

**THE ENGEL CONTINUED FRACTION EXPANSIONS
OVER THE FIELD OF LAURENT SERIES: ALGORITHM,
METRIC PROPERTIES AND FRACTAL DIMENSIONS**

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RÉSUMÉ. Le but de cet article est d'introduire le développement en fraction continue d'Engel sur le corps des séries de Laurent. Nous discutons des propriétés métriques élémentaires des chiffres en polynômes apparus dans le développement. Aussi la dimension fractale des ensembles d'exception est étudiée.

ABSTRACT. The aim of this paper is to introduce the Engel continued fraction expansion over the field of Laurent series. We discuss some basic metric properties of the polynomial digits occurring in the expansion. Also the fractal dimensions of exceptional sets are studied.

1. Introduction. Real numbers have different kinds of representations, such as decimal expansions, continued fraction expansions, Lüroth series, Engel series, Sylvester series expansions and Cantor infinite products (see [7] and [24]). These representations have been proved to be useful tools to study the arithmetic properties of real numbers. Quite recently, Hartono, Kraaikamp and Schweiger [9] introduced a new continued fraction algorithm with non-decreasing partial quotients, called the Engel continued fraction expansion.

Instead of expansions of real numbers, expansions over the field of Laurent series have been studied extensively. Artin [1] and Magnus [19,20] had studied a Laurent series analogue of simple continued fractions of real numbers, involving “digits” $a_1, a_2, \dots$ lying in a polynomial ring $\mathbb{F}[z]$. In addition to sketching elementary properties of an n -dimensional “Jacobi-Perron” variant of this, Paysant-Leroux and Dubois [21,22] also briefly outlined certain “metric” theorems analogous to some of Khintchine’s [12] for real continued fractions. Analogous to the so called Oppenheim expansions of real numbers, discussed in Perron [23] Chapter 4 and Galambos [7], A. Knopfmacher and J. Knopfmacher [13,14] introduced the various unique expansions of formal Laurent series over a field $\mathbb{F}$, as the sums of reciprocals of polynomials. In the case of the Lüroth expansions of Laurent series, ergodic and other metric properties have been investigated by J. Knopfmacher [16] and extended by A. Knopfmacher and J. Knopfmacher [15]. For

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the Engel expansions of Laurent series, the corresponding results have been discussed by Grabner and A. Knopfmacher [8]. Fan and the author [5] studied the metric properties for the general Oppenheim expansion of Laurent series. For the topic of metric Diophantine approximation over the field of Laurent series, please refer to [3], [17] and [6].

In [9], Hartono, Kraaikamp and Schweiger studied the ergodic properties of the Engel continued fraction expansions of real numbers. The main aim of this paper is to introduce the analogous Engel continued fraction expansions over the field of Laurent series. Not only the ergodic properties of the Engel continued fraction expansions over the field of Laurent series are discussed, but also the asymptotic properties of the polynomial digits occurring in the expansions. In addition, we also consider the fractal properties of exceptional sets.

This paper is organized as follows. In section 2, the algorithm which generates the Engel continued fractions over the field of Laurent series is introduced and the basic properties of the expansions are discussed. Section 3 is devoted to study the metric and ergodic properties of the Engel continued fraction expansions and the fractal properties of exceptional sets are investigated in section 4.

2. The Engel continued fraction expansions over the field of Laurent series. Let $\mathcal{L} = \mathbb{F}((z^{-1}))$ denote the field of all formal Laurent series $A = \sum_{n=v}^{\infty} c_n z^{-n}$ in an indeterminate z , with coefficients c_n all lying in a given field $\mathbb{F}$.

We also consider the ring $\mathbb{F}[z]$ of polynomials in z , and the field $\mathbb{F}(z)$ of rational functions in z , with coefficients in $\mathbb{F}$.

If $c_v \neq 0$, we call $v = v(A)$ the order of A above, and define the norm (or valuation) of A to be $\|A\| = q^{-v(A)}$, where initially $q > 1$ may be an arbitrary constant. Letting $v(0) = +\infty$, $\|0\| = 0$, one then has (cf. Jones and Thron [11] Chapter 5):

$$\left\{ \begin{array}{l} \|A\| \geq 0 \text{ with } \|A\| = 0 \text{ iff } A = 0, \\ \|AB\| = \|A\| \cdot \|B\| \text{ and} \\ \|\alpha A + \beta B\| \leq \max(\|A\|, \|B\|) \text{ for non-zero } \alpha, \beta \in \mathbb{F} \\ \text{with equality when } \|A\| \neq \|B\|. \end{array} \right.$$

From above, the norm $\|\cdot\|$ is non-Archimedean, and it is well known that $\mathcal{L}$ forms a complete metric space under the metric ρ such that $\rho(A, B) = \|A - B\|$.

Remark 2.1. Since the metric ρ is non-Archimedean, it follows that each point of a disc may be considered its center and thus if two discs intersect, then one contains the other.

For $A = \sum_{n=v}^{\infty} c_n z^{-n} \in \mathcal{L}$, let $[A] = \sum_{v \leq n \leq 0} c_n z^{-n} \in \mathbb{F}[z]$, and refer to $[A]$ as the integral part of $A \in \mathcal{L}$. Then $-v(A) := -v$ is degree $\deg[A]$ of $[A]$ relative to z . The function $[\cdot]$ was used by Artin [1], Baum and Sweet [2] and Magnus [19,20] for their continued fractions.

Let I denote the valuation ideal $z^{-1}\mathbb{F}[[z^{-1}]]$ in the ring of formal power series $\mathbb{F}[[z^{-1}]]$, i.e. $I = \{x \in \mathcal{L} : \|x\| < 1\}$.

For $x = \sum_{n=1}^{\infty} c_n z^{-n} \in I$, let $x_1 = x$. If $x_n \neq 0$, ($n \geq 1$), is already defined, then let $a_n(x) = \left[\frac{1}{x_n}\right]$, and put

$$x_{n+1} = \left(\frac{1}{x_n} - a_n(x)\right) \frac{1}{a_n(x)}. \quad (1)$$

Then

$$\begin{aligned} x &= \frac{1}{a_1(x) + a_1(x)x_2} = \dots \\ &= \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots + \frac{a_{n-1}(x)}{a_n(x) + a_n(x)x_{n+1}}}}. \end{aligned}$$

This process ends if $x_n = 0$ for some $n \geq 1$.

Theorem 2.2. *Every $x \in I$ has a finite or convergent (relative to ρ) expansion of the form*

$$x = \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots}} := [0; a_1(x), a_2(x), \dots],$$

where $a_n(x) \in \mathbb{F}[z]$ and

$$\deg a_1(x) \geq 1 \text{ and } \deg(a_{n+1}(x)) \geq \deg(a_n(x)) + 1 \text{ for } n \geq 1. \quad (2)$$

The expansion is unique for x subject to the preceding “digit” conditions on the $a_n(x) \in \mathbb{F}[z]$, and is finite iff $x \in \mathbb{F}(z)$, that is, x is a rational function of z . In addition, if we write

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots + \frac{a_{n-1}(x)}{a_n(x)}}}, \quad (3)$$

then

$$\left\| x - \frac{p_n(x)}{q_n(x)} \right\| \leq q^{-\frac{n(n+1)}{2}}. \quad (4)$$

Proof. If $x \in I$ has a finite expansion, it is obvious that $x \in \mathbb{F}(z)$. Let $x = \frac{r(z)}{s(z)} \in \mathbb{F}(z)$, where $r(z), s(z), (s(z) \neq 0)$, are polynomials in $\mathbb{F}[z]$. Then by algorithm (1),

$$x_2 = \left(\frac{s(z)}{r(z)} - \left[\frac{s(z)}{r(z)} \right] \right) \frac{1}{\left[\frac{s(z)}{r(z)} \right]} = \frac{\frac{s(z)}{r(z)} - a_1(x)}{a_1(x)} = \frac{s(z) - a_1(x)r(z)}{a_1(x)r(z)} := \frac{r^{(1)}(z)}{s^{(1)}(z)}.$$

Noting that

$$\frac{s(z)}{r(z)} = a_1(x) + \sum_{k=1}^{\infty} c_k z^{-k},$$

thus

$$s(z) = r(z)a_1(x) + r(z) \sum_{k=1}^{\infty} c_k z^{-k},$$

which implies

$$\nu(s(z) - r(z)a_1(x)) \geq \nu(r(z)) + 1,$$

i.e.

$$\deg r^{(1)}(z) = \deg(s(z) - r(z)a_1(x)) \leq \deg r(z) - 1.$$

Now let

$$x_{n+1} = \frac{r^{(n)}(z)}{s^{(n)}(z)}, n = 1, 2, \dots,$$

then

$$-\infty < \dots < \deg r^{(n)}(z) < \dots < \deg r(z).$$

Since $\deg r^{(n)}(z) \in \mathbb{N} \cup \{0\} \cup \{-\infty\}$, there exists an n such that $x_n = 0$.

If $x_n \neq 0$, $x_{n+1} \neq 0$ for some $n \geq 1$, suppose

$$\frac{1}{x_n} = a_n(x) + \sum_{k=1}^{\infty} c_k z^{-k}.$$

By algorithm (1), we have

$$x_{n+1} = \left(\frac{1}{x_n} - a_n(x) \right) \frac{1}{a_n(x)} = \frac{1}{a_n(x)} \sum_{k=1}^{\infty} c_k z^{-k},$$

which implies

$$-\nu(a_{n+1}(x)) = \nu(x_{n+1}) \geq -\nu(a_n(x)) + 1,$$

i.e.

$$\deg a_{n+1}(x) \geq \deg a_n(x) + 1. \quad (5)$$

Now we prove (4). Without loss of generality, assume x is not in $\mathbb{F}(z)$ and let $\left\{ \frac{p_n(x)}{q_n(x)}, n \geq 0 \right\}$ be the sequence of convergents in the expansion of x , that is,

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots + \frac{a_{n-1}(x)}{a_n(x)}}} := [0; a_1(x), a_2(x), \dots, a_n(x)],$$

where $\frac{p_0(x)}{q_0(x)} = \frac{0}{1}$ and $\frac{p_1(x)}{q_1(x)} = \frac{1}{a_1(x)}$. It is easily seen as in the real case (see [9, Proposition 2.1]), that

$$p_n(x) = a_n(x)p_{n-1}(x) + a_{n-1}(x)p_{n-2}(x) \text{ for } n \geq 2, \quad (6)$$

$$q_n(x) = a_n(x)q_{n-1}(x) + a_{n-1}(x)q_{n-2}(x) \text{ for } n \geq 2. \quad (7)$$

From (6) and (7), one has

$$p_n(x)q_{n-1}(x) - p_{n-1}(x)q_n(x) = (-1)^{n-1} \prod_{j=1}^{n-1} a_j(x) \text{ for } n \geq 2, \quad (8)$$

i.e.

$$\frac{p_n(x)}{q_n(x)} - \frac{p_{n-1}(x)}{q_{n-1}(x)} = \frac{(-1)^{n-1} \prod_{j=1}^{n-1} a_j(x)}{q_{n-1}(x)q_n(x)} \text{ for } n \geq 2. \quad (9)$$

By induction, one has that

$$x = \frac{p_n(x) + a_n(x)x_{n+1}p_{n-1}(x)}{q_n(x) + a_n(x)x_{n+1}q_{n-1}(x)}, \quad n \geq 1. \quad (10)$$

In fact, if $x \in \mathbb{F}(z)$, the special case $x_{n+1} = 0$ gives $x = \frac{p_n(x)}{q_n(x)}$. From (8) and (10),

$$x - \frac{p_{n-1}(x)}{q_{n-1}(x)} = \frac{(-1)^{n-1} \prod_{j=1}^{n-1} a_j(x)}{q_{n-1}(x)(q_n(x) + a_n(x)x_{n+1}q_{n-1}(x))}, \quad n \geq 1. \quad (11)$$

By (5), (6) and (7), it is easy to see that

$$\deg p_n(x) = \deg(a_n(x)p_{n-1}(x)) = \sum_{j=2}^n \deg a_j(x). \quad (12)$$

$$\deg q_n(x) = \deg(a_n(x)q_{n-1}(x)) = \sum_{j=1}^n \deg a_j(x). \quad (13)$$

Thus by (5),

$$\begin{aligned} \left\| x - \frac{p_n(x)}{q_n(x)} \right\| &= q^{\sum_{j=1}^{n-1} \deg a_j(x) - \sum_{j=1}^{n-1} \deg a_j(x) - \sum_{j=1}^n \deg a_j(x)} \\ &= q^{-\sum_{j=1}^n \deg a_j(x)} \leq q^{-\frac{n(n+1)}{2}}. \end{aligned}$$

Now we establish the uniqueness of the “digits” $a_n(x) \in \mathbb{F}[z]$. It is enough to prove the following claim.

Claim. Let $\{b_n, n \geq 1\}$ be a sequence in $\mathbb{F}[z]$ with $\deg b_1 \geq 1$, $\deg b_{n+1} \geq \deg b_n + 1$ for any $n \geq 1$, and let $\{p_n, n \geq 0\}$, $\{q_n, n \geq 0\}$ be given in (6), (7). Then $\frac{p_n}{q_n}$ converges (as $n \rightarrow \infty$) to an element $x \in I$ such that $b_n = a_n(x)$ for $n \geq 1$.

In fact, from (9) and (13),

$$\left\| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right\| = q^{-\sum_{j=1}^n \deg b_j} \leq q^{-\frac{n(n+1)}{2}},$$

and hence $\frac{p_n}{q_n}$ converges to an element $x \in \mathcal{L}$ because $\mathcal{L}$ is complete and $\| \cdot \|$ is non-Archimedean. By (12) and (13), $\nu(\frac{p_n}{q_n}) = \deg b_1 \geq 1$ for any $n \geq 1$. Thus $x \in I$. Now

$$\frac{p_n}{q_n} = [0; b_1, b_2, \dots, b_n] = \frac{1}{b_1 + b_1[0; b_2, \dots, b_n]}, \quad (14)$$

using the same proof as above,

$$[0; b_2, \dots, b_n] \rightarrow x' \in I, \text{ as } n \rightarrow \infty,$$

and $\nu(x') = \deg b_2 \geq \deg b_1 + 1$. This implies $b_1 x' \in I$. By letting $n \rightarrow \infty$ in (14) we find that

$$x = \frac{1}{b_1 + b_1 x'},$$

from which

$$b_1 x' = \frac{1}{x} - b_1 \in I,$$

thus $b_1 = \left\lfloor \frac{1}{x} \right\rfloor = a_1(x)$. Repeating the above argument yields that $a_2(x) = b_2$. By induction one finds that $a_n(x) = b_n$ for $n \geq 1$. The **Claim** has been proved and the proof of Theorem 2.2 is finished. $\square$

Remark 2.3. From Theorem 2.2 and (9), if x is not in $\mathbb{F}(z)$, we have

$$x = \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n a_j(x)}{q_n(x) q_{n+1}(x)}, \quad \left(\prod_{j=1}^0 a_j(x) := 1 \right). \quad (15)$$

3. Metric properties of the Engel continued fraction expansions over the field of Laurent series. From now on we assume $q \geq 2$ is a positive integer and $\mathbb{F} = \mathbb{F}_q$ is a finite field with exactly q elements. In this case, I is compact under the metric ρ . Let $\mathbf{P}$ denote probability measure with respect to the Haar measure on $\mathcal{L}$ normalized by $\mathbf{P}(I) = 1$. The Haar measure on I is the product measure on $\prod_{n=1}^{\infty} \mathbb{F}_q$ defined by $\mathbf{P}(\{a\}) = q^{-1}$ for each factor and any element $a \in \mathbb{F}_q$.

In order to establish metric and ergodic properties of the Engel continued fraction expansions over the field of Laurent series, we need some preliminary results.

Lemma 3.1. *For any $d_1, d_2, \dots, d_n \in \mathbb{F}_q[z]$ satisfying $\deg d_1 \geq 1$ and $\deg d_{j+1} \geq \deg d_j + 1$, $1 \leq j \leq n-1$, we have*

$$\mathbf{P}\{x \in I : a_1(x) = d_1, a_2(x) = d_2, \dots, a_n(x) = d_n\} = q^{-\sum_{j=1}^{n-1} \deg d_j - 2 \deg d_n}.$$

Proof. By Theorem 2.2 and Remark 2.3, the set

$$\{x \in I : a_1(x) = d_1, a_2(x) = d_2, \dots, a_n(x) = d_n\}$$

is a disc with center at

$$\sum_{k=0}^{n-1} \frac{(-1)^k \prod_{j=1}^k d_j}{q_k q_{k+1}},$$

and diameter

$$q^{-\sum_{j=1}^{n-1} \deg d_j - 2 \deg d_n - 1},$$

where $\{q_n, n \geq 0\}$ are defined in (7). Thus

$$\mathbf{P}\{x \in I : a_1(x) = d_1, a_2(x) = d_2, \dots, a_n(x) = d_n\} = q^{-\sum_{j=1}^{n-1} \deg d_j - 2 \deg d_n}. \quad \square$$

Proposition 3.2. For any $d_1, d_2, \dots, d_{n+1} \in \mathbb{F}_q[z]$ such that $\deg d_1 \geq 1$ and $\deg d_{j+1} \geq \deg d_j + 1, 1 \leq j \leq n$, we have

$$\begin{aligned} \mathbf{P}\{a_{n+1}(x) = d_{n+1} | a_n(x) = d_n\} \\ &= \mathbf{P}\{a_{n+1}(x) = d_{n+1} | a_1(x) = d_1, a_2(x) = d_2, \dots, a_n(x) = d_n\} \\ &= \frac{\|d_n\|}{\|d_{n+1}\|^2}, \end{aligned}$$

i.e. $\{a_n(x), n \geq 1\}$ forms a Markov chain with transition probabilities,

$$\mathbf{P}\{a_{n+1}(x) = l_{n+1} | a_n(x) = l_n\} = \frac{\|l_n\|}{\|l_{n+1}\|^2}$$

if $\deg l_{n+1} \geq \deg l_n + 1$, and 0 otherwise.

Proof. By Lemma 3.1,

$$\begin{aligned} \mathbf{P}\{a_{n+1}(x) = d_{n+1} | a_n(x) = d_n, \dots, a_1(x) = d_1\} \\ &= \frac{\mathbf{P}\{a_1(x) = d_1, \dots, a_n(x) = d_n, a_{n+1}(x) = d_{n+1}\}}{\mathbf{P}\{a_1(x) = d_1, \dots, a_{n-1}(x) = d_{n-1}, a_n(x) = d_n\}} \\ &= \frac{q^{-\sum_{j=1}^n \deg d_j - 2 \deg d_{n+1}}}{q^{-\sum_{j=1}^{n-1} \deg d_j - 2 \deg d_n}} \\ &= \frac{\|d_n\|}{\|d_{n+1}\|^2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbf{P}\{a_{n+1}(x) = d_{n+1} | a_n(x) = d_n\} \\ &= \frac{\mathbf{P}\{a_n(x) = d_n, a_{n+1}(x) = d_{n+1}\}}{\mathbf{P}\{a_n(x) = d_n\}} \\ &= \frac{\sum \mathbf{P}\{a_j(x) = l_j, 1 \leq j \leq n-1, a_n(x) = d_n, a_{n+1}(x) = d_{n+1}\}}{\sum \mathbf{P}\{a_j(x) = m_j, 1 \leq j \leq n-1, a_n(x) = d_n\}} \quad (16) \end{aligned}$$

$$\begin{aligned} &= \frac{\sum q^{-\sum_{j=1}^{n-1} \deg l_j - \deg d_n - 2 \deg d_{n+1}}}{\sum q^{-\sum_{j=1}^{n-1} \deg m_j - 2 \deg d_n}} \quad (17) \\ &= \frac{\|d_n\|}{\|d_{n+1}\|^2}, \end{aligned}$$

where both the summations in the numerators in (16) and (17) are over all $l_1, l_2, \dots, l_{n-1} \in \mathbb{F}_q[z]$ satisfying $\deg l_1 \geq 1$, $\deg l_{j+1} \geq \deg l_j + 1, 1 \leq j \leq n-2$, and $\deg d_n \geq \deg l_{n-1} + 1$, and both the summations in the denominators in (16) and (17) are over all $m_1, m_2, \dots, m_{n-1} \in \mathbb{F}_q[z]$ satisfying $\deg m_1 \geq 1$, $\deg m_{j+1} \geq \deg m_j + 1, 1 \leq j \leq n-2$, and $\deg d_n \geq \deg m_{n-1} + 1$. $\square$

Now we are going to prove that $\{\deg a_n(x), n \geq 1\}$ forms a Markov chain.

Lemma 3.3. *For any positive integers $k_1, k_2, \dots, k_n$ satisfying $k_1 \geq 1$ and $k_{j+1} \geq k_j + 1, 1 \leq j \leq n-1$, we have*

$$\mathbf{P}\{x \in I : \deg a_1(x) = k_1, \deg a_2(x) = k_2, \dots, \deg a_n(x) = k_n\} = (q-1)^n q^{-k_n}.$$

Proof. By Lemma 3.1,

$$\begin{aligned} \mathbf{P}\{x \in I : \deg a_1(x) = k_1, \deg a_2(x) = k_2, \dots, \deg a_n(x) = k_n\} \\ = \sum \mathbf{P}\{a_1(x) = l_1, a_2(x) = l_2, \dots, a_n(x) = l_n\} \quad (18) \end{aligned}$$

$$= \sum q^{-\sum_{j=1}^{n-1} \deg l_j - 2 \deg l_n} \quad (19)$$

$$\begin{aligned} &= \sum q^{-\sum_{j=1}^{n-1} k_j - 2k_n} \quad (20) \\ &= (q-1)^n q^{-k_n}, \end{aligned}$$

where the summations in (18), (19) and (20) are over all $l_1, l_2, \dots, l_n \in \mathbb{F}_q[z]$ such that $\deg l_j = k_j, 1 \leq j \leq n$. $\square$

Proposition 3.4. *For any positive integers $k_1, k_2, \dots, k_{n+1}$ satisfying $k_1 \geq 1$ and $k_{j+1} \geq k_j + 1, 1 \leq j \leq n$, we have*

$$\begin{aligned} \mathbf{P}\{\deg a_{n+1}(x) = k_{n+1} | \deg a_n(x) = k_n\} \\ = \mathbf{P}\{\deg a_{n+1}(x) = k_{n+1} | \deg a_1(x) = k_1, \dots, \deg a_n(x) = k_n\} \\ = (q-1)q^{k_n - k_{n+1}}. \end{aligned}$$

Proof. By Lemma 3.3,

$$\begin{aligned} \mathbf{P}\{\deg a_{n+1}(x) = k_{n+1} | \deg a_n(x) = k_n, \dots, \deg a_1(x) = k_1\} \\ = \frac{\mathbf{P}\{\deg a_1(x) = k_1, \dots, \deg a_{n+1}(x) = k_{n+1}\}}{\mathbf{P}\{\deg a_1(x) = k_1, \dots, \deg a_n(x) = k_n\}} \\ = \frac{(q-1)^{n+1} q^{-k_{n+1}}}{(q-1)^n q^{-k_n}} \\ = (q-1)q^{k_n - k_{n+1}}. \end{aligned}$$

On the other hand, write

$$\begin{aligned} A_n &= \{\deg a_n(x) = k_n, \deg a_{n+1}(x) = k_{n+1}\}, \\ B_n &= \{\deg a_n(x) = k_n\}. \end{aligned}$$

Also by Lemma 3.3, we have

$$\begin{aligned}
& \mathbf{P}\{\deg a_{n+1}(x) = k_{n+1} \mid \deg a_n(x) = k_n\} \\
&= \frac{\mathbf{P}\{\deg a_n(x) = k_n, \deg a_{n+1}(x) = k_{n+1}\}}{\mathbf{P}\{\deg a_n(x) = k_n\}} \\
&= \frac{\sum \mathbf{P}(\{\deg a_j(x) = l_j, 1 \leq j \leq n-1\} \cap A)}{\sum \mathbf{P}(\{\deg a_j(x) = m_j, 1 \leq j \leq n-1\} \cap B)} \quad (21) \\
&= \frac{\sum (q-1)^{n+1} q^{-k_{n+1}}}{\sum (q-1)^n q^{-k_n}} \quad (22) \\
&= (q-1)q^{k_n - k_{n+1}},
\end{aligned}$$

where both the summations in the numerators in (21) and (22) are over all positive integers $l_1, l_2, \dots, l_{n-1}$ satisfying $l_1 \geq 1, l_{j+1} \geq l_j + 1, 1 \leq j \leq n-2$, and $k_n \geq l_{n-1} + 1$, and both the summations in the denominators in (21) and (22) are over all positive integers $m_1, m_2, \dots, m_{n-1}$ satisfying $m_1 \geq 1, m_{j+1} \geq m_j + 1, 1 \leq j \leq n-2$, and $k_n \geq m_{n-1} + 1$. $\square$

For any $x \in I$, let $\{\triangle_n(x), n \geq 0\}$ be the sequence of random variables such that $\triangle_0(x) = \deg a_1(x)$, $\triangle_n(x) = \deg a_{n+1}(x) - \deg a_n(x)$ for $n \geq 1$.

Now we prove our key result.

Theorem 3.5. *$\{\triangle_n(x), n \geq 0\}$ is a sequence of independently and identically distributed random variables, and for any $k \geq 1$,*

$$\mathbf{P}\{x \in I : \triangle_n(x) = k\} = \frac{q-1}{q^k}.$$

Proof. For any $n \geq 1$ and $k \geq 1$, by Proposition 3.4,

$$\begin{aligned}
& \mathbf{P}\{x \in I : \triangle_n(x) = k\} \\
&= \sum_{j=1}^{\infty} \mathbf{P}\{\triangle_n(x) = k \mid \deg a_n(x) = j\} \mathbf{P}\{\deg a_n(x) = j\} \\
&= \sum_{j=1}^{\infty} \mathbf{P}\{\deg a_{n+1}(x) = k+j \mid \deg a_n(x) = j\} \mathbf{P}\{\deg a_n(x) = j\} \\
&= \sum_{j=1}^{\infty} \mathbf{P}\{\deg a_n(x) = j\} \cdot \frac{q-1}{q^k} \\
&= \frac{q-1}{q^k}.
\end{aligned}$$

Also it is easy to see, for any $k \geq 1$,

$$\mathbf{P}\{x \in I : \triangle_0(x) = k\} = \frac{q-1}{q^k}.$$

Now we prove that $\triangle_n(x), n = 0, 1, \dots$, are independent. For any positive integers $k_1, k_2, \dots, k_{n+1}$,

$$\begin{aligned}
& \mathbf{P}\{x \in I : \triangle_0(x) = k_1, \triangle_1(x) = k_2, \dots, \triangle_n(x) = k_{n+1}\} \\
&= \mathbf{P}\left\{x \in I : \deg a_1(x) = k_1, \deg a_2(x) = k_1 + k_2, \dots, \deg a_{n+1}(x) = \sum_{j=1}^{n+1} k_j\right\} \\
&= (q-1)^{n+1} q^{-\sum_{j=1}^{n+1} k_j} \\
&= \mathbf{P}\{\triangle_0(x) = k_1\} \mathbf{P}\{\triangle_1(x) = k_2\} \cdots \mathbf{P}\{\triangle_n(x) = k_{n+1}\}. \quad \square
\end{aligned}$$

Theorem 3.6. *For the Engel continued fraction expansions of Laurent series, we have*

$$(i) \lim_{n \rightarrow \infty} \mathbf{P}\left\{x \in I : \frac{\sum_{j=1}^n \deg a_j(x) - \frac{q}{q-1}n}{\sqrt{nq/(q-1)}} < t\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du,$$

(ii) For $\mathbf{P}$ -almost all $x \in I$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \deg a_n(x) = \frac{q}{q-1}, \quad (23)$$

(iii) For $\mathbf{P}$ -almost all $x \in I$,

$$\limsup_{n \rightarrow \infty} \frac{\deg a_n(x) - \frac{q}{q-1}n}{\sqrt{2n \log \log n}} = \frac{\sqrt{q}}{q-1}$$

and

$$\liminf_{n \rightarrow \infty} \frac{\deg a_n(x) - \frac{q}{q-1}n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{q}}{q-1}.$$

Proof. By Theorem 3.5,

$$\begin{aligned}
\mathbf{E}(\triangle_n(x)) &= \sum_{k=1}^{\infty} k \mathbf{P}\{\triangle_n(x) = k\} \\
&= \sum_{k=1}^{\infty} k \cdot \frac{q-1}{q^k} = \frac{q}{q-1}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{E}(\triangle_n(x)^2) &= \sum_{k=1}^{\infty} k^2 \mathbf{P}\{\triangle_n(x) = k\} \\
&= \sum_{k=1}^{\infty} k^2 \cdot \frac{q-1}{q^k} = \frac{q}{q-1} + \frac{2q}{(q-1)^2},
\end{aligned}$$

thus

$$\mathbf{Var}(\triangle_n(x)) = \mathbf{E}(\triangle_n(x)^2) - (\mathbf{E}(\triangle_n(x)))^2 = \frac{q}{(q-1)^2}.$$

Hence by the central limit theorem, (see [25], p.331, Corollary 3), the strong law of large numbers, (see [25], p.366, Theorem 3), and the iterated logarithm law, (see [25], p.372, Theorem 1), we get the desired results of Theorem 3.6. $\square$

If we consider the random variables $\left\| \frac{a_{n+1}(x)}{a_n(x)} \right\| = q^{\triangle_n(x)}$, $n = 0, 1, 2, \dots$, from Theorem 3.5, these are independent and identically distributed with infinite expectation. However, we have the following result:

Theorem 3.7. For any fixed $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ x \in I : \left| \frac{1}{n \log_q n} \sum_{j=1}^n \left\| \frac{a_{j+1}(x)}{a_j(x)} \right\| - (q-1) \right| > \varepsilon \right\} = 0,$$

i.e. $\frac{1}{n \log_q n} \sum_{j=1}^n \left\| \frac{a_{j+1}(x)}{a_j(x)} \right\| \rightarrow (q-1)$ in probability.

Proof. We obtain our result following the idea of Theorem 5 in [15].

By Theorem 3.5, the random variables $\left\| \frac{a_{k+1}(x)}{a_k(x)} \right\| = q^{\triangle_k(x)}$, $k = 1, 2, \dots$, are independent and identically distributed, and it is easy to check that $q^{\triangle_k(x)}$ has infinite expectation. For any $k \leq n$, define

$$U_k(x) = \left\| \frac{a_{k+1}(x)}{a_k(x)} \right\|, \quad V_k(x) = 0 \quad \text{if} \quad \left\| \frac{a_{k+1}(x)}{a_k(x)} \right\| \leq n \log_q n,$$

and

$$V_k(x) = \left\| \frac{a_{k+1}(x)}{a_k(x)} \right\|, \quad U_k(x) = 0 \quad \text{if} \quad \left\| \frac{a_{k+1}(x)}{a_k(x)} \right\| > n \log_q n.$$

Then

$$\begin{aligned} \mathbf{P} \left\{ x \in I : \left| \frac{1}{n \log_q n} \sum_{j=1}^n \left\| \frac{a_{j+1}(x)}{a_j(x)} \right\| - (q-1) \right| > \varepsilon \right\} \\ \leq \mathbf{P}\{x \in I : |U_1(x) + \dots + U_n(x) - (q-1)n \log_q n| > \varepsilon n \log_q n\} \\ + \mathbf{P}\{x \in I : V_1(x) + \dots + V_n(x) \neq 0\}. \end{aligned}$$

By Theorem 3.5,

$$\begin{aligned} \mathbf{P}\{x \in I : V_1(x) + \dots + V_n(x) \neq 0\} &\leq n \mathbf{P}\{x \in I : \left\| \frac{a_2(x)}{a_1(x)} \right\| > n \log_q n\} \\ &= n \sum_{k: q^k > n \log_q n} (q-1)q^{-k} \\ &\leq \frac{q}{\log_q n} = o(1). \end{aligned}$$

Also by Theorem 3.5, we have

$$\begin{aligned} \mathbf{E}(U_1(x) + \dots + U_n(x)) &= n \mathbf{E}(U_1(x)), \\ \mathbf{Var}(U_1(x) + \dots + U_n(x)) &= n \mathbf{Var}(U_1(x)), \end{aligned}$$

where

$$\begin{aligned} \mathbf{E}(U_1(x)) &= \sum_{q^k \leq n \log_q n} q^k \mathbf{P}(\triangle_1(x) = k) \\ &= \sum_{q^k \leq n \log_q n} q^{-k} (q-1)q^k = (q-1) \log_q([n \log_q n]), \end{aligned}$$

and

$$\begin{aligned}\mathbf{Var}(U_1(x)) &\leq \mathbf{E}(U_1(x)^2) \\ &= \sum_{q^k \leq n \log_q n} (q-1)q^k < qn \log_q n.\end{aligned}$$

Chebyshev's inequality then yields

$$\begin{aligned}\mathbf{P}\{x \in I : |U_1(x) + \dots + U_n(x) - n\mathbf{E}(U_1(x))| > \varepsilon n\mathbf{E}(U_1(x))\} \\ \leq \frac{n\mathbf{Var}(U_1(x))}{(\varepsilon n\mathbf{E}(U_1(x)))^2} < \frac{qn^2 \log_q n}{(\varepsilon(q-1)n \log([n \log_q n]))^2} = o(1).\end{aligned}$$

Since $\mathbf{E}(U_1(x)) \sim (q-1) \log_q n$ as $n \rightarrow \infty$, Theorem 3.7 follows. $\square$

For the convergent speed of the Engel continued fraction expansions of Laurent series, we have

Theorem 3.8. *For the Engel continued fraction expansions of Laurent series, for $\mathbf{P}$ -almost all $x \in I$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log_q \left\| x - \frac{p_n(x)}{q_n(x)} \right\| = -\frac{q}{2(q-1)}.$$

Proof. From (11) and (5),

$$\left\| x - \frac{p_n(x)}{q_n(x)} \right\| = q^{-\sum_{j=1}^{n+1} \deg a_j(x)}.$$

By Theorem 3.6, for $\mathbf{P}$ -almost all $x \in I$,

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n^2} \log_q \left\| x - \frac{p_n(x)}{q_n(x)} \right\| &= \lim_{n \rightarrow \infty} \frac{-\sum_{j=1}^{n+1} \deg a_j(x)}{n^2} \\ &= \lim_{n \rightarrow \infty} \frac{-\deg a_{n+2}(x)}{(n+1)^2 - n^2} \\ &= -\frac{q}{2(q-1)}. \quad \square\end{aligned}$$

Remark 3.9. For the Engel continued fraction expansions of real numbers, the results corresponding to Theorem 3.6, Theorem 3.7 and Theorem 3.8 have been established by Kraaikamp and the author [10].

Let $T : I \rightarrow I$ be the transformation defined by

$$\begin{aligned}T(0) &= 0, \\ T(x) &= \left(\frac{1}{x} - \left[\frac{1}{x} \right] \right) \cdot \frac{1}{\left[\frac{1}{x} \right]}, \quad \text{for any } x \neq 0.\end{aligned}$$

By the algorithm generating the Engel continued fraction expansions of Laurent series, we conclude that for any $x \in I$, if the recursive process does not stop, for any $n \geq 1$,

$$x = \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \cdots + \frac{a_{n-1}(x)}{a_n(x) + a_n(x)T^n x}}}. \quad (24)$$

Now we consider the ergodic property of T with respect to $\mathbf{P}$. For any $n \geq 1$ and positive integers $k_1, k_2, \dots, k_n$ satisfying $k_1 \geq 1$ and $k_{j+1} \geq k_j + 1$, $1 \leq j \leq n-1$, we call

$$I_n(k_1, k_2, \dots, k_n) = \{x \in I : \deg a_1(x) = k_1, \dots, \deg a_n(x) = k_n\}$$

a n -order cylinder.

Lemma 3.10. For any Borel set $E \subset I$ and any n -order cylinder $I_n(k_1, k_2, \dots, k_n)$,

$$\mathbf{P}(T^{-n}E \cap I_n(k_1, k_2, \dots, k_n)) = (q-1)^n \mathbf{P}(T^n I_n(k_1, k_2, \dots, k_n) \cap E).$$

Proof. For any $m \geq 1$ and m -order cylinder $I_m(b_1, b_2, \dots, b_m)$, if $b_1 \leq k_n$, by Theorem 2.2,

$$T^{-n}I_m(b_1, b_2, \dots, b_m) \cap I_n(k_1, k_2, \dots, k_n) = \emptyset. \quad (25)$$

If $b_1 \geq k_n + 1$, by Lemma 3.3, we have

$$\begin{aligned} \mathbf{P}(T^{-n}I_m(b_1, b_2, \dots, b_m) \cap I_n(k_1, k_2, \dots, k_n)) \\ &= \mathbf{P}\{x \in I : \deg a_1(x) = k_1, \dots, \deg a_n(x) = k_n, \\ &\quad \deg a_{n+1}(x) = b_1, \dots, \deg a_{m+n}(x) = b_m\} \\ &= (q-1)^{m+n} q^{-b_m} \\ &= (q-1)^n \mathbf{P}(I_m(b_1, b_2, \dots, b_m)). \end{aligned} \quad (26)$$

Combining (11) and (12), using standard measure theory, we have for any Borel set $E \subset I$,

$$\mathbf{P}(T^{-n}E \cap I_n(k_1, k_2, \dots, k_n)) = (q-1)^n \mathbf{P}(T^n I_n(k_1, k_2, \dots, k_n) \cap E). \quad \square$$

Theorem 3.11. T is ergodic with respect to $\mathbf{P}$.

Proof. Suppose E is a Borel set in I such that $T^{-1}E = E$.

For any $n \geq 1$ and n -order cylinder $I_n(k_1, k_2, \dots, k_n)$, by Lemma 3.10, we have

$$\begin{aligned} \mathbf{P}(E \cap I_n(k_1, k_2, \dots, k_n)) &= \mathbf{P}(T^{-n}E \cap I_n(k_1, k_2, \dots, k_n)) \\ &= (q-1)^n \mathbf{P}(T^n I_n(k_1, k_2, \dots, k_n) \cap E). \end{aligned}$$

Noting that $T^n(I_n(k_1, k_2, \dots, k_n)) = \{x \in I : \deg a_1(x) \geq k_n + 1\}$, we have

$$\mathbf{P}(E \cap I_n(k_1, k_2, \dots, k_n)) = (q-1)^n \mathbf{P}(\{x \in I : \deg a_1(x) \geq k_n + 1\} \cap E).$$

By Lemma 3.3, we have

$$\frac{\mathbf{P}(E \cap I_n(k_1, k_2, \dots, k_n))}{\mathbf{P}(I_n(k_1, k_2, \dots, k_n))} = q^{k_n} \mathbf{P}(\{x \in I : \deg a_1(x) \geq k_n + 1\} \cap E).$$

For any $k \geq 0$, let

$$D(k) := q^k \mathbf{P}(\{x \in I : \deg a_1(x) \geq k + 1\} \cap E).$$

Since $T^{-1}E = E$, by Lemma 3.10, we have

$$\begin{aligned} D(k) &= q^k \mathbf{P}(\{x \in I : \deg a_1(x) \geq k + 1\} \cap E) \\ &= q^k \sum_{s=k+1}^{\infty} \mathbf{P}(\{x \in I : \deg a_1(x) = s\} \cap E) \\ &= q^k \sum_{s=k+1}^{\infty} \mathbf{P}(\{x \in I : \deg a_1(x) = s\} \cap T^{-1}E) \\ &= q^k (q - 1) \sum_{s=k+1}^{\infty} \mathbf{P}(\{x \in I : \deg a_1(x) \geq s + 1\} \cap E) \\ &= \sum_{j=1}^{\infty} \frac{q-1}{q^j} D(k+j). \end{aligned}$$

In the same way,

$$D(k+1) = \sum_{j=1}^{\infty} \frac{q-1}{q^j} D(k+j+1).$$

Thus

$$D(k) - \frac{D(k+1)}{q} = \frac{q-1}{q} D(k+1),$$

which implies that for any $k \geq 0$,

$$D(k) = D(k+1).$$

Since $D(0) = \mathbf{P}(E)$, we have for n -order cylinder $I_n(k_1, k_2, \dots, k_n)$,

$$\frac{\mathbf{P}(E \cap I_n(k_1, k_2, \dots, k_n))}{\mathbf{P}(I_n(k_1, k_2, \dots, k_n))} = \mathbf{P}(E).$$

By standard measure theory, for any Borel set $A \subset I$,

$$\mathbf{P}(E \cap A) = \mathbf{P}(E) \cdot \mathbf{P}(A).$$

Choosing $A = E^c$, we have

$$0 = \mathbf{P}(E \cap E^c) = \mathbf{P}(E) \cdot \mathbf{P}(E^c).$$

This implies $\mathbf{P}(E) = 0$ or $\mathbf{P}(E) = 1$. $\square$

Now we consider the invariant measures for T . Let δ_0 be the Dirac measure concentrated at $\{0\}$.

Theorem 3.12. δ_0 is the only probability measure on I which is preserved by T .

Proof. Since I is compact and T is continuous on I , thus there exists a probability measure $\mathbf{Q}$ on I which is preserved by T .

For any $k \geq 1$, noting that $T^{-n}(I_1(k)) = \emptyset$ for $n \geq k$, we have

$$\mathbf{Q}(I_1(k)) = \mathbf{Q}(T^{-n}(I_1(k))) = 0,$$

therefore

$$\mathbf{Q}(I \setminus \{0\}) = \sum_{k=1}^{\infty} \mathbf{Q}(I_1(k)) = 0.$$

So

$$\mathbf{Q} = \delta_0. \quad \square$$

Remark 3.13. From the proof of Theorem 3.12, we know, for Engel continued fraction expansions of Laurent series, any non null measure on I which is preserved by T is concentrated at $\{0\}$.

4. Hausdorff dimensions of the exceptional sets. The definition of Hausdorff measure on I is the same as that on $\mathbb{R}^n$. For $s > 0$ and a subset E of I , the Hausdorff s -measure is given by

$$\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \left\{ \inf \sum_j (\text{diam } D_j)^s \right\},$$

where the infimum is over all covers of E by discs D_j of diameter (in the metric ρ) at most δ and diam denotes the diameter of a set. The Hausdorff dimension of E is defined by

$$\dim E = \inf \{s : \mathcal{H}^s(E) = 0\}.$$

Remark 4.1. From the definition of Hausdorff dimension, it is easy to see that for any Borel subset E of I , if $\mathbf{P}(E) > 0$, then $\dim E = 1$.

From Theorem 3.6, we know that for $\mathbf{P}$ -almost all $x \in I$,

$$\frac{1}{n} \deg a_n(x) \rightarrow \frac{q}{q-1} \text{ as } n \rightarrow \infty.$$

Also note that $\deg a_n(x) \geq n$, so it is natural to consider the following set

$$A(\alpha) = \{x \in I : \lim_{n \rightarrow \infty} \frac{1}{n} \deg a_n(x) = \alpha\} \quad (27)$$

for any $\alpha \geq 1$.

We first state the mass distribution principle (see [4], Proposition 4.2) that will be used later.

Lemma 4.2. Suppose $E \subset I$ and μ is a measure with $\mu(E) > 0$. If there exist constants $c > 0$ and $\delta > 0$ such that

$$\mu(D) \leq c(\text{diam } D)^s$$

for all disc D with diameter $\text{diam } D \leq \delta$, then

$$\dim E \geq s.$$

Theorem 4.3. For any $\alpha \geq 1$, $\dim A(\alpha) = 1$.

Proof. For any $n \geq 1$, let $\mathbb{F}_q^{(n)}[z]$ denote the polynomials in $\mathbb{F}_q[z]$ with degree n , that is

$$\mathbb{F}_q^{(n)}[z] = \left\{ x \in \mathbb{F}_q[z] : x = \sum_{k=0}^n c_k z^k, c_i \in \mathbb{F}_q, (1 \leq i \leq n), \text{ and } c_n \neq 0 \right\}.$$

For any $n \geq 1$ and $b_1 \in \mathbb{F}_q^{(\text{int}(\alpha))}[z]$, $b_2 \in \mathbb{F}_q^{(\text{int}(2\alpha))}[z]$, $\dots$, $b_n \in \mathbb{F}_q^{(\text{int}(n\alpha))}[z]$, where $\text{int}(a)$ denotes the integer part for any real number a , define

$$J(b_1, b_2, \dots, b_n) = \{x \in I : a_1(x) = b_1, a_2(x) = b_2, \dots, a_n(x) = b_n\}.$$

We call $J(b_1, b_2, \dots, b_n)$ a n -order basic disc. Noting that $\text{int}((n+1)\alpha) \geq \text{int}(n\alpha) + 1$ for any $n \geq 1$, by Theorem 2.2 and Remark 2.1, we have that $J(b_1, b_2, \dots, b_n)$ is a disc with center at

$$\sum_{k=0}^{n-1} \frac{(-1)^k \prod_{j=1}^k b_j}{q_k q_{k+1}}$$

and diameter

$$q^{-\text{int}(\alpha) - \text{int}(2\alpha) - \dots - \text{int}(n\alpha) - \text{int}(n\alpha) - 1}.$$

Also by Theorem 2.2, we have

- (i) If $(b_1, b_2, \dots, b_n) \neq (b'_1, b'_2, \dots, b'_n)$, $J(b_1, b_2, \dots, b_n) \cap J(b'_1, b'_2, \dots, b'_n) = \emptyset$.
- (ii) $J(b_1, b_2, \dots, b_n, b_{n+1}) \subset J(b_1, b_2, \dots, b_n)$ for any $n \geq 1$.

Let

$$E_n = \cup J(b_1, b_2, \dots, b_n),$$

where the union is over all $b_1 \in \mathbb{F}_q^{(\text{int}(\alpha))}[z]$, $b_2 \in \mathbb{F}_q^{(\text{int}(2\alpha))}[z]$, $\dots$, $b_n \in \mathbb{F}_q^{(\text{int}(n\alpha))}[z]$. Then

$$E_n = \{x \in I : \deg a_1(x) = \text{int}(\alpha), \dots, \deg a_n(x) = \text{int}(n\alpha)\}, \quad (28)$$

and E_n consists of $(q-1)^n q^{\text{int}(\alpha) + \text{int}(2\alpha) + \dots + \text{int}(n\alpha)}$ disjoint discs with diameter

$$q^{-\text{int}(\alpha) - \text{int}(2\alpha) - \dots - \text{int}(n\alpha) - \text{int}(n\alpha) - 1}.$$

Define

$$E = \bigcap_{n=1}^{\infty} E_n.$$

It is obvious that

$$E = \{x \in I : \deg a_k(x) = \text{int}(k\alpha) \text{ for any } k \geq 1\}. \quad (29)$$

Thus $E \subset A(\alpha)$. Now we estimate the Hausdorff dimension of E .

Let μ be a mass distribution support on E such that for any $n \geq 1$ and $b_1 \in \mathbb{F}_q^{(\text{int}(\alpha))}[z]$, $b_2 \in \mathbb{F}_q^{(\text{int}(2\alpha))}[z]$, $\dots$, $b_n \in \mathbb{F}_q^{(\text{int}(n\alpha))}[z]$,

$$\mu(J(b_1, b_2, \dots, b_n)) = (q-1)^{-n} q^{-\text{int}(\alpha) - \text{int}(2\alpha) - \dots - \text{int}(n\alpha)}. \quad (30)$$

For any $\varepsilon > 0$, choose n_0 large enough such that for any $n \geq n_0$,

$$n^2 - 3n + 2 \geq (n^2 + 3n + 2)(1 - \varepsilon). \quad (31)$$

For any $x \in I$ and $m \geq \text{int}(\alpha) + \text{int}(2\alpha) + \cdots + \text{int}(n_0\alpha)$, choose $k \geq n_0$ such that

$$\text{int}(\alpha) + \cdots + \text{int}(k\alpha) \leq m < \text{int}(\alpha) + \cdots + \text{int}((k+1)\alpha),$$

which implies

$$\begin{aligned} -\text{int}(\alpha) - \cdots - \text{int}((k+1)\alpha) &< -m \\ &\leq -\text{int}(\alpha) - \cdots - \text{int}((k-1)\alpha) - \text{int}((k-1)\alpha) - 1, \end{aligned} \quad (32)$$

thus $B(x, q^{-m}) := \{y \in I : \|y - x\| \leq q^{-m}\}$ can intersect at most one $(k-1)$ -order basic disc. In fact, if there exist $b_i, b'_i \in \mathbb{F}_q^{(\text{int}(i\alpha))}[z]$, $1 \leq i \leq k-1$ such that $(b_1, b_2, \dots, b_{k-1}) \neq (b'_1, b'_2, \dots, b'_{k-1})$, $B(x, q^{-m}) \cap J(b_1, b_2, \dots, b_{k-1}) \neq \emptyset$ and $B(x, q^{-m}) \cap J(b'_1, b'_2, \dots, b'_{k-1}) \neq \emptyset$, then $B(x, q^{-m}) \subset J(b_1, b_2, \dots, b_{k-1})$ and $B(x, q^{-m}) \subset J(b'_1, b'_2, \dots, b'_{k-1})$. Thus by Remark 2.1, $J(b'_1, b'_2, \dots, b'_{k-1}) \subset J(b_1, b_2, \dots, b_{k-1})$ or $J(b_1, b_2, \dots, b_{k-1}) \subset J(b'_1, b'_2, \dots, b'_{k-1})$, and this contradicts

$$J(b_1, b_2, \dots, b_{k-1}) \cap J(b'_1, b'_2, \dots, b'_{k-1}) = \emptyset.$$

Therefore, by (31) and (32),

$$\begin{aligned} \mu(B(x, q^{-m})) &\leq (q-1)^{-(k-1)} q^{-\text{int}(\alpha) - \text{int}(2\alpha) - \cdots - \text{int}((k-1)\alpha)} \\ &\leq q^{-\text{int}(\alpha) - \text{int}(2\alpha) - \cdots - \text{int}((k-1)\alpha)} \\ &\leq q^{-\frac{k(k-1)}{2}\alpha + k-1} \leq q^{-\frac{k(k-1)}{2}\alpha + (k-1)\alpha} \\ &= q^{-\frac{k^2+3k-2}{2}\alpha} \leq q^{-\frac{(k+1)(k+2)}{2}(1-\varepsilon)\alpha} \\ &\leq (\text{diam} B(x, q^{-m}))^{1-\varepsilon}. \end{aligned}$$

By Lemma 4.2, we have $\dim E \geq 1 - \varepsilon$. Since ε is arbitrary, we have $\dim E = 1$. Note that $E \subset A(\alpha)$, thus $\dim A(\alpha) = 1$. $\square$

If α is an integer in Theorem 4.3, we can even get the following quite strong result.

For any positive integer m , let

$$B(m) = \{x \in I : \deg a_{n+1}(x) - \deg a_n(x) = m \text{ for any } n \geq 1\}. \quad (33)$$

Theorem 4.4. For any positive integer m , $\dim B(m) = 1$.

Proof. For any positive integer m , let $\alpha = m$ and E_n, E be constructed in the same way as in the proof of Theorem 4.3. Then $E \subset B(m)$ and $\dim E = 1$ by the proof of Theorem 4.3, thus $\dim B(m) = 1$ and we complete the proof of Theorem 4.4. $\square$

By Theorem 4.4, we can get the following corollaries immediately.

Corollary 4.5. The Hausdorff dimension of the set where (ii) and (iii) of Theorem 3.6 fail is 1.

Corollary 4.6. *For any positive integer m , let*

$$C(m) = \left\{ x \in I : \left\| x - \frac{p_n(x)}{q_n(x)} \right\| = q^{-\frac{m(n+1)(n+2)}{2}} \text{ for any } n \geq 1 \right\}.$$

Then

$$\dim C(m) = 1.$$

In particular, the Hausdorff dimension of the set where Theorem 3.8 fails is 1.

Remark 4.7. Using the same idea, the author [26] got the corresponding results of Theorem 4.3 and Theorem 4.4 for the Engel expansions of Laurent series.

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Résumé substantiel en français. Nous introduisons, sur le corps des séries de Laurent, le développement d'Engel en fractions continues et nous obtenons les propriétés métriques des polynômes qu'on rencontre dans ce développement. De plus, nous étudions la dimension fractale pour certains ensembles exceptionnels.

Soient p un nombre premier, q une puissance de p , et $\mathbb{F}_q$ un corps fini à q éléments. Soit $\mathbb{F}_q((z^{-1}))$ le corps des séries formelles de Laurent en z de la forme $A = \sum_{n=v}^{\infty} c_n z^{-n}$ avec coefficients $c_n \in \mathbb{F}_q$. Comme d'habitude, $\mathbb{F}_q[z]$ désigne l'anneau des polynômes en z sur $\mathbb{F}_q$ et $\mathbb{F}_q(z)$ le corps des fonctions rationnelles en z à coefficients dans $\mathbb{F}_q$.

Si $c_v \neq 0$ dans la définition de A , alors l'ordre de A est donné par $v = v(A)$ et sa norme (ou valuation) par $\|A\| = q^{-v(A)}$. Rappelons que la valuation $\|\cdot\|$ est non-archimédienne sur le corps $\mathbb{F}_q((z^{-1}))$ et que $\mathbb{F}_q((z^{-1}))$ muni de la distance $\rho(A, B) = \|A - B\|$ est un espace métrique complet.

Pour $A = \sum_{n=v}^{\infty} c_n z^{-n} \in \mathcal{L}$, on pose $[A] = \sum_{v \leq n \leq 0} c_n z^{-n} \in \mathbb{F}_q[z]$ qu'on appelle partie intégrale de A . Évidemment, le nombre $-v(A) := -v$ est précisément le degré $\deg[A]$ du polynôme $[A]$ en z .

Représentons par I l'idéal $z^{-1}\mathbb{F}_q[[z^{-1}]]$ dans l'anneau $\mathbb{F}_q[[z^{-1}]]$ des séries formelles en puissances de z^{-1} . Donc, les éléments de I sont précisément toutes les séries formelles $\sum_{n=1}^{\infty} c_n z^{-n}$. De plus I est isomorphe à $\prod_{n=1}^{\infty} \mathbb{F}_q$ et donc compact. Nous avons donc la mesure de Haar normalisée sur I que nous désignons par $\mathbf{P}$.

Soit $x = \sum_{n=1}^{\infty} c_n z^{-n} \in I$ et posons $x_1 = x$. Soit x_n (pour $n \geq 1$) défini par récurrence comme suit. Pour $x_n \neq 0$ on pose $a_n(x) = \lfloor \frac{1}{x_n} \rfloor$ et

$$x_{n+1} = \left(\frac{1}{x_n} - a_n(x) \right) \frac{1}{a_n(x)}. \quad (1)$$

Alors

$$\begin{aligned} x &= \frac{1}{a_1(x) + a_1(x)x_2} = \dots \\ &= \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots + \frac{a_{n-1}(x)}{a_n(x) + a_n(x)x_{n+1}}}}. \end{aligned}$$

Cependant, si $x_n = 0$, alors on cesse l'algorithme.

Théorème 2.2. *Tout $x \in I$ admet un développement fini ou infini (mais convergent par rapport à ρ) de la forme*

$$x = \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots}} := [0; a_1(x), a_2(x), \dots],$$

avec $a_n(x) \in \mathbb{F}_q[z]$ et

$$\deg a_1(x) \geq 1 \text{ et } \deg(a_{n+1}(x)) \geq \deg(a_n(x)) + 1 \text{ pour } n \geq 1. \quad (2)$$

Si ces conditions sur les $a_n(x) \in \mathbb{F}_q[z]$ sont vérifiées, alors le développement est unique et il est fini si et seulement si $x \in \mathbb{F}_q(z)$ (c'est-à-dire lorsque x est une fonction rationnelle en z). De plus, si on pose

$$\frac{p_n(x)}{q_n(x)} = \frac{1}{a_1(x) + \frac{a_1(x)}{a_2(x) + \dots + \frac{a_{n-1}(x)}{a_n(x)}}}, \quad (3)$$

alors on a

$$\left\| x - \frac{p_n(x)}{q_n(x)} \right\| \leq q^{-\frac{n(n+1)}{2}}. \quad (4)$$

Théorème 3.6. *Par rapport au développement d'Engel des séries de Laurent en fractions continues, on a*

(i)

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ x \in I : \frac{\sum_{j=1}^n \deg a_j(x) - \frac{q}{q-1}n}{\sqrt{nq}/(q-1)} < t \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-u^2/2} du;$$

(ii)

$$\lim_{n \rightarrow \infty} \frac{1}{n} \deg a_n(x) = \frac{q}{q-1} \quad (23)$$

$\mathbf{P}$ -presque partout sur I , et

(iii)

$$\limsup_{n \rightarrow \infty} \frac{\deg a_n(x) - \frac{q}{q-1}n}{\sqrt{2n \log \log n}} = \frac{\sqrt{q}}{q-1}$$

et

$$\liminf_{n \rightarrow \infty} \frac{\deg a_n(x) - \frac{q}{q-1}n}{\sqrt{2n \log \log n}} = -\frac{\sqrt{q}}{q-1}$$

$\mathbf{P}$ -presque partout sur I .

Pour les variables aléatoires $\left\| \frac{a_{n+1}(x)}{a_n(x)} \right\|$ ($n = 0, 1, 2, \dots$), nous avons le résultat suivant :

Théorème 3.7. *Pour tout $\varepsilon > 0$, on a*

$$\lim_{n \rightarrow \infty} \mathbf{P} \left\{ x \in I : \left| \frac{1}{n \log_q n} \sum_{j=1}^n \left\| \frac{a_{j+1}(x)}{a_j(x)} \right\| - (q-1) \right| > \varepsilon \right\} = 0$$

et donc $\frac{1}{n \log_q n} \sum_{j=1}^n \left\| \frac{a_{j+1}(x)}{a_j(x)} \right\| \rightarrow (q-1)$ en probabilité.

Pour le taux de convergence du développement d'Engel d'une série de Laurent en fractions continues, nous avons le résultat suivant :

Théorème 3.8. *Pour le développement d'Engel d'une série de Laurent en fractions continues, on a*

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \log_q \left\| x - \frac{p_n(x)}{q_n(x)} \right\| = -\frac{q}{2(q-1)}$$

P-presque partout sur I .

À partir du théorème 3.6, on déduit que lorsque $n \rightarrow \infty$, on a

$$\frac{1}{n} \deg a_n(x) \rightarrow \frac{q}{q-1}$$

P-presque partout sur I . De plus, on a $\deg a_n(x) \geq n$. Donc il est naturel d'étudier les ensembles

$$A(\alpha) = \left\{ x \in I : \lim_{n \rightarrow \infty} \frac{1}{n} \deg a_n(x) = \alpha \right\} \quad (27)$$

pour $\alpha \geq 1$.

Théorème 4.3. *On a $\dim A(\alpha) = 1$ pour tout $\alpha \geq 1$.*

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