

A REMARK ON THE EXTENSION OF WEIERSTRASS FACTORIZATION THEOREM

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RÉSUMÉ. Nous montrons que l'extension du théorème de factorisation de Weierstrass à certaines suites $(a_n)_{n \in \mathbb{N}}$ d'éléments d'une algèbre de Banach $(A, \|\cdot\|)$ commutative et unitaire sur le corps $\mathbb{C}$ établie en [4] peut s'obtenir seulement à l'aide de la semi-norme spectrale ρ de cette algèbre.

ABSTRACT. We show that the extension of Weierstrass factorization theorem for some sequence of elements of a commutative Banach algebra with unit element over the field of complex numbers $\mathbb{C}$ given in [4] can be obtained only by using the spectral semi-norm of the algebra and its properties.

1. Introduction. Let A be a commutative Banach algebra over the field of complex numbers $\mathbb{C}$ with unit element denoted 1; $\|\cdot\|$ is the norm of A . For $a \in A$, the spectrum $\text{sp}(a)$ of a is a compact nonempty set in $\mathbb{C}$. It is well known that the mapping $z \in \mathbb{C} - \text{sp}(a) \mapsto (z - a)^{-1} \in A$, is analytic. For $|z| > \|a\|$, its Laurent development is then given by

$$f(z) = (z - a)^{-1} = \sum_{n \geq 0} \frac{a^n}{z^{n+1}}.$$

The singularities of f lie on the spectrum of a . As f has no zero in $\mathbb{C} - \text{sp}(a)$ its inverse $g(z) = z - a$ is also holomorphic in $\mathbb{C} - \text{sp}(a)$; g has a holomorphic extension to the whole $\mathbb{C}$ if and only if the resolvent function $(z - a)^{-1}$ is a rational function. In this case $\text{sp}(a)$ is a finite set and each one of its point is a zero of finite order of g . The holomorphic extension $\tilde{g}$ of g to the whole $\mathbb{C}$ is uniquely given by

$$\tilde{g}(z) = \begin{cases} g(z), & z \in \mathbb{C} - \text{sp}(a), \\ 0, & z \in \text{sp}(a). \end{cases}$$

By the holomorphic functional calculus, we can define $\tilde{g}[b]$ for any element $b \in A$. In particular, we have $\tilde{g}[a] = 0$; a is then algebraic. Otherwise, g has a unique C^∞ extension $\tilde{g}$ to the whole $\mathbb{C}$ defined also by

$$\tilde{g}(z) = \begin{cases} g(z), & z \in \mathbb{C} - \text{sp}(a), \\ 0, & z \in \text{sp}(a) \end{cases}$$

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$\tilde{g}$ is weakly holomorphic in $\mathbb{C}$ and we can define $\tilde{g}[a] = 0 \in A$ (see [7,8]).

In [4], the extension of Weierstrass factorization theorem was given by using the norm of the Banach algebra A and its properties.

In this note, we show that it is possible to give this extension by using only the spectral semi-norm of the Banach algebra A and its properties.

In what follows, A denotes a commutative Banach algebra with unit element over the field of complex numbers $\mathbb{C}$; $1, \|\cdot\|$ and ρ denote respectively the unit element, the norm and the spectral semi-norm of the Banach algebra A .

2. Infinite product. Weierstrass introduced the following sequence of functions defined by:

$$\begin{cases} E_0(z) = 1 - z \\ E_m(z) = (1 - z) \exp \left\{ z + \frac{z^2}{2} + \cdots + \frac{z^m}{m} \right\}, \quad m \geq 1. \end{cases}$$

These functions are holomorphic in $\mathbb{C}$, with zeros only at $z = 1$. For $m \in \mathbb{N}^*$ large enough and $|z| < 1$, $E_m(z)$ are close to 1. By the holomorphic functional calculus, we can define $E_m[a]$ for any positive integer m and any element a of a Banach algebra A with unit element. The infinite product $\prod_m E_m[z]$ defines a holomorphic function in the open unit ball $B(0, 1)$ of $\mathbb{C}$, so for any element $a \in A$ such that $\rho(a) < 1$, the spectrum of a is contained in the open unit ball $B(0, 1)$ of $\mathbb{C}$. So we can define $\prod_m E_m[a]$.

Proposition 2.1. *Let $a \in A$ such that $\rho(a) < 1$. The series $\sum_n a^n$ converges to $(1 - a)^{-1}$.*

Proof. Let $\alpha > 0$ be such that $\rho(a) \leq \alpha < 1$. As $\rho(a) = \lim_n \|a^n\|^{1/n}$, there exists $N \in \mathbb{N}$ such that

$$\|a^n\| \leq \alpha^n < 1, \quad \forall n \geq N.$$

It follows that the sequence $(1 - a) \cdot \sum_{p=0}^n a^p = 1 - a^{n+1}$ converges to 1. So the series $\sum_n a^n$ converges to $(1 - a)^{-1}$. $\square$

Remark 2.2. The series $\sum_n a^n/n$ converges too and its sum is $\text{Log}(1 - a)^{-1}$ because $(1 - a)^{-1}$ belongs to the principal component of invertible elements which is precisely the maximal connected subgroup of invertible elements that have logarithms.

Proposition 2.3. *Let m be a positive integer and $a \in A$ such that $\rho(a) < 1$. Then*

$$\rho(E_m[a]) \leq \rho(1 - a)(1 - \rho(a))^{-1}.$$

Proof. If $m = 0$ then $E_0[a] = 1 - a$ and we have

$$\rho(E_0[a]) = \rho(1 - a) \leq \rho(1 - a)(1 - \rho(a))^{-1}$$

because $(1 - \rho(a))^{-1} > 1$. For $m > 0$, we have

$$E_m[a] = (1 - a) \exp \left\{ a + \frac{a^2}{2} + \cdots + \frac{a^m}{m} \right\}.$$

So

$$\rho(E_m[a]) \leq \rho(1-a) \exp \left\{ \rho(a) + \frac{\rho(a)^2}{2} + \cdots + \frac{\rho(a)^m}{m} \right\}.$$

Since $\rho(a) < 1$, we have

$$\exp \left\{ \rho(a) + \frac{\rho(a)^2}{2} + \cdots + \frac{\rho(a)^m}{m} \right\} \leq (1 - \rho(a))^{-1}, \quad \forall m > 0.$$

Therefore

$$\rho(E_m[a]) \leq \rho(1-a)(1 - \rho(a))^{-1}. \quad \square$$

Theorem 2.4. *Let S be a non empty set, and $(A, \|\cdot\|)$ be a commutative Banach algebra over the field of complex numbers $\mathbb{C}$, with unit element. The set $B(S, A)$ of all bounded mappings $f : S \rightarrow A$ normed by*

$$\|f\|_B = \sup_S \|f(x)\|$$

is a Banach algebra. If $(u_n)_{n \in \mathbb{N}}$ is a sequence of elements in the unit open ball of $B(S, A)$ such that the series

$$\sum_n \|u_n(s)\| \tag{2.1}$$

converges simply on S , then the infinite product

$$\prod_n (1 + u_n) \tag{2.2}$$

converges simply on S . Moreover the infinite product is null at a point $s \in S$ if and only if there exists a positive integer $n \in \mathbb{N}$ such that $u_n(s) = -1$.

Proof. It is obvious that $B(S, A)$ is a commutative Banach algebra with unit element over the field of complex numbers $\mathbb{C}$. Let

$$p_n(s) = \prod_{q=0}^n (1 + u_q(s))$$

be the partial product of (2.2). First, for all $n \in \mathbb{N}$, p_n is invertible and we have

$$\|1 + u_q(s)\| \leq 1 + \|u_q(s)\| \leq \exp \left\{ \|u_q(s)\| \right\}.$$

So for any point $s \in S$, we have

$$\|p_n(s)\| \leq \exp \left\{ \sum_q \|u_q(s)\| \right\} \leq M_s, \quad \forall n \in \mathbb{N}$$

because the series $\sum_q \|u_q(s)\|$ converges simply on the set S , so it is simply bounded.

Let $0 < \varepsilon < 1/2$ and $s \in S$, then there exists a positive integer $N_s \in \mathbb{N}$ such that

$$\sum_{q \geq N_s} \|u_q(s)\| \leq \varepsilon. \quad (2.3)$$

Let $\{\eta_1, \eta_2, \dots\}$ be any permutation of the set $\{1, 2, 3, \dots\}$. Let $N \geq N_s$ and M be so large that

$$\{1, 2, 3, \dots, N\} \subseteq \{\eta_1, \eta_2, \dots, \eta_M\}. \quad (2.4)$$

If $q_M(s)$ denotes the M^{th} partial product of (2.2), then

$$q_M - p_N = p_N \left\{ \prod_k (1 + u_{\eta_k}) - 1 \right\}. \quad (2.5)$$

The η_k which occur in (2.5) are all distinct and are larger than N_s . Therefore

$$\|q_M - p_N\| \leq \|p_N\| (e^\varepsilon - 1) \leq 2 \|p_N\| \varepsilon \leq 2C_\varepsilon. \quad (2.6)$$

If $\eta_k = k$ ($k = 1, 2, 3, \dots$), then $q_M = p_M$, and (2.6) shows that $\{p_M\}$ converges simply to a limit f . Also, (2.6) shows that

$$\|p_M(s) - p_{N_s}(s)\| \leq 2 \|p_{N_s}(s)\|, \quad M > N_s, \quad (2.7)$$

so that $(1 - 2\varepsilon)\|p_{N_s}(s)\| \leq \|p_M(s)\|$. Hence

$$(1 - 2\varepsilon)\|p_{N_s}\| \leq \|f(s)\|. \quad (2.8)$$

which shows that $f(s) = 0$ if and only if $p_{N_s}(s) = 0$. $\square$

Remark 2.5. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of elements in A such that for each positive integer n , $\rho(u_n) < 1$. If the infinite product $\prod_n (1 + u_n)$ converges it is invertible.

Lemma 2.6. Let m be a positive integer sufficiently large and a in A such that $\rho(a) < 1$, then

$$\|1 - E_m[a]\| \leq \|a^{m+1}\| \sup_{t \in [0,1]} \|(1 - ta)^{-1}\|.$$

Proof. From Proposition 2.1, the series $\sum_n a^n = (1 - a)^{-1}$ because $\rho(a) < 1$. For $m \geq 1$, consider the mapping of one real variable defined by

$$t \in \mathbb{R} \rightarrow E_m[ta] = (1 - ta) \exp \left\{ ta + \frac{(ta)^2}{2} + \dots + \frac{(ta)^m}{m} \right\} \in A.$$

This mapping is differentiable and its derivative at any point $t \in \mathbb{R}$ is

$$(E_m[ta])' = E'_m[ta].a = (-ta)^m.a \exp \left\{ ta + \frac{(ta)^2}{2} + \dots + \frac{(ta)^m}{m} \right\}.$$

Apply the mean value theorem on the compact interval $[0, 1]$ we get

$$\|1 - E_m[a]\| \leq \|a^{m+1}\| \sup_{t \in [0,1]} \left\| \exp \left\{ ta + \frac{(ta)^2}{2} + \cdots + \frac{(ta)^m}{m} \right\} \right\|.$$

But

$$\lim_{m \rightarrow \infty} \exp \left\{ ta + \frac{(ta)^2}{2} + \cdots + \frac{(ta)^m}{m} \right\} = (1 - ta)^{-1}$$

so for sufficiently large positive integer m , we have

$$\|1 - E_m[a]\| \leq \|a^{m+1}\| \sup_{t \in [0,1]} \|(1 - ta)^{-1}\|. \quad \square$$

Theorem 2.7. *Let A be a commutative Banach algebra with unit, $(a_n)_{n \in \mathbb{N}}$ be a sequence of invertible elements in A and $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of positive integers such that for any positive real number $r > 0$ the series*

$$\sum_{n \geq 0} \|(r \cdot a_n^{-1})^{\alpha_n+1}\| \quad (2.9)$$

converges.

- a) *The infinite product $\prod_n E_{\alpha_n}[z \cdot a_n^{-1}]$ defines a continuous mapping f from $\mathbb{C}$ to A .*
- b) *$f(z)$ is invertible if and only if $z \notin \bigcup_n \text{sp}(a_n)$.*

Proof. a) From Lemma 2.6, we have:

$$\|1 - E_{\alpha_n}[z \cdot a_n^{-1}]\| \leq \|(z \cdot a_n^{-1})^{\alpha_n+1}\| \sup_{t \in [0,1]} \|(1 - t \cdot z \cdot a_n^{-1})^{-1}\|.$$

From the convergence of the series (2.9) and the invertibility of elements a_n , we have $\lim_n a_n^{-1} = 0$. Now, for sufficiently large positive integer n , we have

$$\|1 - E_{\alpha_n}[z \cdot a_n^{-1}]\| \leq \|(z \cdot a_n^{-1})^{\alpha_n+1}\|.$$

From this inequality we deduce that the infinite product converges absolutely and uniformly on compact sets of $\mathbb{C}$ to a continuous mapping $f : \mathbb{C} \rightarrow A$ since each one of its terms is a continuous mapping of the variable z .

- b) Obviously, $f(z)$ is invertible if and only if $z \notin \bigcup_{n \in \mathbb{N}} \text{sp}(a_n)$. $\square$

Remarks 2.8.

1. The mapping f in Theorem 2.7 is weakly holomorphic in the variable z , in particular, for each complex homomorphism φ on A , we have

$$\varphi \circ f(z) = \prod_n E_{\alpha_n} \left[\frac{z}{\varphi(a_n)} \right] = F_\varphi(z)$$

and $F_\varphi^{-1}\{0\} = \{\varphi(a_p) : p \in \mathbb{N}\}$ is a closed set without any point limit in $\mathbb{C}$, so necessarily $\lim_n |\varphi(a_n)| = +\infty$.

2. If $\mathcal{X}(A)$ is the space of all complex homomorphisms on A then

$$\bigcup_{\varphi \in \mathcal{X}(A)} F_\varphi^{-1}\{0\} = \bigcup_{n \in \mathbb{N}} \text{sp}(a_n).$$

Theorem 2.9. *Let A be a commutative Banach algebra with unit element over the field of complex numbers $\mathbb{C}$, Ω be a non empty open set in $\mathbb{C}$ and $a \in A$ such that $\text{sp}(a) \subset \Omega$. If f is a non null analytic mapping in Ω such that $f[a] = 0$. Then $\text{sp}(a)$ is a finite set.*

Proof. As f is non null analytic mapping in Ω , we can define $f[a]$ by the holomorphic functional calculus. But $f[a] = 0 \Rightarrow \text{sp}(a) \subset f^{-1}\{0\}$ which is a discrete set. But $\text{sp}(a)$ is a non empty compact set of $\mathbb{C}$ so it is necessarily finite and each one of its point is a zero of f of finite order. $\square$

Theorem 2.10. *Assume the hypotheses of Theorem 2.7 hold. Then the infinite product $\prod_n E_{\alpha_n}[z \cdot a_n^{-1}]$ is a non null analytic mapping f in the variable z if and only if each one of its terms $E_{\alpha_n}[z \cdot a_n^{-1}]$ is a non null analytic mapping of the variable z .*

Proof. It is immediate from Theorem 2.7, since the non null uniform limit of non null analytic mappings is a non null analytic mapping. $\square$

Remarks 2.11.

1. If the infinite product $\prod_n E_{\alpha_n}[z \cdot a_n^{-1}]$ is a non null analytic mapping f in $\mathbb{C}$, we can define $f[a]$ for any element $a \in A$. In particular, for each positive integer n , $f[a_n] = 0$. So $\text{sp}(a_n)$ is a finite set and the set $\bigcup_n \text{sp}(a_n)$ is necessarily closed without any limit point in $\mathbb{C}$. The sequence of finite sets $(\text{sp}(a_n))_{n \in \mathbb{N}}$ verifies necessarily the following property: every bounded set B in $\mathbb{C}$ meets at most a finite number of elements of the sequence $(\text{sp}(a_n))_{n \in \mathbb{N}}$, that is

$$\forall B \text{ (bounded set)} \subset \mathbb{C} \Rightarrow \exists N_B \in \mathbb{N} : B \cap \text{sp}(a_n) = \emptyset, \quad \forall n \geq N_B$$

(this means that $\lim_n \text{sp}(a_n) = \infty$ relative to the filter basis formed with the complements of bounded sets in $\mathbb{C}$). This property is a natural extension of the definition of a sequence of complex numbers that converges to infinity.

2. If $\lim_n \text{sp}(a_n) = \infty$ then $\lim_n \rho(a_n) = +\infty$. The converse is not true; for example, consider a sequence of self-adjoint operators $(T_n)_{n \in \mathbb{N}}$ on a Hilbert space H such that for each positive integer n , the spectrum of T_n is the compact interval $[0, n]$. Obviously, $\lim_n \rho(T_n) = +\infty$ but $\lim_n \text{sp}(T_n) \neq \infty$ in the sense described above.
3. The proof of Theorem 2.3 can be adapted to any open set $\Omega \subset \mathbb{C}$ containing the closed set $\bigcup_n \text{sp}(a_n)$.

Now we state the Weierstrass extension factorization theorem.

Theorem 2.12. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of invertible elements of a commutative Banach algebra A over the field of complex numbers $\mathbb{C}$. Let f be a non constant holomorphic function in $\mathbb{C}$ such that for each positive integer $n \in \mathbb{N}$, $f[a_n] = 0$ and f has no other zeros in $\mathbb{C}$. If $\lim_n a_n^{-1} = 0$, then there exist a holomorphic function g in $\mathbb{C}$ and a sequence of positive integers $(\alpha_n)_{n \in \mathbb{N}}$ such that*

$$f[a] = \exp \{g[a]\} \prod_n E_{\alpha_n}[a \cdot a_n^{-1}] \quad (2.10)$$

for all $a \in A$.

Proof. As f is a non constant holomorphic function in $\mathbb{C}$, we can define $f[a]$ for any element $a \in A$. But for each positive integer $n \in \mathbb{N}$, $f[a_n] = 0$, so for each positive integer $n \in \mathbb{N}$, $\text{sp}(a_n)$ is a finite set. Necessarily the sequence $(\text{sp}(a_n))_{n \in \mathbb{N}}$ verifies $\lim_n \text{sp}(a_n) = \infty$ in the sense described above. So $\lim_n \rho(a_n) = +\infty$ which implies that $\lim_n \|a_n\| = \infty$.

From the spectral mapping theorem and the invertibility of a_n we have

$$\text{sp}(a_n^{-1}) = [\text{sp}(a_n)]^{-1}$$

so $\lim_n \text{sp}(a_n^{-1}) = \{0\}$ as $\lim_n a_n^{-1} = 0$. Therefore there exists an increasing sequence $(\alpha_n)_{n \in \mathbb{N}}$ of positive integers such that for any positive real number $r > 0$ the series

$$\sum_{n \geq 0} \|(r \cdot a_n^{-1})^{\alpha_n+1}\|$$

converges. From the lemma above, the infinite product

$$\prod_n E_{\alpha_n}[z \cdot a_n^{-1}]$$

converges absolutely and uniformly on compact sets of $\mathbb{C}$, so it defines a non null analytic mapping $h : \mathbb{C} \rightarrow A$ since each one of its terms is a non null analytic mapping in $\mathbb{C}$ (the infinite product $\prod_n E_{\alpha_n}[z \cdot a_n^{-1}]$ is formed only with all zeros $(a_n)_{n \in \mathbb{N}}$ of the holomorphic mapping f).

Consider the mapping defined from $\mathbb{C} \setminus \bigcup_{n \in \mathbb{N}} \text{sp}(a_n)$ to A by

$$(f \cdot h^{-1})(z) = f(z) \cdot h^{-1}(z).$$

It is a meromorphic mapping in $\mathbb{C}$. All its singularities are removable, so it can be extended holomorphically to the whole $\mathbb{C}$. Moreover $f \cdot h^{-1}$ has no zero in $\mathbb{C}$ which is simply connected, so there exists a non null analytic mapping g in $\mathbb{C}$ such that

$$f(z) \cdot h^{-1}(z) = \exp \{g(z)\}.$$

Thus for every element $a \in A$, we can define

$$f[a] = \exp \{g[a]\} \prod_n E_{\alpha_n}[a \cdot a_n^{-1}]. \quad \square$$

Remarks 2.13.

1. By “ f has no other zeros in $\mathbb{C}$ ” we mean that if there exists $a \in A$ such that $f[a] = 0$ then necessarily $\text{sp}(a)$ lies in the closed set $\bigcup_n \text{sp}(a_n)$.
2. The decomposition (2.10) is not necessarily unique; $f[a]$ is invertible if and only if $\text{sp}(a) \subset \mathbb{C} \setminus \bigcup_{n \in \mathbb{N}} \text{sp}(a_n)$.
3. If the mapping f has a pole α of order m in $\mathbb{C}$, we consider the mapping

$$g(z) = (z - \alpha)^m f(z).$$

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Résumé substantiel en français. Dans cette note, nous montrons que l'extension du théorème de factorisation de Weierstrass à certaines suites $(a_n)_{n \in \mathbb{N}}$ d'éléments d'une algèbre de Banach $(A, \|\cdot\|)$ commutative et unitaire sur le corps $\mathbb{C}$ établie en [4] peut s'obtenir seulement à l'aide de la semi-norme spectrale ρ de cette algèbre.

Nous établissons d'abord quelques résultats préliminaires simples relatifs à la semi-norme spectrale ρ qui permettront d'obtenir cette extension. L'intérêt d'utiliser la semi-norme spectrale ρ réside dans la remarque suivante : si $a \in A$ est tel que $\rho(a) < 1$ alors $\text{sp}(a) \subset B(0, 1)$, la boule unité ouverte de $\mathbb{C}$. Ceci nous permet d'appliquer le résultat à une large classe d'éléments de l'algèbre de Banach A .

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