

HOMOMORPHISM DUALITY IN EDGE-COLOURED GRAPHS

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RÉSUMÉ. Soient G et H des graphes aux arêtes coloriées. Un *homomorphisme* de G vers H , noté $G \rightarrow H$, est une fonction $f : V(G) \rightarrow V(H)$ qui préserve les arêtes, et leurs couleurs. On pose $\rightarrow H = \{G : G \rightarrow H\}$. De même, pour une famille de graphes aux arêtes coloriées $\mathcal{F}$, on pose $\mathcal{F} \not\rightarrow := \{G : \text{pour tout } F \in \mathcal{F}, F \not\rightarrow G\}$. Pour tout graphe avec arêtes coloriées H de multiplicité deux et d'ordre au plus deux, on décrit une classe de graphes aux arêtes coloriées $\mathcal{F}_H$ telle que

$$\mathcal{F}_H \not\rightarrow = \rightarrow H$$

Nous employons ces caractérisations pour identifier une structure des voisins dans la catégorie des chemins rouges-bleus, accentuant un raccordement aux chemins orientés.

ABSTRACT. Given edge-coloured graphs G and H , a homomorphism of G to H , denoted $G \rightarrow H$, is a mapping of $V(G)$ to $V(H)$ that preserves edges and their colours. We define $\rightarrow H = \{G : G \rightarrow H\}$. Similarly, for a family of edge-coloured graphs $\mathcal{F}$, we define $\mathcal{F} \not\rightarrow := \{G : \text{for all } F \in \mathcal{F}, F \not\rightarrow G\}$. For each edge-coloured graph H with two edge-colours and at most two vertices, we describe a class $\mathcal{F}_H$ such that

$$\mathcal{F}_H \not\rightarrow = \rightarrow H$$

We use these characterizations to identify a gap structure in the category of red-blue paths, highlighting a connection to oriented paths.

1. Introduction. The concept of a homomorphism, i.e. a structure preserving map, is well developed in many areas of mathematics. In graph theory the preserved structure is edges. In contrast, for example to groups, testing for the existence of a homomorphism from one graph to another is a notoriously difficult problem, even when the target graph is fixed. Specifically, given a fixed graph H , the HOM- H problem asks whether an input graph G admits a homomorphism to H . If H contains a loop the problem is trivial – the answer is always yes. If H does not contain a loop, the problem is solvable in polynomial time if H is bipartite; and it is NP-complete otherwise, see [6]. In the case H is bipartite, i.e. HOM- H is (known to be) polynomial, it is well known how to characterize those G such that $G \rightarrow H$:

- if H contains no edges, then $G \rightarrow H$ if and only if G contains no edge; and
- if H contains an edge, then $G \rightarrow H$ if and only if G contains no odd cycle.

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In other words we can describe which graphs G map to H via the absence of some forbidden structure mapping to G .

In this paper we examine edge-coloured graphs with the goal of producing similar *duality theorems*. We again focus on HOM- H problems solvable in polynomial time, specifically red-blue graphs H with order at most two.

We present definitions below, followed by the duality theorems. We conclude the paper by applying these theorems to study the structure of the category of red-blue graphs. We highlight similarities with the category of oriented graphs.

An *edge-coloured graph* $G = (V(G), E_1(G), E_2(G), \dots, E_k(G))$ is a set of vertices $V(G)$ together with k edge sets consisting of unordered pairs of vertices. We use the simpler notations V and E_i when the graph G is clear from the context. We say that G has *multiplicity* k , and we call E_i the edges of colour i . The *order* of G is $|V|$. We note that (V, E_i) is a (classical, undirected) graph for each i . We allow loops, but parallel edges of the same colour are uninteresting to us, and thus not considered. However, there may be edges of different colours between a given pair of vertices. Given we allow parallel edges (of different colours), the term *edge-coloured multigraph* may be more appropriate; however, the term edge-coloured graph is established in the literature. The edge $\{u, v\}$ will normally be denoted uv . In this paper we restrict our attention to finite edge-coloured graphs.

The *underlying graph* of an edge-coloured graph $G = (V, E_1, \dots, E_k)$ is the classical graph $(V, E_1 \cup \dots \cup E_k)$, i.e. the graph obtained by ignoring edge-colours and multiple edges. We denote the underlying graph by $\underline{G}$. To say that an edge-coloured graph has some property means the underlying graph has the property. For example, an edge-coloured *clique* is an edge-coloured graph where between every pair of distinct vertices there is at least one edge of some colour.

Given edge-coloured graphs G and H of multiplicity k , a *homomorphism* of G to H is a function $f : V(G) \rightarrow V(H)$ which preserves edges, i.e., if $uv \in E_i(G)$ for some i , $1 \leq i \leq k$, then $f(u)f(v) \in E_i(H)$. We write $G \rightarrow H$ to indicate the existence of a homomorphism of G to H .

Let H be a fixed edge-coloured graph of multiplicity k . The decision problem HOM- H is defined as follows:

HOM- H

Instance: An edge-coloured graph G of multiplicity k .

Question: Does G admit a homomorphism to H ?

For this paper we focus on the situation that H has multiplicity two and order at most two. While HOM- H is NP-complete in general, in this restricted situation HOM- H is, in fact, solvable in polynomial time [2, 3]. As is often the case for polynomial solvable HOM- H problems, we are able to provide a *duality theorem* describing which edge-coloured graphs G admit a homomorphism to H . Given H , define $\rightarrow H := \{G : G \rightarrow H\}$. For a family of edge-coloured graphs $\mathcal{F}$, define $\mathcal{F} \not\rightarrow := \{G : \text{for all } F \in \mathcal{F}, F \not\rightarrow G\}$. When $\mathcal{F} = \{F\}$ we use the shorter $F \not\rightarrow$. Finally, let $\mathcal{F} \rightarrow := \{G : \text{for some } F \in \mathcal{F}, F \rightarrow G\}$.

For each edge-coloured graph H of multiplicity two and order at most two, we

describe a class of edge-coloured graphs $\mathcal{F}_H$ such that

$$\mathcal{F}_H \not\rightarrow = \rightarrow H$$

That is, we characterize which edge-coloured graphs G admit a homomorphism to H , in terms of forbidden homomorphisms from the class $\mathcal{F}_H$. Such a description is referred to as a *duality characterization*. Moreover, for each such H , $\mathcal{F}_H$ has an algorithmically well behaved description, indeed in many of our examples $\mathcal{F}_H$ is a single edge-coloured graph. Thus, the duality characterizations are *good characterizations* in the sense of Edmonds. We call the set $\mathcal{F}_H$ the *obstruction set* for H . Duality characterizations where $\mathcal{F}(= \{F\})$ is a single edge-coloured graph are studied in [9]. In such situation, we call (F, H) a *duality pair*.

The proof that HOM- H is polynomial time solvable in [2, 3] is via a reduction to 2-SAT. We remark that the proof holds for H of order at most two and *any* multiplicity. (An extension of our duality results to higher multiplicity is straightforward, although notationally cumbersome.) Also, we remark that the obstruction sets (especially the odd figure eights in Subsection 3.2) are fairly natural given the reduction to 2-SAT.

Two edge-coloured graphs A and B are *homomorphically equivalent* if $A \rightarrow B$ and $B \rightarrow A$. Clearly, for given equivalent edge-coloured graphs A and B , for any edge-coloured graph G , $G \rightarrow A$ if and only if $G \rightarrow B$. We remark that the category of edge-coloured graphs (with homomorphisms) can be viewed as a semi-order. We apply the usual operation of creating a partial order by considering nonequivalent objects, i.e. we have a partial order on the equivalence classes under the homomorphically equivalent relation. (See [7].) (Readers familiar with the concept of a *core* will recognize that each equivalence class has a natural representative, namely the unique core in the class, [5].)

Observe that the statement (F, H) is a duality pair is simply that the principal ideal generated by H (in the partial order) is equal to the complement of the principal filter generated by F . Such pairs turn out to be connected to density in the above partial order in the following way. A pair (A, B) of edge-coloured graphs such that $A \rightarrow B$ and $B \not\rightarrow A$ is a *gap pair* if for every Z such that $A \rightarrow Z \rightarrow B$, either $Z \rightarrow A$ or $B \rightarrow Z$. That is, A covers B in the partial order. In [9] a correspondence between gap pairs and duality pairs is presented.

We conclude the paper by examining the gap structure in the category of edge-coloured paths. We note the similarity to the gap structure for oriented paths, cf. [1,8].

For the remainder of the paper, all edge-coloured graphs have multiplicity two. We will refer to such edge-coloured graphs as *red-blue graphs*. We will refer to colour 0 as blue and colour 1 as red. In diagrams we will draw blue edges as solid lines and red edges as dotted lines.

Edge-coloured paths play an important role in our work. We will use a string of R 's and B 's to describe such a path. The k^{th} character of the string is R (respectively B) if the k^{th} edge of the path is red (respectively blue). We will use the obvious exponential shorthand, e.g. RB^iR denotes a path of length $i + 2$ whose first and last edges are red, and whose remaining edges are blue. Note R denotes a two vertex edge-coloured graph with a single red edge.

2. Targets of order 1. There are four red-blue graphs of order 1. The convention for naming the graphs is as follows: each graph is named H_α^1 ; the superscript 1 refers to the order of H ; and the subscript α indicates which loops are present. When α is '-' there are no loops. When α is b (respectively r) there is a single loop of colour blue (respectively red). Finally, if α is rb , there are two loops: one blue and one red.

Every red-blue graph admits a homomorphism to H_{rb}^1 thus the obstruction set for H_{rb}^1 is empty. Only red-blue graphs with no red (respectively blue) edges admit a homomorphism to H_b^1 (respectively H_r^1). Finally a red-blue graph admits a homomorphism to H_-^1 if and only if it has no edges. The following proposition is immediate.

Proposition 2.1. *For the red-blue graphs of order one, the following hold:*

$$\begin{aligned}\{R, B\} \not\rightarrow &= \rightarrow H_-^1 \\ \{R\} \not\rightarrow &= \rightarrow H_b^1 \\ \{B\} \not\rightarrow &= \rightarrow H_r^1 \\ \{\} \not\rightarrow &= \rightarrow H_{rb}^1\end{aligned}$$

3. Targets of order two. Throughout this section we assume our target has vertex set $\{0, 1\}$. We will not consider targets which are homomorphically equivalent to an edge-coloured graph on one vertex, since the obstruction sets for these targets are already identified. Specifically, we may assume that neither 0 nor 1 has a loop of colour blue and a loop of colour red, for such a target is homomorphically equivalent to H_{rb}^1 .

We will use the notation $H_{\beta, \gamma}^{2\alpha}$ to describe the target. The string α can be $-$, r , b , or d to indicate there is no edge, a red edge, a blue edge, or both a red edge and a blue edge from 0 to 1 respectively. The string β (respectively γ) can take the value $-$, r , or b to indicate there is a no loop, a red loop, or a blue loop on vertex 0 (respectively 1). When $\alpha = d$ we say that H contains a *digon*.

3.1. Digon free targets. There are five digon free targets up to reversal of edge-colours.

Proposition 3.1. *Let $H = H_{b,r}^{2-}$ and $F = RB$. Then $\rightarrow H = F \not\rightarrow$.*

Proof. Let G be an edge-coloured graph. Then $G \rightarrow H$ if and only if no vertex of G is incident with both a red and a blue edge. $\square$

Proposition 3.2. *Let $H = H_{-, -}^{2b}$. Let $\mathcal{F} = \{C : C \text{ is an odd cycle with only blue edges}\} \cup \{R\}$. Then $\rightarrow H = \mathcal{F} \not\rightarrow$.*

Proof. An edge-coloured graph G admits a homomorphism to H if and only if G contains no red edges, and G is bipartite. $\square$

Proposition 3.3. *Let $H = H_{r,r}^{2b}$ and $\mathcal{F} = \{C : C \text{ is a cycle containing an odd number of blue edges}\}$. Then $\rightarrow H = \mathcal{F} \not\rightarrow$.*

Proof. Let G be a red-blue graph. Clearly under any homomorphism of G to H , the ends of each red edge must receive the same image. Thus let G_c be the red-blue graph obtained by contracting each red edge of G . We have $G \rightarrow H$ if and only if $G_c \rightarrow H$. The latter condition holds if and only if the blue subgraph of G_c , i.e. $(V(G_c), E_0(G_c))$, is bipartite. The result follows. $\square$

Proposition 3.4. Let $H = H_{r,b}^{2b}$ and $F = RBR$. Then $\rightarrow H = F \not\rightarrow$

Proof. Let G be a red-blue graph. Map all the vertices incident with a red edge to 0, and map all others to 1. This is a homomorphism unless two vertices mapped to 0 are joined by a blue edge. Clearly in this case $F \rightarrow G$. $\square$

Theorem 3.5. Let $H = H_{-,r}^{2b}$ and let $\mathcal{F} = \{C : C \text{ is an odd cycle with only blue edges}\} \cup \{RB^{2k+1}R : k \geq 0 \text{ is an integer}\}$. Then $\rightarrow H = \mathcal{F} \not\rightarrow$.

Proof. Let G be a connected red-blue graph. (For disconnected graphs we can apply the following algorithm to each component.) Clearly, if $G \in \mathcal{F} \rightarrow$, then $G \not\rightarrow H$. Conversely, suppose $G \in \mathcal{F} \not\rightarrow$. Let r be a vertex incident with both a blue edge and a red edge. (If no such vertex exists, then G contains only blue edges, and is bipartite. Hence $G \rightarrow H$.) Label r with 1. We proceed with a breadth first labeling from r with the following rules:

- If $\text{parent}(u)$ is labeled 1 and the edge $\text{parent}(u)u$ is red, then label u with 1.
- If $\text{parent}(u)$ is labeled 1 and the edge $\text{parent}(u)u$ is blue, then label u with 0.
- If $\text{parent}(u)$ is labeled 0 and the edge $\text{parent}(u)u$ is blue, then label u with 1.

The *expected fourth* rule for the case “ $\text{parent}(u)$ is labeled 0 and $\text{parent}(u)u$ is red” is not required since throughout the labeling the following invariant is true: “ u is labeled 0 (respectively 1) if and only if there is a walk from r to u containing an odd (respectively even) number of blue edges.” Thus under the assumption that $G \in \mathcal{F} \not\rightarrow$, we cannot have $\text{parent}(u)$ labeled with 0 and the edge $\text{parent}(u)u$ red. Hence the rules above define a labeling for the entire graph G . We claim this labeling is a homomorphism $G \rightarrow H$. Suppose to the contrary this is not the case. Then using the invariant it is easy to see there exists $F \in \mathcal{F}$ such that $F \rightarrow G$, a contradiction. $\square$

3.2. Targets with digons. We now define the obstruction sets for all two vertex edge-coloured graphs which contain a digon (a pair of edges between 0 and 1: one of each colour). The category of all targets with digons is shown in Figure 1.

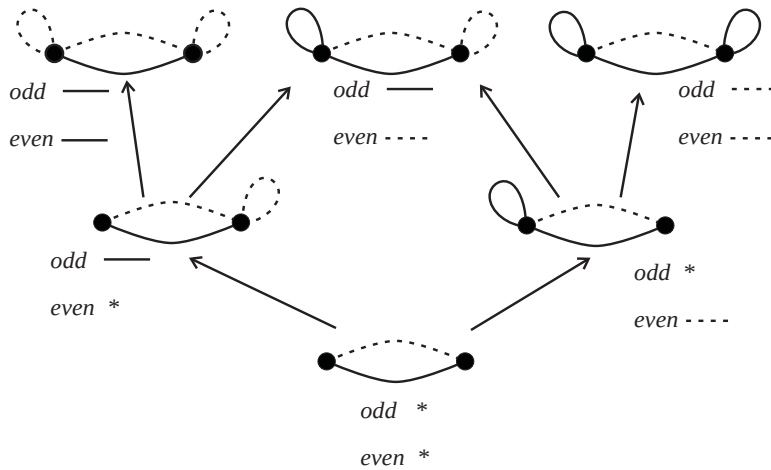


Fig. 1 – Targets with digons.

Definition 3.6. An *odd figure eight* is a closed walk of distinct vertices $v_0 v_1 \cdots v_{2j} v_0 v_{2j+2} \cdots v_{2k-1} v_0$, that is, a closed walk consisting of two odd cycles sharing the vertex

v_0 . The edges $v_{i-1}v_i$ are the *odd* (respectively *even*) edges where i is odd (respectively even). The set of all odd figure eights where the odd edges are of colour α and the even edges are of colour β is denoted $\mathcal{F}8_{\alpha,\beta}$. Note that α and β can take the values r for red, b for blue, and $*$. The value $*$ denotes “do not care”; that is, no restriction is placed on the edge colour.

An example of an odd figure eight is shown in Figure 2 below. In the example, the odd edges are blue, and the even edges are a mix of blue and red edges. Thus, the example belongs to both $\mathcal{F}8_{b,*}$ and $\mathcal{F}8_{*,*}$.

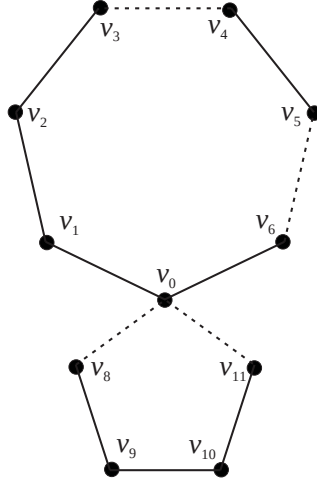


Fig. 2 – An odd figure eight.

The obstruction set for each target in Figure 1 is a collection of odd figure eights. In the top row, both the odd and the even edges have their colours specified. In the middle row the obstructions have the colour of only one of the odd or even edges specified. Finally for the bottom row there is no specification of the edge colours, (cf. Proposition 4.4). The obstruction set for each target is indicated below the target.

Theorem 3.7. *The following equalities hold:*

$$\begin{aligned} \rightarrow H_{-,-}^{2d} &= \mathcal{F}8_{*,*} \not\rightarrow \\ \rightarrow H_{r,-}^{2d} &= \mathcal{F}8_{b,*} \not\rightarrow \\ \rightarrow H_{r,b}^{2d} &= \mathcal{F}8_{b,r} \not\rightarrow \\ \rightarrow H_{r,r}^{2d} &= \mathcal{F}8_{b,b} \not\rightarrow \end{aligned}$$

Proof. First consider $H = H_{-,-}^{2d}$. It is clear that $G \rightarrow H$ if and only if the underlying graph of G , $\underline{G}$, is bipartite. We claim this is the case if and only if $G \in \mathcal{F}8_{*,*} \not\rightarrow$. Suppose $\underline{G}$ is not bipartite. Then $\underline{G}$ contains some odd cycle C . Let u be a vertex in C . Construct F by taking two copies of C and identifying u in the first copy with u in the second copy. We have $F \rightarrow G$ and $F \in \mathcal{F}8_{*,*}$. On the other hand if some $F' \rightarrow G$ and $F' \in \mathcal{F}8_{*,*}$, then $\underline{F'}$ is not bipartite and consequently $\underline{G}$ is not bipartite.

A similar argument shows $\rightarrow H_{r,r}^{2d} = \mathcal{F}8_{b,b} \not\rightarrow$. This follows from the observation that $G \in H_{r,r}^{2d}$ if and only if the blue subgraph, i.e. $(V(G), E_0(G))$, is bipartite.

Suppose $H = H_{r,b}^{2d}$. Let G be an edge-coloured graph, and let r be a vertex in G . Label r with 0. Notice that the vertices adjacent to r by a blue edge must map to 1 under any homomorphism. This motivates the following labeling rules for the construction of a breadth first search tree. However, unlike the examples above, these rules will not, in general, label all of G . Rather we will label a subgraph S of G such that one of the following holds: either the labeling is a homomorphism $f : S \rightarrow H$, and moreover any homomorphism of $g : G \setminus S \rightarrow H$ can be used to extend f to a homomorphism of $G \rightarrow H$; or we discover $F \in \mathcal{F}8_{b,r}$ such that $F \rightarrow G$.

- Label the root, r , of the BFS tree with 0.
- Label u with 0 if $\text{parent}(u)$ is labeled 1, and the edge $\text{parent}(u)u$ is red.
- Label u with 1 if $\text{parent}(u)$ is labeled 0, and the edge $\text{parent}(u)u$ is blue.

The following invariant remains true throughout the labeling: if u is labeled 0 (respectively 1), then there is an even (respectively odd) length walk from r to u whose first edge is blue, and whose edges alternate blue/red.

Let S be the maximal set of vertices labeled by these rules, and let $f : S \rightarrow \{0, 1\}$ be the labeling. The mapping f is a homomorphism unless there is a blue (respectively red) edge whose ends are both labeled 0 (respectively 1). In either case, using the invariant, one can easily find a walk of odd length which starts and ends with blue edges (incident at r) and whose edges alternate blue/red. Clearly we can construct an odd cycle C whose edges alternate blue/red (starting and ending with a blue edge at vertex x) such that $C \rightarrow G$ and $x \mapsto r$.

If such a walk exists, we construct a new DFS tree using the rules above with the exception that we label r with 1. Again either the labeling is a homomorphism of the labeled vertices to H or there is an odd length cycle, say C' , whose edges alternate red/blue (again starting at vertex x), with $C' \rightarrow G$ and $x \mapsto r$.

If in both cases we discover odd walks, then let F be the edge-coloured graph constructed by taking a copy of C and a copy of C' and identifying x in the two copies. Clearly $F \rightarrow G$ and $F \in \mathcal{F}8_{b,r}$. Hence $G \not\rightarrow H$.

On the other hand, suppose one of the labelings does define a homomorphism, say from the subgraph S to H . Further suppose w is a vertex of G adjacent to a labeled vertex, but w is not labeled. Observe, if w is adjacent to v and v is labeled 0 (respectively 1), then the edge wv must be red (respectively blue). Notice there are two extensions of the homomorphism $S \rightarrow H$ to the subgraph on $V(S) \cup \{w\}$: one where w is labeled 0 and the other where w is labeled 1. Both extensions are indeed homomorphisms. Hence, any homomorphism of $g : G \setminus S \rightarrow H$ can be combined with $f : S \rightarrow H$ to give a homomorphism $(f + g) : G \rightarrow H$. (Here $(f + g)(u) = f(u)$ if $u \in S$ and $g(u)$ otherwise.) Thus we choose any unlabeled vertex, and begin the DFS labeling process at that vertex. Either we discover $G \not\rightarrow H$, or we extend the labeling to a homomorphism $G \rightarrow H$.

Finally, consider $H = H_{r,-}^{2d}$. We label G using the following labeling rules:

- Label the root, r , of the BFS tree with 0.
- Label u with 0 if $\text{parent}(u)$ is labeled 1 (the edge $\text{parent}(u)u$ is either red or

blue).

- Label u with 1 if $\text{parent}(u)$ is labeled 0, and the edge $\text{parent}(u)u$ is blue.

In this case the following invariant remains true throughout the labeling: u is labeled 0 (respectively 1) if and only if there is an even (respectively odd) length walk from r to u whose edges first, third, fifth, etc. (respectively second, fourth, sixth, etc.) edges are blue.

As above, either the labeling defined a homomorphism on a subgraph of G , or we discover an odd cycle starting at r such that odd numbered edges (starting with edge number one) are blue. In this latter case, label r with 1 and repeat the DFS labeling. Either we have a homomorphism of the labeled subgraph to H , or we discover an odd cycle starting at r whose even edges are blue. This cycle, together with the first cycle, shows that there is $F \in \mathcal{F}8_{b,*}$ such that $F \rightarrow G$, and thus $G \not\rightarrow H$. In the former case, we may extend the homomorphism as described above. $\square$

4. Density in red-blue graphs. Initially our duality results were motivated by algorithmic considerations of $\text{HOM-}H$. However the following theorem from [9] connects duality pairs to gap pairs. The product used below is the *categorical product*: $V(G \times H) = V(G) \times V(H)$; and $(u_1, v_1)(u_2, v_2) \in E_i(G \times H)$ if and only if $u_1u_2 \in E_i(G)$ and $v_1v_2 \in E_i(H)$.

Theorem 4.1 (NEŠETŘIL AND TARDIF). *If (A, B) is a duality pair, then $(A \times B, B)$ is a gap pair.*

We now explore some questions in the category of red-blue graphs. We introduce an infinite family of duality pairs, and we examine the corresponding infinite family of gap pairs. This gap structure is similar to the structure found in the category of oriented paths, see below. Finally, we apply our duality results to compute a natural categorical limit.

Definition 4.2. Let $i \geq 1$. The red-blue graph H_i has vertex set $\{0, 1, \dots, i\}$, blue edge set $\{01, 12, 23, \dots, (i-1)i, ii\}$ and red edge set $\{00\}$. That is, H_i is a blue path of length i with a blue loop on one end, and a red loop on the other. The red-blue graph F_i is $RB^{2i-1}R$.

Theorem 4.3. *For each i , the pair (F_i, H_i) is a duality pair, and the pair (F_{i+1}, F_i) is a gap pair.*

Proof. It is easy to verify that $F_i \not\rightarrow H_i$. Thus if $F_i \rightarrow G$, then $G \not\rightarrow H_i$. Hence suppose $F_i \not\rightarrow G$. For each vertex v in G , let $d(v)$ be the minimum distance from v to a vertex incident with a red edge. (We use the convention that $d(v)$ is ∞ if v is in a component with no red edges.) Label each vertex v with $\min\{d(v), i\}$. In particular, each vertex incident with a red edge is labeled with zero. Also, any two vertices adjacent in blue must receive labels that differ by at most 1. Thus the labeling defines a homomorphism to H_i unless two vertices adjacent in blue, say x, y , receive the same label, say j , (with $0 \leq j < i$). However, in this case there is a walk of length $2j + 3$ in G whose first edge and last edge are red, and whose interior edges are blue. This walk is a homomorphic image of F_i , a contradiction. This completes the proof that (F_i, H_i) is a duality pair.

Given Theorem 4.1 above, we could complete the proof by proving that F_{i+1} is homomorphically equivalent to $H_i \times F_i$, and thus (F_{i+1}, F_i) is a gap. However, we proceed with a direct proof of the gap pair statement.

Let Z be a red-blue graph, and $f : F_{i+1} \rightarrow Z$, $g : Z \rightarrow F_i$ be two homomorphisms. We will show that $Z \rightarrow F_{i+1}$ or $F_i \rightarrow Z$. In fact, we may assume that Z is connected. (Otherwise we can apply the argument below to each component of Z . Either all components of Z map to F_{i+1} or F_i maps to some component of Z , from which the result follows.) Also, we may assume Z contains a vertex incident with both a blue and a red edge. (Otherwise Z is a bipartite edge-coloured graph containing only blue edges or only red edges, and in this case it maps to a single edge of F_{i+1} .) Let the vertices of F_i be $v_0, v_1, \dots, v_{2i+1}$, and the vertices of F_{i+1} be $u_0, u_1, \dots, u_{2i+3}$ (in the natural order).

We define a partition $\{P_j\}_{j=0}^{2i+1}$ on the vertices of Z as follows: $P_j = g^{-1}(v_j)$, for $j = 0, 1, 2i$ and $2i+1$. Thus each red edge in Z has one end in P_0 and one end in P_1 , or one end in each of P_{2i} and P_{2i+1} . Call a vertex (of Z) *mixed* if it is incident with both a blue edge and a red edge. At this point each mixed vertex in Z has been placed in P_1 or P_{2i} . It remains to assign the vertices incident with only blue edges to some set P_j . Call these vertices *blue only vertices*. (Since the homomorphism $g \circ f : F_{i+1} \rightarrow F_i$ must be onto, we have $g : Z \rightarrow F_i$ is onto, and $P_1 \neq \emptyset$.)

Let P_j be the blue only vertices at distance $j - 1$ from a mixed vertex in P_1 for $j = 2, 3, \dots, 2i - 1$. Finally, for each blue only vertex w at distance at least $2i - 1$ from some mixed vertex in P_1 we do the following: If the distance is odd, then add w to P_{2i} ; otherwise, add w to P_{2i-1} . We note that a blue only vertex, say x , may be mapped to v_1 by g . In this case, we remove x from P_1 and assign x to P_j for some $j \geq 2$, based on the distance from x to a mixed vertex in P_1 . Hence after assigning all the blue only vertices to some P_j , the set P_1 will not contain any blue only vertices.

Each P_j is independent. This follows from the fact that $g : Z \rightarrow F_i$ is a homomorphism. In fact, it is easy to see that the function $g' : Z \rightarrow F_i$ defined by $g'(w) = v_j$ if $w \in P_j$ is a homomorphism $Z \rightarrow F_i$.

Let X be the vertices in P_{2i} that are distance exactly $2i - 1$ from a mixed vertex in P_1 . (Since $Z \rightarrow F_i$ is onto, the set X is not empty.) If there is a mixed vertex in X , then clearly $F_i \rightarrow Z$. On the other hand, if there are no mixed vertices in X , then let Y be the vertices in P_{2i-1} that are distance (exactly) $2i - 2$ from P_1 . There are no edges from a mixed vertex in P_{2i} to a vertex in Y . Hence the mapping $h : Z \rightarrow F_{i+1}$ defined by

$$h(w) = \begin{cases} u_j & \text{if } w \in P_j \text{ for } j = 0, 1, \dots, 2i - 2 \\ u_{2i-1} & \text{if } w \in Y \\ u_{2i} & \text{if } w \in X \\ u_{2i+1} & \text{if } w \in P_{2i-1} \setminus Y \\ u_{2i+2} & \text{if } w \in P_{2i} \setminus X \\ u_{2i+3} & \text{if } w \in P_{2i+1} \end{cases}$$

is a homomorphism. The key observation required to verify that h is indeed a homomorphism, is that two blue only vertices both of which are an even distance from a mixed vertex in P_1 cannot be adjacent since $g(Z)$ would contain an odd length cycle, a

contradiction. Similarly for two blue only vertices at an odd distance. $\square$

The homomorphism theory for directed graphs and for red-blue graphs is very similar. This has caused some to speculate there must be a natural translation from one theory to the other, and vice versa. As further evidence, we note that the gaps between the red-blue paths correspond in a very natural way to the gaps between oriented paths of net-length three, see [10]. In a companion work, [4], we show these are essentially the only gaps in the category of red-blue paths, mirroring the results for oriented paths. In fact, we show there exists a correspondence between bipartite oriented graphs and bipartite red-blue graphs which preserves the density structure.

To conclude the paper we examine again the family of duality pairs from Theorem 4.3. We use the following proposition to construct a greatest lower bound for the family.

Proposition 4.4. *Let X and Y be edge-coloured graphs such that $X \rightarrow Y$. Suppose that $\rightarrow X = \mathcal{F}_X \not\rightarrow$ and $\rightarrow Y = \mathcal{F}_Y \not\rightarrow$. Then $\mathcal{F}_Y \rightarrow \subseteq \mathcal{F}_X \rightarrow$.*

Proof. Let $G \in \mathcal{F}_Y \rightarrow$. Then $G \not\rightarrow Y$. Hence, $G \not\rightarrow X$, and $G \in \mathcal{F}_X \rightarrow$. $\square$

The families introduced in Theorem 4.3 have the properties that $\cdots, H_3 \rightarrow H_2 \rightarrow H_1$ and by Proposition 4.4 we have $F_1 \rightarrow \subseteq F_2 \rightarrow \subseteq F_3 \rightarrow \subseteq \cdots$. Let H_∞ be the edge-coloured graph on three vertices say $\{x, y, z\}$ with a blue loop on x , a blue edge between y and z and a red loop on z . We claim that H_∞ is the greatest lower bound for the family $\{H_i\}$ in the homomorphism order.

Proposition 4.5. *The edge-coloured graph H_∞ is such that $H_\infty \rightarrow H_i$ for $i = 1, 2, 3, \dots$. Moreover if $X \rightarrow H_i$ for all i , then $X \rightarrow H_\infty$.*

Proof. The fact that $H_\infty \rightarrow H_i$ for all i is clear. Suppose $X \rightarrow H_i$ for all i . Then $F_i \not\rightarrow X$ for all i , i.e. $RB^{2i-1}R \not\rightarrow X$ for all $i \geq 1$. Thus there are not two mixed vertices in X joined by an odd length path of blue edges. However, $X \rightarrow H_\infty$ if and only if X contains only blue edges, or all paths of blue edges between mixed vertices have even length. $\square$

The least upper bound of the families $\{F_i \rightarrow\}$ is the union of the families. Thus the obstruction set for the greatest lower bound of $\{H_i\}$ is the least upper bound of the obstruction sets.

Corollary 4.6. *Let H_∞ be defined as above. Then*

$$\rightarrow H_\infty = \bigcup_{i=1}^{\infty} (F_i \not\rightarrow)$$

Of course the above is really a discussion of categorical limits and colimits. To see H_∞ is the limit of the H_i 's define a functor $F : \mathbb{N} \rightarrow \{H_i\}$ where $f_i : i \rightarrow (i-1)$ and $F(f_i) = \rho_i$. Thus,

Corollary 4.7. *Let $\rho_i : H_i \rightarrow H_{i-1}$ be homomorphisms. Then there exists $\phi_i : H_\infty \rightarrow H_i$ such that $\rho_i \phi_i = \phi_{i-1}$. Moreover for any X such that there are homomorphisms $\psi_i : X \rightarrow H_i$ such that $\rho_i \psi_i = \psi_{i-1}$, we have a homomorphism $\alpha : X \rightarrow H_\infty$ such that $\psi_i = \phi_i \alpha$.*

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Résumé substantiel en français. Le concept d'homomorphisme, c'est-à-dire de fonction préservant les structures, est standard en mathématiques. En théorie des graphes, la structure préservée est celle des arêtes. Déterminer l'existence d'un homomorphisme entre deux graphes est un problème notoirement difficile. Ceci contraste avec les catégories algébriques pour lesquelles il existe toujours un homomorphisme trivial entre deux objets.

Dans les cas où déterminer l'existence d'un homomorphisme se fait en un temps polynômial, on peut souvent trouver un théorème dit de *dualité*, c'est-à-dire une description des graphes G pour lesquels il existe un homomorphisme vers un graphe H fixé, en fonction de certaines structures pour lesquelles il n'existe pas d'homomorphismes vers G . Par exemple, il existe un homomorphisme de G vers K_2 si et seulement si il n'existe pas d'homomorphisme d'un cycle impair vers G . Dans cet article, nous donnons des théorèmes de dualité pour les graphes bicolores H d'ordre au plus deux, et étudions les liens de ces résultats avec la densité dans la catégorie des graphes bicolores.

Un *graphe aux arêtes coloriées* $G = (V(G), E_1(G), E_2(G), \dots, E_k(G))$ est un ensemble de sommets $V(G)$ ainsi que k ensembles d'arêtes données par des paires non ordonnées de sommets. On dit que G est de *multiplicité* k , et on appelle les éléments de E_i les arêtes de couleur i . L'ordre de G est $|V(G)|$. Si $k = 2$, on dit que G est un graphe *rouge-bleu*. Les boucles sont admises, mais les arêtes parallèles de la même couleur sont sans intérêt pour nous. Tous les graphes sont supposés finis.

Soient G et H des graphes aux arêtes coloriées de multiplicité k . Un *homomorphisme* de G vers H est une fonction $f : V(G) \rightarrow V(H)$ qui préserve les arêtes, c'est-à-dire que si $uv \in E_i(G)$ pour un certain i , $1 \leq i \leq k$, alors $f(u)f(v) \in E_i(H)$. Nous écrivons $G \rightarrow H$ pour indiquer l'existence d'un homomorphisme de G vers H .

Étant donné H , on pose $\rightarrow H := \{G : G \rightarrow H\}$. De même, pour une famille de graphes aux arêtes coloriées $\mathcal{F}$, on pose $\mathcal{F} \not\rightarrow := \{G : \text{pour tout } F \in \mathcal{F}, F \not\rightarrow G\}$. Dans le cas où $\mathcal{F} = \{F\}$ on utilise la notation plus simple $F \not\rightarrow$. En outre, $\mathcal{F} \rightarrow := \{G : \text{il existe } F \in \mathcal{F} \text{ tel que } F \rightarrow G\}$.

Pour tout graphe avec arêtes coloriées H de multiplicité deux et d'ordre au plus deux, on décrit une classe de graphes aux arêtes coloriées $\mathcal{F}_H$ telle que

$$\mathcal{F}_H \not\rightarrow = \rightarrow H.$$

Des caractérisations de dualité où $\mathcal{F} (= \{F\})$ consiste en un seul graphe avec arêtes coloriées ont été étudiées par Nešetřil et Tardif. Dans cette situation, on appelle (F, H) une paire de *dualité*. Notons que la catégorie des graphes aux arêtes coloriées (avec les homomorphismes tels que décrits ci-haut) peut être vue comme un semi-ordre. On applique l'opération usuelle pour créer un ordre partiel en considérant les objets non équivalents. Une paire (A, B) de graphes aux arêtes coloriées telle que $A \rightarrow B$ et $B \not\rightarrow A$ est une paire *voisine* si pour chaque Z tel que $A \rightarrow Z \rightarrow B$, $Z \rightarrow A$ ou $B \rightarrow Z$. C'est-à-dire que A couvre B dans l'ordre partiel. Nešetřil et Tardif ont découvert une

correspondance entre les paires voisines et les paires de dualité (dans la catégorie des systèmes relationnels). Nous explorons ce lien dans les graphes rouges-bleus.

Les chemins aux arêtes colorées sont décrits par les mots de l'ensemble $\{R, B\}$. La k^e lettre du mot est R (respectivement B) si la k^e arête du chemin est rouge (respectivement bleue).

Résultats sur la dualité. Il y a quatre graphes rouges-bleus d'ordre 1. On note ces graphes de la façon suivante: chaque graphe est noté H_α^1 où 1 réfère à l'ordre de H et α indique quelles boucles sont présentes dans H . Si α est $-$, il n'y a pas de boucles. Si α est b (respectivement r) alors il y a une unique boucle de couleur bleue (respectivement rouge) et si α est rb , il y a deux boucles, une rouge et une bleue.

Proposition 2.1. *Pour les graphes rouges-bleus d'ordre un, on a:*

$$\begin{aligned}\{R, B\} \not\rightarrow &= \rightarrow H_-^1 \\ \{R\} \not\rightarrow &= \rightarrow H_b^1 \\ \{B\} \not\rightarrow &= \rightarrow H_r^1 \\ \{\} \not\rightarrow &= \rightarrow H_{rb}^1\end{aligned}$$

Dans le cas des graphes rouges-bleus d'ordre deux, l'ensemble des sommets est $\{0, 1\}$. On note les graphes $H_{\beta, \gamma}^{2\alpha}$ où α est $-$, r , b ou d pour indiquer qu'il n'y a pas d'arêtes, une arête rouge, une arête bleue ou une arête bleue et une arête rouge entre 0 et 1 respectivement. La variable β (respectivement γ) peut prendre les valeurs $-$, r ou b pour indiquer qu'il n'y a pas de boucle, une boucle rouge, ou une boucle bleue au sommet 0 (respectivement 1).

Théorème. *Pour un graphe rouge-bleu sans arêtes parallèles (entre deux sommets distincts), on a*

$$\begin{aligned}\rightarrow H_{b,r}^{2-} &= \{RB\} \not\rightarrow \\ \rightarrow H_{-, -}^{2b} &= \{C : C \text{ est un cycle impair d'arêtes bleues seulement}\} \cup \{R\} \not\rightarrow \\ \rightarrow H_{r,r}^{2b} &= \{C : C \text{ est un cycle contenant un nombre impair d'arêtes bleues}\} \not\rightarrow \\ \rightarrow H_{r,b}^{2b} &= \{RBR\} \not\rightarrow \\ \rightarrow H_{-, r}^{2b} &= \{C : C \text{ est un cycle impair d'arêtes bleues seulement}\} \cup \{RB^{2k+1}R\} \not\rightarrow\end{aligned}$$

Définition 3.6. Un *double-cycle impair* est une marche fermée de sommets distincts $v_0 v_1 \dots v_{2j} v_0 v_{2j+2} \dots v_{2k-1} v_0$, c'est-à-dire une marche fermée consistant en deux cycles impairs ayant en commun le sommet v_0 . Les arêtes $v_{i-1} v_i$ sont dites *impaires* (respectivement *paires*) si i est impair (respectivement pair). L'ensemble de tous les double-cycles impairs dont les arêtes impaires sont de couleur α et les arêtes paires sont de couleurs β est noté $\mathcal{F}8_{\alpha, \beta}$. Notons que α et β peuvent prendre les valeurs r pour rouge, b pour bleu et $*$. La valeur $*$ signifie qu'aucune restriction n'est imposée à la couleur des arêtes.

Théorème 3.7. *Pour un graphe rouge-bleu avec arêtes parallèles (entre deux sommets distincts), on a*

$$\begin{aligned}\rightarrow H_{-,-}^{2d} &= \mathcal{F}8_{*,*} \not\vdash \\ \rightarrow H_{r,-}^{2d} &= \mathcal{F}8_{b,*} \not\vdash \\ \rightarrow H_{r,b}^{2d} &= \mathcal{F}8_{b,r} \not\vdash \\ \rightarrow H_{r,r}^{2d} &= \mathcal{F}8_{b,b} \not\vdash .\end{aligned}$$

Densité dans les graphes rouges-bleus.

Théorème 4.1 (NEŠETŘIL AND TARDIF). *Si (A, B) est une paire de dualité, alors $(A \times B, B)$ est une paire voisine.*

Définition 4.2. Soit $i \geq 1$. Le graphe rouge-bleu H_i a pour ensemble de sommets $\{0, 1, \dots, i\}$, ensemble d'arêtes bleues $\{01, 12, 23, \dots, (i-1)i, ii\}$ et ensemble d'arêtes rouges $\{00\}$. Le graphe rouge-bleu F_i est $RB^{2i-1}R$.

Théorème 4.3. *Pour chaque i , la paire (F_i, H_i) est une paire de dualité, et la paire (F_{i+1}, F_i) est une paire voisine.*

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