

INVARIANT FACTORS OF AN ENDOMORPHISM OF A PROJECTIVE MODULE

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RÉSUMÉ. Soit R un domaine commutatif intégralement clos. Pour tout endomorphisme u d'un R -module projectif M de rang constant nous définissons un polynôme caractéristique et donnons une suite finie de polynômes unitaires $\{s_i(u, X)\}_{1 \leq i \leq t}$ dont le produit est le polynôme caractéristique de u et $s_i(u, X)$ divise $s_{i+1}(u, X)$. Comme pour les facteurs invariants dans le cas des espaces vectoriels, ces polynômes sont les générateurs unitaires des idéaux $[F_{t-i}(M_u) : F_{t-i+1}(M_u)]$, où $F_i(M_u)$ est le i^{e} idéal de Fitting du $R[X]$ -module M_u défini sur M via u . Nous donnons aussi le lien entre les $s_i(u, X)$ et les facteurs invariants de M_u , tels que définis en [3].

ABSTRACT. Let R be a commutative integrally closed domain. For any endomorphism u of a projective R -module M of constant rank, we define its characteristic polynomial and we give a finite sequence of monic polynomials $\{s_i(u, X)\}_{1 \leq i \leq t}$ such that their product is the characteristic polynomial of u and $s_i(u, X)$ divides $s_{i+1}(u, X)$. Like the invariant factors in the case of vector spaces, these polynomials are the monic generators of the ideals $[F_{t-i}(M_u) : F_{t-i+1}(M_u)]$, where $F_i(M_u)$ is the i^{th} Fitting ideal of the $R[X]$ -module M_u defined on M via u . We also give the link between the $s_i(u, X)$'s and the invariant factors of M_u , as defined in [3].

1. Introduction. Let R be a commutative ring with unity, and M a finitely generated R -module. In this paper, we are concerned with the classical problem of classification of R -endomorphisms of a projective constant rank R -module M . We shall assign to each endomorphism u of M a finite set of monic polynomials such that when $R = K$ is a field and $M = E$ is a finite dimensional vector space, these polynomials are the classical invariant factors associated to u .

Let M be an R -module and $u : M \longrightarrow M$ an R -endomorphism. The endomorphism u converts M into an $R[X]$ -module by $X \cdot x = u(x)$ for each $x \in M$. We will denote M by M_u when we view M as an $R[X]$ -module via u .

In the classical case, i.e. when $R = K$ is a field and $M = E$ is a finite dimensional vector space, the invariant factors theorem implies

$$M_u = \bigoplus_{i=1}^t K[X]/(s_i(u, X)),$$

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where $\{s_i(u, X)\}_{1 \leq i \leq t}$ are the invariant factors of u , i.e. are monic generators of the ideals $[F_{t-i}(E_u) : F_{t-i+1}(E_u)]_{1 \leq i \leq t}$, where $F_i(E_u)$ is the i^{th} Fitting ideal of E_u . Two endomorphisms u and v are similar if and only if $s_i(u, X) = s_i(v, X)$ for each i , or equivalently, if and only if $F_i(E_u) = F_i(E_v)$ for each i . In Section 2, we recall the notion of the Fitting ideals of a finitely presented module over a commutative ring and describe some of their properties. We also give the characteristic exact sequence which allows us to compute the Fitting ideals of M_u when M is a finite free R -module and u is an endomorphism of M . In Section 3, we prove Lemma 3.1 and Proposition 3.2 which are crucial tools in the proof of our main result, Theorem 3.3, in which we give the construction of the invariant factors $\{s_i(u, X)\}_{1 \leq i \leq t}$ associated to an endomorphism u of a projective R -module of constant rank. We show that the invariant factors of M_u , according to the definition in [3], are the smallest principal ideals of $R[X]$ generated by monic polynomials that contain the $s_i(u, X)$. In the fourth section, we give the construction of the invariant factors $\{s_i(u, X)\}_{1 \leq i \leq t}$ associated to an endomorphism u of a projective module of constant rank.

2. Preliminaries and notation. Let R be a commutative ring with unity, and M a finitely presented module. We denote by $F_i(M)$ the i^{th} Fitting ideal of M (see [1]), i.e, if

$$R^m \xrightarrow{u} R^n \longrightarrow M \longrightarrow 0$$

is a presentation of M , and A is the matrix of u relatively to some R -bases of R^m and R^n , then

$$F_k(M) = \begin{cases} \mathcal{U}_{n-k}(A) & \text{if } k = 0, 1, 2, \dots, n-1, \\ R & \text{if } k \geq n, \end{cases}$$

where $\mathcal{U}_{n-k}(A)$ is the ideal generated by the $(n-k) \times (n-k)$ minors of A . The Fitting ideals satisfy the following properties:

- (1) $0 \subseteq F_0(M) \subseteq F_1(M) \subseteq \dots$.
- (2) $(\text{Ann}_R(M))^n \subseteq F_0(M) \subseteq \text{Ann}_R(M)$.
- (3) Let $u : R \longrightarrow R'$ be a homomorphism of commutative rings. Then for $k \geq 0$,

$$F_k(R' \otimes_R M) = F_k(M) \cdot R'.$$

In particular, if S is a multiplicatively closed subset of R , and I is an ideal of R , then for any positive integer k ,

$$F_k(S^{-1}M) = S^{-1}F_k(M) \text{ and } F_k(M/IM) = F_k(M) \cdot R/I.$$

Let M be a finitely generated R -module and $u : M \longrightarrow M$ an R -endomorphism. We have the characteristic exact sequence

$$0 \longrightarrow R[X] \otimes_R M \xrightarrow{\psi_u} R[X] \otimes_R M \xrightarrow{\varphi} M_u \longrightarrow 0$$

where

$$\psi_u(f(X) \otimes m) = X \cdot f(X) \otimes m - f(X) \otimes u(m)$$

and

$$\varphi(f(X) \otimes m) = f(X) \cdot m = f(u)(m)$$

for each $(m, f(X)) \in M \times R[X]$ (see [1]). Therefore if M is a finite free R -module and A is the matrix of u , then $XI_n - A$ is a matrix of ψ_u , where I_n is the $n \times n$ identity matrix. So $F_k(M_u) = \mathcal{U}_{n-k}(XI_n - A)$ for any positive integer k . In consequence, the ideal $F_0(M_u)$ is the principal ideal generated by the characteristic polynomial $\chi_u(X)$ of u , while the ideals $F_k(M_u)$ for $k \geq 1$ are not principal in general. Therefore in this case, we can talk about the characteristic polynomial but not about the invariant factors.

In the third section, we show that for an endomorphism of a finite free module over an integrally closed domain, we can construct a sequence of monic polynomials analogous to the invariant factors in the case of vector spaces.

3. Invariant factors of an endomorphism. In this section, R is an integrally closed domain and u is a R -endomorphism of a finite free R -module M of rank n . Let K be the quotient field of R , and set $E = K \otimes_R M$ and $u(K) = 1_K \otimes u$. Then

$$E_{u(K)} \simeq K[X] \otimes_{R[X]} M_u \text{ and } F_k(E_{u(K)}) = F_k(M_u) \cdot K[X]$$

for any positive integer k (see [1, Proposition 1.2]).

Lemma 3.1. *Let J be an ideal of $R[X]$, which contains a monic polynomial of $R[X]$. Set $K \cdot J = f(X)K[X]$, where $f(X) \in K[X]$ is monic. Then we have:*

- (i) $f(X) \in R[X]$,
- (ii) $J \subseteq f(X)R[X]$,
- (iii) $f(X)R[X]$ is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains J .

Proof. (i) Let $g(X)$ be a monic polynomial in J . Then there exists $h(X) \in K[X]$ such that $g(X) = f(X)h(X)$. Hence every root of $f(X)$ is a root of $g(X)$, and every root of $f(X)$ is integral over R . Since $f(X)$ is monic, its coefficients are integral over R . The fact that R is integrally closed implies that $f(X) \in R[X]$.

(ii) Let $g(X) \in J$. Then $g(X) = f(X)h(X)$ with $h(X) \in K[X]$. Moreover, since $f(X)$ is monic, the Euclidian algorithm implies that there exist polynomials $q(X)$ and $r(X)$ in $R[X]$ such that $g(X) = f(X)q(X) + r(X)$, where the degree of $r(X)$ is smaller than the degree of $f(X)$. Hence $r(X) = 0$ and $h(X) = q(X) \in R[X]$. So $g(X) \in f(X)R[X]$ and then $J \subseteq f(X)R[X]$.

(iii) Suppose that there exist another monic polynomial $g(X)$ in $R[X]$ such that $J \subseteq g(X)R[X]$. Then $K \cdot J \subseteq g(X)K[X]$. So $f(X)K[X] \subseteq g(X)K[X]$. Hence $f(X) = g(X)h(X)$, where $h(X) \in K[X]$. Then, by Gauss Lemma, $h(X) \in R[X]$ ($f(X)$ and $g(X)$ are monic). Hence $f(X)R[X] \subseteq g(X)R[X]$. $\square$

For two ideals I and J of R , we denote by $[I : J]$ the quotient ideal of I and J , i.e.,

$$[I : J] = \{a \in R \mid aJ \subseteq I\}.$$

Proposition 3.2. *For any positive integer k , there exists a monic polynomial $q_k(X)$ (resp. $q'_k(X)$) in $R[X]$ such that $q_k(X)R[X]$ (resp. $q'_k(X)R[X]$) is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $F_k(M_u)$ (resp. $[F_k(M_u) : F_{k+1}(M_u)]$).*

Proof. Let k be a positive integer. Set

$$J = F_k(M_u) \text{ and } J' = [F_k(M_u) : F_{k+1}(M_u)].$$

On the one hand, we know that $F_r(E_{u(K)}) = F_r(M_u)K[X]$ for any positive integer r . Then $F_k(E_{u(K)}) = J \cdot K[X]$. So, by [2, corollary 3.15, p. 43],

$$[F_k(E_{u(K)}) : F_{k+1}(E_{u(K)})] = J' \cdot K[X].$$

Set

$$J \cdot K[X] = q_k(X)K[X] \text{ and } J' = q'_k(X)K[X],$$

where $q_k(X)$ and $q'_k(X)$ are two monic polynomials of $K[X]$. On the other hand, $J = F_k(M_u) = \mathcal{U}_{n-k}(XI_n - A)$, where n is the rank of M , and A is a matrix of u . Then J and J' contain monic polynomials. So the result holds by Lemma 3.1 (iii). $\square$

The following result gives the construction of the “invariant factors” of an endomorphism of a finite free module.

Theorem 3.3. *Suppose R is an integrally closed domain, and u a R -endomorphism of a finite free R -module M . There exist unique monic polynomials $s_1(u, X), \dots, s_t(u, X)$ of positive degree verifying:*

- (i) $\chi_u(X) = s_1(u, X) \cdots s_t(u, X)$, where $\chi_u(X)$ is the characteristic polynomial of u .
- (ii) For $i = 1, \dots, t-1$, $s_i(u, X)$ divides $s_{i+1}(u, X)$.
- (iii) For each positive integer $k \leq t-1$, the ideal $(\prod_{i=1}^{t-k} s_i(u, X))$ is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $F_k(M_u)$.
- (iv) For each positive integer $k \geq t$, the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $F_k(M_u)$ is $R[X]$.

Proof. Let $t = \min \{k \geq 0 \mid F_k(M_u) = R[X]\}$. For any positive integer k , there exists a monic polynomial $q'_k(X)$ of $R[X]$ such that $q'_k(X)R[X]$ is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $[F_k(M_u) : F_{k+1}(M_u)]$ (by Proposition 3.2). Then we put $s_i(u, X) = q'_{t-i}(X)$ for $i = 1, \dots, t$. So we have a sequence of monic polynomials of $R[X]$. The uniqueness of these polynomials is insured by the definition of the $q'_i(X)$. Since the polynomials $s_1(u, X), \dots, s_t(u, X)$ are the invariant factors of $u(K)$, then (i) and (ii) holds. Finally (iii) and (iv) are consequences of Proposition 3.2. $\square$

Definition 3.4. Let the assumptions and notation be as in Theorem 3.3. We define the invariant factors of u as the polynomials $\{s_i(u, X)\}_{1 \leq i \leq t}$ given in Theorem 3.3.

The following corollary makes the link between the invariant factors of u and the invariant factors $\{\delta_k(M_u)\}_{k \geq 1}$ of M_u defined in [3] by

$$\delta_k(M_u) = \text{Ann}_{R[X]} \left(\bigwedge^k (M_u) \right),$$

where $\bigwedge^k(M_u)$ is the k^{th} exterior power of M_u .

Corollary 3.5. *Let k be a positive integer $\leq t-1$. Then $s_{t-k}(u, X)R[X]$ is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $\delta_{k+1}(M_u)$.*

Proof. Let k be a positive integer $\leq t-1$. Then on the one hand,

$$\delta_k(E_{u(K)}) = \text{Ann}_{K[X]} \left(\bigwedge^k(E_{u(K)}) \right).$$

On the other hand,

$$\bigwedge^{k+1}(E_{u(K)}) = \bigwedge^{k+1}(K[X] \otimes_{R[X]} M_u) \cong K[X] \otimes_{R[X]} \bigwedge^{k+1}(M_u)$$

(see [9, p. 204]). Then

$$\begin{aligned} \text{Ann}_{K[X]} \left(\bigwedge^{k+1}(E_{u(K)}) \right) &= \text{Ann}_{K[X]} \left(K[X] \otimes_{R[X]} \bigwedge^{k+1}(M_u) \right) \\ &= S^{-1} \left(\text{Ann}_{R[X]} \left(R[X] \otimes_{R[X]} \bigwedge^{k+1}(M_u) \right) \right), \end{aligned}$$

where $S = R \setminus \{0\}$. So

$$\text{Ann}_{K[X]} \left(\bigwedge^{k+1}(E_{u(K)}) \right) = \text{Ann}_{R[X]} \left(\bigwedge^{k+1}(M_u) \right) K[X].$$

Hence $\delta_{k+1}(E_{u(K)}) = \delta_{k+1}(M_u)K[X]$. Since

$$\begin{aligned} \delta_{k+1}(E_{u(K)}) &= [F_k(E_{u(K)}) : F_{k+1}(E_{u(K)})] \\ &= [F_k(M_u) : F_{k+1}(M_u)]K[X] = s_{t-k}(u, X)K[X], \end{aligned}$$

then $\delta_{k+1}(M_u)K[X] = s_{t-k}(u, X)K[X]$. By [5, corollary 1.4],

$$\chi_u(X)R[X] = F_0(M_u) \subseteq \delta_{k+1}(M_u),$$

so $\delta_{k+1}(M_u)$ contains a monic polynomial of $R[X]$. The result holds by Lemma 3.1. $\square$

4. The case of a projective module of constant rank. In this section, our aim is to extend the main result, Theorem 3.3, of the previous section, which allows us to define invariant factors of an endomorphism of a projective module of constant rank. First, we define the characteristic polynomial of u . Let $\mathcal{P}$ be a maximal ideal of R , $u_{\mathcal{P}}$ the $R_{\mathcal{P}}$ -endomorphism of $M_{\mathcal{P}}$, which extends u , and $\chi_{\mathcal{P},u}(X)$ the characteristic polynomial of $u_{\mathcal{P}}$.

Lemma 4.1. *If R is a domain, then $\chi_{\mathcal{P},u}(X)$ is a monic polynomial in $R[X]$ of degree n , defined independently of the choice of the maximal ideal $\mathcal{P}$ of R .*

Proof. Let K be the quotient field of R and $\mathcal{P}$ a maximal ideal of R and $(e_1, \dots, e_n)$ a $R_{\mathcal{P}}$ -basis of $M_{\mathcal{P}}$. Then $(e_1, \dots, e_n)$ is a K -basis of $K \otimes_R M$. Since the $R_{\mathcal{P}}$ -endomorphism $u_{\mathcal{P}}$ and the K -endomorphism $u(K)$, induced by u , operate in the same manner over $(e_1, \dots, e_n)$, then they have the same characteristic polynomial. Consequently,

$$\chi_{u(K)}(X) \in \bigcap_{\mathcal{P} \in \max(R)} R_{\mathcal{P}}[X]$$

and then $\chi_{\mathcal{P},u}(X)$ is a monic polynomial in $R[X]$ of degree n which is defined independently of the choice of the maximal ideal $\mathcal{P}$ of R . $\square$

Set $I(u) = \{f \in R[X] \mid f(u) = 0\}$ the annihilator ideal of M_u . Then we have the natural extension of McCoy's Theorem.

Proposition 4.2. *Let R be a domain, M a projective R -module of constant rank and u a R -endomorphism M . Then*

$$I(u) = [(\chi_u(X)) : F_1(M_u)].$$

Proof. Since $\bigoplus_{\mathcal{P} \in \max(R)} R_{\mathcal{P}}$ is a faithfully-flat R -algebra, it suffices to check the result for localization. Since $(F_1(M_u))_{\mathcal{P}} = F_1(M_{u_{\mathcal{P}}})$, $\chi_{u_{\mathcal{P}}}(X) = \chi_u(X)$ and $[I : J]_{\mathcal{P}} = [I_{\mathcal{P}} : J_{\mathcal{P}}]$ for all ideals I and J of R , we can use the classical theorem of McCoy. $\square$

Corollary 4.3. *Let R be a domain, M be a projective R -module of constant rank and u a R -endomorphism M . Then $\chi_u(u) = 0$. In particular, u is integral over R .*

Remark 4.4. Let u be a R -endomorphism of a projective R -module of constant rank. Since u is integral over R , if R is an integrally closed domain, the minimal polynomial $\mu_u(X)$ of u lies in $R[X]$, i.e., $I(u)$ is a principal ideal which is generated by a monic polynomial of $R[X]$. Consequently, $F_1(M_u)$ is a principal ideal of $R[X]$, which is generated by $\chi_u(X)/\mu_u(X)$.

For any maximal ideal $\mathcal{P}$ of R , $M_{\mathcal{P}}$ is a free $R_{\mathcal{P}}$ -module. Let $s_1(u_{\mathcal{P}}, X), \dots, s_t(u_{\mathcal{P}}, X)$ be the invariant factors of $u_{\mathcal{P}}$, which do not depend upon the choice of the maximal ideal $\mathcal{P}$ of R . Define the invariant factors of u as the invariant factors of $u_{\mathcal{P}}$ for some maximal ideal $\mathcal{P}$ of R . Since $R_{\mathcal{P}}$ is an integrally closed domain, we have the following result:

Theorem 4.5. *Under the previous hypotheses, the following properties hold true.*

- (i) $\chi_u(X) = s_1(u, X) \cdots s_t(u, X)$, where $\chi_u(X)$ and $\mu_u(X)$ are respectively the characteristic polynomial and the minimal polynomial of u .
- (ii) $s_i(u, X)$ divides $s_{i+1}(u, X)$ for $i = 1, \dots, t-1$.
- (iii) For each positive integer $k \leq t-1$ the ideal $\left(\prod_{i=1}^{t-k} s_i(u, X)\right)$ is the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $F_k(M_u)$.
- (iv) For each positive integer $k \geq t$, the smallest principal ideal of $R[X]$ generated by a monic polynomial that contains $F_k(M_u)$ is $R[X]$.

Résumé substantiel en français. Soient R un anneau commutatif unitaire et M un R -module. La donnée d'un R -endomorphisme u de M permet de munir M d'une structure de $R[X]$ -module : $X \cdot m = u(m)$ pour tout élément m de M . Le $R[X]$ -module obtenu sera noté M_u .

Considérons le cas classique, c'est-à-dire le cas où $R = K$ est un corps et $M = E$ est un K -espace vectoriel de dimension finie. Dans ce cas, on sait que les facteurs invariants $(d_i(u, X))_{1 \leq i \leq s}$ de u , où $d_i(u, X)$ divise $d_{i+1}(u, X)$ pour $i = 1, \dots, s-1$, sont les générateurs des idéaux $([F_{s-i}(E_u) : (F_{s-i+1}(E_u))])_{1 \leq i \leq s}$, où $F_i(E_u)$ est le i^e idéal de Fitting de E_u . Les idéaux de Fitting de E_u sont des idéaux de $K[X]$ et $F_0(E_u)$ est engendré par le polynôme caractéristique de u .

Nous allons nous inspirer de ce lien entre les facteurs invariants et les idéaux de Fitting, dans le cas classique, pour construire une suite analogue de polynômes unitaires dans le cas des modules projectifs de rang fini sur un anneau intégralement clos. D'abord, soit M un R -module libre de rang fini n ; on a la suite exacte

$$R[X] \otimes_R M \xrightarrow{\psi_u} R[X] \otimes_R M \xrightarrow{\varphi_u} M_u \longrightarrow 0,$$

où

$$\psi(f(X) \otimes m) = X \cdot f(X) \otimes m - f(X) \otimes u(m) \text{ et } \varphi(f(X) \otimes m) = f(X) \cdot m = f(u)(m)$$

pour tout élément m de M et tout polynôme $f(X)$ de R . Donc si on désigne par A une matrice de u alors $XI_n - A$ est une matrice de ψ_u , où I_n est la matrice unité d'ordre n . Par conséquent, $F_k(M_u) = \mathcal{U}_{n-k}(XI_n - A)$ pour tout entier naturel k . Ainsi $F_0(M_u)$ est engendré par le polynôme caractéristique $\chi_u(X)$ de u , tandis que les autres idéaux de Fitting ne sont pas nécessairement principaux. Par suite, on peut toujours parler du polynôme caractéristique de u mais pas de ses facteurs invariants.

Expliquons comment sur un anneau intégralement clos on peut construire une suite de polynômes unitaires de R analogue à celle des facteurs invariants dans le cas des espaces vectoriels. On suppose que R est un anneau commutatif intégralement clos et M est un R -module libre de rang n muni d'un R -endomorphisme u . Soit K le corps des fractions de R . On pose $E = K \otimes_R M$ et $u(K) = 1_K \otimes u$. Alors

$$E_{u(K)} \cong K[X] \otimes_{R[X]} M_u \text{ et } F_k(E_{u(K)}) = F_k(M_u) \cdot K[X]$$

pour tout entier naturel k .

Proposition 3.2. *Pour tout entier naturel k , il existe un polynôme unitaire $q_k(X)$ (resp. $q'_k(X)$) de R tel que $q_k(X)R[X]$ (resp. $q'_k(X)R[X]$) est le plus petit idéal de $R[X]$ contenant $F_k(M_u)$ (resp. $[F_k(M_u) : F_{k+1}(M_u)]$).*

Le résultat suivant donne la construction des facteurs invariants d'un endomorphisme de module libre sur un anneau intégralement clos.

Théorème 3.3. *Supposons que R est intégralement clos. Soit M un R -module libre de rang fini muni d'un R -endomorphisme u . On a une suite de polynômes unitaires uniques $s_1(u, X), \dots, s_t(u, X)$ de R qui vérifient les propriétés suivantes :*

- (i) $\chi_u(X) = s_1(u, X) \cdots s_t(u, X)$, où $\chi_u(X)$ est le polynôme caractéristique de u ;

- (ii) $s_i(u, X)$ divise $s_{i+1}(u, X)$, pour $i = 1, \dots, t-1$;
- (iii) Pour tout entier positif $k \leq t-1$, l'idéal $(\prod_{i=1}^{t-k} s_i(u, X))$ est le plus petit idéal de $R[X]$ contenant $F_k(M_u)$ et engendré par un polynôme unitaire;
- (iv) Pour tout entier positif $k \geq t$, le plus petit idéal de $R[X]$ contenant $F_k(M_u)$ et engendré par un polynôme unitaire est $R[X]$.

Définition 3.4. On définit les facteurs invariants de u comme étant les polynômes

$$(s_i(u, X))_{1 \leq i \leq t}$$

construits dans le théorème précédent.

Dans le cas d'un R -module M projectif de rang constant, on donne une généralisation du polynôme caractéristique d'un endomorphisme u . On montre que le R -endomorphisme u et le K -endomorphisme $u(K)$ ont le même polynôme caractéristique. On montre que le théorème de Cayley-Hamilton reste aussi vrai dans ce cas. Par suite, le R -endomorphisme u est entier sur l'anneau intégralement clos R et admet donc un polynôme minimal dans $R[X]$. On termine avec un théorème analogue au précédent, dans le cas des modules projectifs de rang fini sur un anneau intégralement clos, qui nous permet de définir la suite des facteurs invariants de u , dans le cas des modules projectifs de rang fini sur un anneau intégralement clos.

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