

NONSINGULAR ZEROS OF CUBIC FORMS DEFINED OVER P -ADIC FIELDS

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RÉSUMÉ. Soit C une forme cubique non-dégénérée définie sur un corps p -adique K d'ordre au moins 10. Nous prouvons que C possède un zéro non-singulier défini sur K n'appartenant pas à l'hyperplan H défini sur K pourvu que le cardinal du corps résiduel de K soit supérieur ou égal à 5.

ABSTRACT. In this note, we show that if C is a nondegenerate cubic form defined over a p -adic field K of order at least 10, then C has a nonsingular zero defined over K that does not lie on a hyperplane H defined over K as long as the cardinality of the residue field of K is at least 5.

1. Introduction. Several authors [Le-S, L-Y] have treated the problem of finding a nonsingular zero of a homogeneous form F that is not a zero of another homogeneous form G . In this note, we prove that if $C \in K[X_1, \dots, X_n]$ is a nondegenerate cubic form defined over a p -adic field K of order at least 10, and H is a hyperplane defined over K given by a nonzero linear form $L \in K[X_1, \dots, X_n]$, then C has a nonsingular zero defined over K that does not lie on H as long as the cardinality of the residue field of K is at least 5.

Let k be a field. If $F \in k[X_1, \dots, X_n]$ is a nonhomogeneous polynomial, then we denote by $F^* \in k[X_1, \dots, X_n, X_{n+1}]$, the homogenization of F with respect to the new variable X_{n+1} . Note that if F^* has a nonsingular zero $(a_1, \dots, a_n, a_{n+1})$ with $a_i \in k$, and $a_i a_{n+1} \neq 0$ for some $i \leq n$, then F has a nonzero nonsingular zero defined over k .

For the remainder of this note, let K denote a p -adic field with ring of integers R , and let (π) be the maximal ideal of R . Let $k = R/(\pi)$ denote the residue field of K , and let $|k|$ denote the cardinality of k .

2. On cubic forms. Let us first prove the following theorem.

Theorem 2.1. *Let $F \in K[X_1, \dots, X_n]$ be a cubic polynomial in at least 9 variables that is not homogeneous. Then F has a nonzero nonsingular zero in K provided $|k| \geq 5$.*

Proof. Without loss of generality, we may assume that $F \in R[X_1, \dots, X_n]$. Let $\overline{F}^*$ denote the reduction of F^* to the residue field k . Let $m = \text{order}(\overline{F}^*)$. Since we may assume F^* reduced, it then follows from Lemma B of [B-Le] that $m \geq 4$. Thus

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$\overline{F}^*$ depends on exactly m variables $X_1, \dots, X_m$ after an invertible linear change of variables over k . Thereby $\overline{F}^*$ is nondegenerate. By Theorem 3 of [Le-S], $\overline{F}^*$ has a nontrivial nonsingular zero $(a_1, \dots, a_m)$ (in the projective sense) that does not lie on the hyperplane given by $X_{n+1} = 0$.

Assume that X_{n+1} does not appear in $\overline{F}^*$. In this case, Hensel's lemma implies that F^* has a nontrivial nonsingular zero $(b_1, \dots, b_{n+1})$ (in the projective sense) that does not lie on the hyperplane given by $X_{n+1} = 0$ and such that $(b_1, \dots, b_n)$ is different from $(0, \dots, 0)$. Then $(b_1, b_2, \dots, b_n)$ is a nonzero nonsingular zero of F .

Assume now that X_{n+1} appears in $\overline{F}^*$ with a nonzero coefficient. We can suppose that X_{n+1} is X_m . First suppose $m - 1 \geq 6$, and let $f(X_1, \dots, X_{m-1})$ denote the polynomial $\overline{F}^*(X_1, \dots, X_{m-1}, 1)$. With a little work, we may assume that the nonsingular zero of $\overline{F}^*$ has the form $(a_1, \dots, a_{m-1}, 1)$ where $(a_1, \dots, a_{m-1}) \neq (0, 0, \dots, 0)$. Then F^* has a p -adic nonsingular zero $(b_1, b_2, \dots, b_{n+1})$ (in the projective sense) by Hensel's lemma that does not lie on the hyperplane given by $X_{n+1} = 0$ and such that $(b_1, \dots, b_n)$ is different from $(0, \dots, 0)$. Therefore $(b_1, b_2, \dots, b_n)$ is a nonzero nonsingular zero of F .

Now assume $m - 1 \leq 5$. Then the nonsingular zero $(a_1, \dots, a_m)$ of $\overline{F}^*$, guaranteed to exist by Theorem 3 of [Le-S], can be refined by Hensel's lemma to a local point of F^* , say $(b_1, \dots, b_{n+1})$, that does not lie on the hyperplane given by $X_{n+1} = 0$ and such that $(b_1, \dots, b_n)$ is different from $(0, \dots, 0)$. This is achieved by setting at least one of the variables that do not appear explicitly in $\overline{F}^*$ to be nonzero. The conclusion follows then easily. $\square$

We can now deduce the main result of this note.

Corollary 2.2. *Let $C \in K[X_1, \dots, X_n]$ be a nondegenerate cubic form of order at least 10. Let H be a hyperplane defined over K and given by a nonzero linear form $L \in K[X_1, \dots, X_n]$. Assume that $|k| \geq 5$. Then C has a nonsingular zero defined over K that does not lie on H .*

Proof. After a linear change of variables we can assume that H is given by $X_n = 0$. Since C is nondegenerate of order n , X_n appears explicitly in C .

First suppose that X_n divides C . Then $C = X_n Q(X_1, \dots, X_n)$, where Q is a quadratic form of order at least 9. In this case we know that Q has a nonsingular zero that does not belong to H , and so does C .

Now assume that X_n does not divide C and let $F = C(X_1, \dots, X_{n-1}, 1)$. Then F is a cubic polynomial that is not homogeneous. By Theorem 2.1, F has a zero, which gives a zero of C outside the hyperplane H . $\square$

Résumé substantiel en français. Nous nous intéressons au problème de trouver un zéro non-singulier d'une forme F définie sur un corps p -adique K qui n'est pas un zéro d'une autre forme G . Appelons *forme* sur un corps commutatif K tout polynôme homogène à coefficients dans K . Soit $f \in K[X_1, \dots, X_n]$ une forme à n variables définie sur K ; nous dirons que $x = (x_1, \dots, x_n) \in K^n$ est un *zéro non-singulier* de f si x est un zéro de f et n'est pas zéro d'au moins une dérivée partielle de f .

Nous démontrons un résultat dans le cas où F est une forme cubique et G une forme linéaire : Si C est une forme cubique non-dégénérée définie sur un corps p -adique K

d'ordre au moins 10, alors C possède un zéro non-singulier défini sur K n'appartenant pas à l'hyperplan H défini sur K pourvu que le cardinal du corps résiduel de K soit supérieur ou égal à 5.

Nous démontrons préalablement le résultat suivant : Soit $F \in K[X_1, \dots, X_n]$ un polynôme de degré 3 et d'au moins 9 variables qui n'est pas homogène; soit k le corps résiduel de K . Alors F a un zéro non-singulier non nul dans K si $|k| \geq 5$.

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