

ON THE CANONICAL FORMS OF FINITELY GENERATED MODULES OVER COMMUTATIVE RINGS

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RÉSUMÉ. Soit R un anneau commutatif et M un R -module de type fini. Alors M est un CF-module si $M \cong \bigoplus_{i=1}^n R/I_i$, où les I_i sont des idéaux de R et $I_1 \subseteq \cdots \subseteq I_n \neq R$. Le module M est dit NCF si les idéaux I_i sont principaux. Dans cet article, nous revoyons les résultats que nous avons établis dans [3] et [2] concernant la décomposition de modules de type fini sur des anneaux commutatifs. Nous étudions les propriétés des CF-modules en termes d'idéaux de Fitting et nous examinons les NCF-modules sur des anneaux locaux et des anneaux de valuation. En particulier, nous donnons des conditions nécessaires et suffisantes pour qu'un module de type fini soit un NCF-module.

ABSTRACT. Let R be a commutative ring and let M be a finitely generated R -module. Then M is a CF-module if $M \cong \bigoplus_{i=1}^n R/I_i$, where the I_i 's are ideals of R and $I_1 \subseteq \cdots \subseteq I_n \neq R$. The module M is said to be a NCF-module if the I_i 's are cyclic. In this paper, we review the results we established in [3] and [2] concerning the decomposition of finitely generated modules over commutative rings. We study the properties of CF-modules in terms of Fitting ideals, and we investigate the NCF-modules over local rings and valuation rings. In particular, we give necessary and sufficient conditions for a finitely generated module to be a NCF-module.

1. Introduction. Throughout this note, R will denote a commutative ring and M a finitely generated R -module. A canonical form for M is a decomposition

$$M \cong R/I_1 \oplus \cdots \oplus R/I_n,$$

where the I_i 's are ideals of R and $I_1 \subseteq \cdots \subseteq I_n \neq R$. If M has a canonical form, the ideals I_i are uniquely determined (see [8]), and we say that M is a *CF-module*. When the ideals I_i are principal, the decomposition is called the *normal canonical form* of M and M is called a *NCF-module*.

In this note we give a survey of the results published in [3] and [2] on the decomposition of finitely generated modules over commutative rings. We exhibit conditions forcing finitely generated modules over local rings and valuation rings to be NCF-modules.

Reçu le 10 septembre 2003 et, sous forme définitive, le 6 mai 2005.

The Fitting ideals are fundamental tools in this investigation. These invariants are defined as follows: Let $X = \{x_1, \dots, x_n\}$ be a set of generators of M . A *relation* in M is a vector $(a_1, \dots, a_n)$ in R^n such that $\sum_{i=1}^n a_i x_i = 0$. For positive integers $k = 0, \dots, n-1$, the k^{th} Fitting ideal of M is defined to be the ideal $F_k(M)$ generated by the determinants of all $(n-k) \times (n-k)$ -submatrices of the matrix

$$K(M) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

where the vectors $(a_{i1}, \dots, a_{in})$ are the relations in M . If $k \geq n$, we define $F_k(M) = R$. These ideals form an ascending sequence of invariant ideals for M independently of the choice of X (see [6] or [10]). Most of our computations on Fitting ideals are based on the following proposition; see [2].

Proposition 1.1. *Let $M = M_1 \oplus M_2$ be a direct sum of two finitely generated R -modules. Then the following properties hold true:*

- (i) *For any positive integer k , $F_k(M_1) \subseteq F_{n_2+k}(M)$ and $F_k(M_2) \subseteq F_{n_1+k}(M)$ where n_1 and n_2 are the sizes of the sets of generators of M_1 and M_2 respectively.*
- (ii) *$F_k(M_i) \subseteq F_{k+1}(M_i) \cdot F_{n-1}(M)$, where $k \leq n_i - 1$, $i = 1, 2$ and $n = n_1 + n_2$.*

If M is a CF-module such that $M \cong \oplus_{i=1}^n R/I_i$, then by [2] we have

$$F_k(M) = I_{k+1} \dots I_n, \text{ for } k = 0, \dots, n-1.$$

There are two numerical invariants attached to the sequence of Fitting ideals of M , namely

$$\nu(M) := \min\{k \mid F_k(M) = R\} \quad \text{and} \quad \omega(M) := \min\{k \mid F_k(M) \neq 0\}.$$

Denote by $I(M)$ the ideal $F_{\omega(M)}(M)$ and by $\mu(M)$ the minimal number of generators of M . In the case of a finitely generated R -module M where R is a local ring, we have $\mu(M) = \nu(M)$; see [3] and [1].

Proposition 1.2. *Let $I_1 \subseteq \dots \subseteq I_n$ be an increasing sequence of proper ideals of R . Let $M = \oplus_{i=1}^n R/I_i$. Then we have:*

- (i) $\mu(M) = \nu(M) = n$,
- (ii) $\omega(M)$ is the greatest positive integer k verifying $I_k \subseteq (0 : I_{k+1} \dots I_n)$.

2. NCF-modules over local rings. A normal canonical form for a R -module M is a decomposition

$$M \cong \oplus_{i=1}^n R/(a_i),$$

where a_i divides a_{i+1} for $i = 1, \dots, n-1$ and $(a_1) \neq R$; we say that M is a NCF-module. In this case, the ideals (a_i) are uniquely determined.

Proposition 2.1. Let $M = \bigoplus_{i=1}^n R/(a_i)$ be a NCF-module M over R . Then we have

- (i) $F_k(M) = (a_1 \dots a_{n-k})$ for $k = 0, \dots, n-1$,
- (ii) If $F_0(M)$ contains a non-zero divisor, then $(a_{n-k+1}) = (F_{k-1}(M) : F_k(M))$, for $k = 1, \dots, n$.

Proposition 2.2. Suppose R is a local ring. Let M be a finitely generated R -module and $\mu(M) = n$. Suppose that $F_{n-1}(M)$ is a principal ideal. Then $R/F_{n-1}(M)$ is a direct summand of M .

The following main theorem of [2] gives conditions for a finitely generated R -module to be a NCF-module.

Theorem 2.3. Suppose R is a local ring. Let M be a finitely generated R -module. Let $\mu(M) = r$. Then the following assertions are equivalent:

- (i) For $k \geq \omega(M)$, the ideal $F_k(M)$ is principal and can be canceled in the set of non-zero ideals of R .
- (ii) The module M verifies $M \cong (\bigoplus_{i=1}^{r-s} R/(a_i)) \oplus R^s$, where a_i divides a_{i+1} and (a_i) can be canceled for $1 \leq i \leq r-s-1$, $s = \omega(M)$ and $a_1 R \neq R$.

Corollary 2.4. Let M be a finitely generated module over a local ring R and let $\mu(M) = r$. Suppose that $I(M)$ is a regular ideal (so $I(M)$ contains a non-zero divisor element). Then the following assertions are equivalent:

- (i) The module M is a NCF-module.
- (ii) The Fitting ideals of M are principal ideals.

Corollary 2.5. Suppose R is a local ring. Let M and M' be two finitely generated R -modules whose Fitting ideals are principal ideals which can be canceled. If $F_k(M) = F_k(M')$ for all positive integers k , then M and M' are isomorphic NCF-modules.

3. The case of valuation rings. The main result of this section gives a condition under which any finitely generated module over a valuation ring is a NCF-module.

Proposition 3.1. Suppose R is a valuation ring. Let M be a finitely generated R -module whose Fitting ideals are principal ideals, and let $n = \mu(M)$. If N is a direct summand of M such that $M = N \oplus R/F_{n-1}(M)$, then the Fitting ideals of N are also principal ideals.

Theorem 3.2. Suppose R is a valuation ring. Let M be a finitely generated R -module whose Fitting ideals are principal ideals, and let $n = \mu(M)$. Then there exist elements $a_1, \dots, a_n$ in R such that $M \cong \bigoplus_{i=1}^n R/(a_i)$, where a_i divides a_{i+1} for $i = 1, \dots, n-1$.

As a corollary of this theorem we obtain a result of R.B. Warfield [8].

Corollary 3.3. Every finitely presented module over a valuation ring is a NCF-module.

Corollary 3.4. Suppose R is a valuation ring. Let M be a finitely generated R -module whose Fitting ideals are principal ideals. Then M is finite free if and only if there exists a positive integer r such that $F_{r-1}(M) = 0$ and $F_r(M) = R$.

In virtue of this corollary, the computation of the Fitting ideals of a finitely generated module M allows us to decide whether M is free or not. For two ideals I and J of R , define the quotient ideal of I by J to be $(I : J) = \{a \in R \mid aI \subseteq J\}$.

Lemma 3.5. Suppose R is a valuation ring. Let $a_1, \dots, a_n$ be elements in R such that a_i divides a_{i+1} for $i = 1, \dots, n-1$. Let $M = \bigoplus_{i=1}^n R/(a_i)$, and let $p = \omega(M)$ and $q = \nu(M)$. Then

$$(F_k(M) : F_{k+1}(M)) = \begin{cases} R & \text{if } k < p-1, \\ \text{Ann}_R((a_1 \dots a_{n-p})) & \text{if } k = p-1, \\ (a_{n-k}) & \text{if } p \leq k \leq q-1, \\ (a_1 \dots a_{n-q+1}) & \text{if } k = q-1, \\ R & \text{if } k \geq q. \end{cases}$$

Corollary 3.6. Let R be a valuation ring. Let M be a finitely generated R -module whose Fitting ideals are non-zero principal ideals. Then $M \cong \bigoplus_{k=0}^{n-1} R/(F_k(M) : F_{k+1}(M))$, where $n = \mu(M)$.

Corollary 3.7. Suppose R is a valuation ring. Let M and M' be two finitely generated R -modules. Suppose that the Fitting ideals of M are non-zero principal ideals. Then, $M \cong M'$ if and only if $F_k(M) = F_k(M')$ for all positive integers k .

Résumé substantiel en français. Soit R un anneau commutatif (unitaire) et M un R -module de type fini. Une forme canonique de M est une décomposition

$$M \cong R/I_1 \oplus \dots \oplus R/I_n,$$

où I_i est un idéal de R et $I_1 \subseteq \dots \subseteq I_n \neq R$. Si elle existe, cette décomposition est unique [8]. Si les idéaux I_i sont principaux, cette décomposition s'appelle *forme canonique normale* de M .

Les idéaux de Fitting interviennent au cours de notre investigation. Voici comment ils sont définis. Soit $X = \{x_1, \dots, x_n\}$ un ensemble de générateurs de M . Une *relation* dans M est un vecteur $(a_1, \dots, a_n)$ de R^n tel que $\sum_{i=1}^n a_i x_i = 0$. Pour $k = 0, \dots, n-1$, les idéaux de Fitting $F_k(M)$ de M sont définis comme étant les idéaux engendrés par les déterminants des $(n-k) \times (n-k)$ -sous-matrices de la matrice

$$K(M) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & \vdots & \vdots \end{pmatrix},$$

où les vecteurs $(a_{i1}, \dots, a_{in})$ sont les relations dans M . Si $k \geq n$, nous définissons $F_k(M) = R$. Ces idéaux forment une suite ascendante d'idéaux invariants pour M indépendamment du choix de X (voir [6] ou [10]).

Il existe deux invariants assez importants pour le test de la liberté et de la liberté locale des modules de type fini, à savoir

$$\nu(M) := \min\{k \mid F_k(M) = R\} \quad \text{et} \quad \omega(M) := \min\{k \mid F_k(M) \neq 0\}.$$

Théorème 2.3. Soit R un anneau local et soit M un R -module de type fini. Posons $\mu(M) = r$. Les deux assertions suivantes sont équivalentes :

- (i) Pour tout $k \geq \omega(M)$, l'idéal $F_k(M)$ est principal simplifiable parmi les idéaux non nuls de R .
- (ii) Nous avons l'isomorphisme

$$M \cong (\oplus_{i=1}^{r-s} R/(a_i)) \oplus R^s,$$

où a_i divise a_{i+1} et (a_i) est simplifiable pour $1 \leq i \leq r-s-1$, avec $s = \omega(M)$ et $a_1 R \neq R$.

Théorème 3.2. Soit R un anneau de valuation et M un R -module de type fini dont les idéaux de Fitting sont principaux. Posons $n = \mu(M)$ le nombre minimal de générateurs de M . Alors il existe $a_1, \dots, a_n$ dans R tels que $M \cong \oplus_{i=1}^n R/(a_i)$, où a_i divise a_{i+1} pour $i = 1, \dots, n-1$.

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