $(V, (|))$ be an irreducible unitary comodule. Suppose that there exists another inner product $(|)'$ in V such that $(V, (|)')$ is also a unitary comodule. Then there exists $a \in \mathbb{R}^+$ such that $(|)' = a(|)$.*

Definition 4.21. If C is a $\circ$ -coalgebra we define its $\circ$ -coradical $\text{co}_\circ(C)$ as the sum of all $\text{Coeff}(V)$, with V an irreducible unitary comodule.

Then $\text{co}_\circ(C)$ is a cosemisimple $\circ$ -subcoalgebra of C . Since every unitary comodule is the sum of its irreducible subcomodules, we moreover get that $\text{co}_\circ(C)$ is the sum of all $\text{Coeff}(V)$, where V is a unitary comodule.

If we consider $\{M_l \mid l \in I\}$, a set of representatives of the isomorphism classes of irreducible unitary comodules, and set $C_l := \text{Coeff}(M_l)$, with $l \in I$, then Corollary 3.7 implies that $\text{co}_\circ(C) = \bigoplus_{l \in I} C_l$, where C_l is a simple subcoalgebra which is also a $\circ$ -subcoalgebra.

For each $l \in I$, let $\{e_1^l, \dots, e_{m_l}^l\}$ be an orthonormal basis of M_l . Also, let $\delta(e_i^l) = \sum_{j=1}^{m_l} e_j^l \otimes t_{ji}^l$. We have $C_l = \mathbb{C}\{t_{ij}^l \mid i, j = 1, \dots, m_l\}$, and the entries of the matrix $T^l = (t_{ij}^l) \in M_{m_l}(C)$ verify

$$\Delta(t_{ij}^l) = \sum_{k=1}^{m_l} t_{ik}^l \otimes t_{kj}^l, \quad \langle \varepsilon, t_{ij}^l \rangle = \delta_{ij} \quad \text{and} \quad (t_{ij}^l)^\circ = t_{ji}^l. \quad (1)$$

If $R_i^l := \mathbb{C}\{t_{ij}^l \mid j = 1, \dots, m_l\}$, then $C_l = \bigoplus_{i=1}^{m_l} R_i^l$ and R_i^l is an irreducible right coideal of C_l . In this situation we can take $\mathbb{C}^{m_l}$ as M_l , with the usual inner product, with $\{e_1^l, \dots, e_{m_l}^l\}$ as the canonical basis of $\mathbb{C}^{m_l}$, and δ defined as above.

It is easily seen that if we define an inner product in $\text{co}_\circ(C)$ in such a way that $\{t_{ij}^l \mid i, j = 1, \dots, m_l, l \in I\}$ is an orthonormal basis, then $\text{co}_\circ(C)$ is a unitary comodule by the regular corepresentation and $R_i^l \cong M_l$, for all $i = 1, \dots, m_l$, as unitary comodules.

Remark 4.22. If C is an arbitrary $\circ$ -coalgebra, then its $\circ$ -coradical is contained in the coradical since any simple $\circ$ -subcoalgebra is a sum of simple subcoalgebras. In general this is a proper inclusion: in Example 4.8, we have $\text{co}_\circ(C) = 0$ but C is cosemisimple.

Definition 4.23. We say that a C -comodule V is *unitarizable* if there exists an inner product in V such that V is a unitary comodule. A $\circ$ -coalgebra is called *$\circ$ -cosemisimple* if it coincides with its $\circ$ -coradical.

Proposition 4.24. C is $\circ$ -cosemisimple if and only if every C -comodule is unitarizable.

Proof. Suppose that $C = \bigoplus_{l \in I} \text{Coeff}(M_l)$, where each M_l is an irreducible unitary C -comodule. If V is a C -comodule, since C is cosemisimple, we have $V = \bigoplus_{k \in K} V_k$, where each V_k is an irreducible subcomodule. Each irreducible C -comodule is isomorphic to an M_l , for some $l \in I$. Hence by Proposition 4.18 we have that for each $k \in K$ there exists an inner product in V_k for which V_k is a unitary comodule. If we extend these inner products to V in such a way that $V_h \perp V_k$ for all $h \neq k$, then V becomes a unitary comodule.

Conversely, if every C -comodule is unitarizable, then Corollaries 4.16 and 3.5, Theorem 3.1 and Proposition 4.13 imply that $C = \bigoplus_{i \in I} \text{Coeff}(V_i)$, where each V_i is an irreducible unitary comodule. $\square$

5. $\circ$ -Hopf algebras. A $\circ$ -Hopf algebra (or a $\circ$ -bialgebra) is a pair $(H, \circ)$ such that H is a Hopf algebra (or a bialgebra, respectively) and $\circ : H \rightarrow H$ is a conjugated linear involution verifying

$$(x^\circ)_{(1)} \otimes (x^\circ)_{(2)} = (x_{(2)})^\circ \otimes (x_{(1)})^\circ \quad \text{and} \quad (xy)^\circ = x^\circ y^\circ \quad (2)$$

for all $x, y \in C$. In particular, H is a $\circ$ -coalgebra.

A $*$ -Hopf algebra (or a $*$ -bialgebra) is a pair $(H, *)$ such that H is a Hopf algebra (or a bialgebra, respectively) and $*$: $H \rightarrow H$ is a conjugated linear involution verifying

$$(x^*)_{(1)} \otimes (x^*)_{(2)} = (x_{(1)})^* \otimes (x_{(2)})^* \quad \text{and} \quad (xy)^* = y^* x^*, \quad (3)$$

for all $x, y \in H$.

In each case we respectively have

$$\begin{cases} 1^\circ = 1, \text{ and } \langle \varepsilon, x^\circ \rangle = \overline{\langle \varepsilon, x \rangle} \text{ for all } x \in H; \\ 1^* = 1, \text{ and } \langle \varepsilon, x^* \rangle = \overline{\langle \varepsilon, x \rangle} \text{ for all } x \in H. \end{cases}$$

If H has an antipode $\mathcal{S}$, then $\mathcal{S} \circ \mathcal{S} \circ = \text{Id}$ or $\mathcal{S} * \mathcal{S} * = \text{Id}$. Hence $\mathcal{S}$ is bijective and $\mathcal{S}^{-1} = \circ \mathcal{S} \circ$ or $\mathcal{S}^{-1} = * \mathcal{S} *$ according to the case. If $*$ is such that $(H, *)$ is a $*$ -Hopf algebra, then we say that $*$ is a *compact involution* in H .

Lemma 5.1. [8] *In a Hopf algebra H there exists a bijection between $*$ -structures and $\circ$ -structures.*

Proof. Explicitly, the correspondence is as follows. To $*$: $H \rightarrow H$ corresponds the structure $\circ : H \rightarrow H$ given by $x^\circ := (\mathcal{S}(x))^*$ for all $x \in H$, and to $\circ : H \rightarrow H$ corresponds the structure $*$: $H \rightarrow H$ given by $x^* := \mathcal{S}(x^\circ)$ for all $x \in H$. As $\mathcal{S}$ is an antialgebra and anticoalgebra map, $*$ satisfies (3) if and only if $\circ$ satisfies (2). $\square$

Remark 5.2. If $\circ$ verifies relations (2) then, for each $n \in \mathbb{Z}$, $\circ_n := S^n \circ S^{-n}$ also verifies relations (2). In particular, $\circ_1 = \circ$ if and only if $S^2 = \text{Id}$. The same holds for $*_n := S^n * S^{-n}$, with $n \in \mathbb{Z}$. If H is cosemisimple, this is a particular case of the following.

Theorem 5.3. [4, (2.6)] *Two compact involutions of a cosemisimple Hopf algebra are necessarily conjugated by a Hopf algebra automorphism.*

The problem of existence of compact involutions on cosemisimple Hopf algebras remains open. There is however the following restriction:

Theorem 5.4. [2] *If a cosemisimple Hopf algebra H has a compact involution, then S^2 has real eigenvalues.*

6. Compact quantum groups.

Definition 6.1. We say that a $\circ$ -bialgebra H is a *compact quantum semigroup* (CQSG) if it is $\circ$ -cosemisimple. If H has also an antipode, then we say that H is a *compact quantum group* (CQG).

Remark 6.2. Let H be a $\circ$ -bialgebra and $(V_1, (\cdot | \cdot)), \dots, (V_n, (\cdot | \cdot))$ be unitary H -comodules. We consider the H -comodule $V_1 \otimes \dots \otimes V_n$ and the inner product defined by

$$(v_1 \otimes \dots \otimes v_n | u_1 \otimes \dots \otimes u_n) = (v_1 | u_1) \dots (v_n | u_n).$$

It is easy to see that $V_1 \otimes \dots \otimes V_n$, with this inner product, becomes a unitary H -comodule. Assume that each V_l is finite dimensional. Let $B_l = \{v_{i_l}^l\}_{i_l}$ be a basis of V_l , and $\delta(v_{i_l}^l) = \sum_{j_l} v_{j_l}^l \otimes t_{j_l i_l}^l$. Then the space of matrix coefficients of $V_1 \otimes \dots \otimes V_n$ is

$$\text{Coeff}(V_1 \otimes \dots \otimes V_n) = \mathbb{K} \{t_{i_1 j_1}^1 \dots t_{i_n j_n}^n \mid i_l, j_l = 1, \dots, m_l, l = 1, \dots, n\} \subset H.$$

Remark 6.3. By Proposition 4.4 we have that a $\circ$ -bialgebra is a CQSG if and only if it is spanned by a family of matrix coefficients of finite dimensional unitary comodules, that is if it is spanned by a family $\{t_{ij}^l \mid 1 \leq i, j \leq m_l, l \in I\}$ satisfying the relations (1). By the above remark we can replace “spanned” by “generated”.

Example 6.4. If C is a $\circ$ -coalgebra, then $T(C)$ (the tensor algebra over C) has a unique structure of $\circ$ -bialgebra for which $C \subset T(C)$ is a $\circ$ -subcoalgebra. Hence, by Remark 6.2, we get that $T(C)$ is a CQSG if and only if C is $\circ$ -cosemisimple.

Suppose that H is a CQSG. We have $H = \bigoplus_{l \in I} H_l$, where $H_l = \mathbb{C}\{t_{ij}^l \mid i, j\}$ is a simple subcoalgebra and a $\circ$ -subcoalgebra, and $T^l = (t_{ij}^l)$ verifies the relations (1). If $M_l = \mathbb{C}^{m_l}$ with the unitary comodule structure induced by T^l , then $\{M_l \mid l \in I\}$ is a set of representatives of the isomorphism classes of irreducible H -comodules.

From Proposition 4.24 we obtain:

Proposition 6.5. *A $\circ$ -bialgebra H is a CQSG if and only if every H -comodule is unitarizable.*

Following [5] we call a compact quantum (semi)group a *compact matrix quantum (semi)group* if it is finitely generated as an algebra.

Proposition 6.6. *Let H be a $\circ$ -bialgebra. The following conditions are equivalent:*

- (1) H is a compact matrix quantum semigroup;
- (2) There exists a finite dimensional unitary comodule V such that H is generated as an algebra by the matrix coefficients of V ;
- (3) There exist finite dimensional unitary comodules $V_1, \dots, V_l$ such that H is generated as an algebra by the matrix coefficients of $V_1, \dots, V_l$;
- (4) There exists a matrix $T = (t_{ij}) \in M_m(H)$ which verifies the relations (1) (Section 4) and such that H is generated by $\{t_{ij} \mid i, j = 1, \dots, m\}$ as an algebra;
- (5) There exists a finite set of matrices $\{T^l = (t_{ij}^l) \in M_{m_l}(H) \mid l = 1, \dots, s\}$ which verify the relations (1) (Section 4) and such that H is generated by $\{t_{ij}^l \mid i, j = 1, \dots, m_l, l = 1, \dots, s\}$ as an algebra.

Proof. We first show that (1) implies (2). Let $\{x_1, \dots, x_n\} \subset H = \bigoplus_{l \in I} H_l$ be a set of generators. There exists $\{l_1, \dots, l_r\} \subset I$ such that $\{x_1, \dots, x_n\} \subset \bigoplus_{i=1}^r H_{l_i}$. Then $V = \bigoplus_{i=1}^r M_{l_i}$ is a finite dimensional unitary comodule and Proposition 3.2 implies $\text{Coeff}(V) = \sum_{i=1}^r \text{Coeff}(M_{l_i}) = \sum_{i=1}^r H_{l_i}$. Hence H is generated as an algebra by $\text{Coeff}(V)$. This proves (2).

Moreover, (2) implies (3) by Remark 6.3, while it is obvious that (2) and (4) are equivalent, and (3) and (5) are equivalent.

Finally, (2) implies (3) because the sum of unitary comodules is a unitary comodule. $\square$

Remark 6.7. The “if” part of Proposition 6.5 is Proposition 3.9 in [5] when H is a CQG, while Proposition 6.6 is Proposition 2.4 in [5] when H is $\ast$ -Hopf algebra.

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