

**ON THE DISTRIBUTION OF EIGENVALUES OF DEGENERATE
ELLIPTIC DIFFERENTIAL OPERATORS FAR FROM
SELFADJOINT ONES FOR GENERAL BOUNDARY CONDITIONS**

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RÉSUMÉ. Dans cet article, on étudie les propriétés spectrales d'un opérateur différentiel elliptique non-auto-adjoint $(Pu)(t) = -(h(t)q(t)u'(t))'(t)$, défini sur l'espace $H_\ell = L^2(0, 1)^\ell$ sous des conditions frontières de type Dirichlet. Ici $h(t) \in C^1(0, 1)$, $q(t) \in C^2([0, 1], \text{End } \mathbb{C}^\ell)$.

Dans ce travail les conditions imposées sur la fonction $h(t)$ sont plus générales que celles que l'on retrouve dans l'article de Boimatov et Seddighi [4]. Nous supposons de plus que, pour chaque $t \in [0, 1]$, les valeurs propres de la fonction matricielle $q(t)$ sont simples, non-nulles et disposées dans certaines régions du plan complexe en relation avec le secteur fermé $\Phi = \{z \in \mathbb{C} : |\arg z| \leq \varphi\}$, $\varphi \in (0, \pi)$.

On y aborde différentes questions telles la complétude des fonctions propres généralisées de l'opérateur P dans H_ℓ ; la sommabilité au sens des méthodes d'Abel avec crochets de la série de Fourier des éléments $f \in H_\ell$ représentés comme combinaison linéaire des fonctions propres généralisées de P ; un estimé de la norme de la résolvante de P dans différents cas ; la distribution asymptotique des valeurs propres de P dans le cas des conditions de type Dirichlet ainsi que pour d'autres types de conditions aux frontières.

ABSTRACT. In this article, we investigate the spectral properties of a non-selfadjoint elliptic differential operator $(Pu)(t) = -(h(t)q(t)u'(t))'(t)$, defined on space $H_\ell = L^2(0, 1)^\ell$, with Dirichlet-type boundary conditions. Here $h(t) \in C^1(0, 1)$, $q(t) \in C^2([0, 1], \text{End } \mathbb{C}^\ell)$.

In view of the paper by Boimatov and Seddighi [4], let the conditions made on the function $h(t)$ be sufficiently more general than [4]. Assume that for every $t \in [0, 1]$ the matrix function $q(t)$ has simple non-zero eigenvalues arranged in the complex plane in different locations in view of $\Phi \subset \mathbb{C}$, where $\Phi = \{z \in \mathbb{C} : |\arg z| \leq \varphi\}$, $\varphi \in (0, \pi)$.

We consider such questions as the completeness of the system of root vector functions of the operator P in H_ℓ ; summability of the Fourier series of elements $f \in H_\ell$ written as a linear combination of root vectors of P by Abelian methods with brackets; resolvent estimate of the operator P in the different cases; asymptotic distribution of eigenvalues of P which satisfies Dirichlet-type boundary conditions, and also other boundary conditions.

1. Introduction. We consider the weighted Sobolev space

$$\mathcal{H}_\ell = W_{2,h}^2(0,1)^\ell = W_{2,h}^2(0,1) \times \cdots \times W_{2,h}^2(0,1)$$

(ℓ times) as the space of vector functions $u(t) = (u_1(t), \dots, u_\ell(t))$ defined on $(0,1)$ with the finite norm

$$|u|_+ = \left(\int_0^1 h(t) \left| \frac{du(t)}{dt} \right|_{\mathbb{C}^\ell}^2 dt + \int_0^1 |u(t)|_{\mathbb{C}^\ell}^2 dt \right)^{1/2}.$$

Here the notation $|\cdot|_{\mathbb{C}^\ell}$ stands for the norm in space $\mathbb{C}^\ell$. The above definition of norm has been previously used. (See [9], [10].) Of course this could also be done in matrix language, but at the cost of greater notational complexity. By $\widehat{\mathcal{H}}_\ell$ we denote the closure of $C_0^\infty(0,1)^\ell$ with respect to the above norm (i.e., $\widehat{\mathcal{H}}_\ell$ is closure of $C_0^\infty(0,1)^\ell$ in the $\mathcal{H}_\ell$). $C_0^\infty(0,1)$ denotes the space of infinitely differentiable functions with compact support in $(0,1)$. If $\ell = 1$, then, $H = H_1$, $\mathcal{H} = \mathcal{H}_1$, and $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_1$.

We now consider a non-selfadjoint elliptic differential operator of type

$$(Pu)(t) = -\frac{d}{dt} \left(h(t)q(t) \frac{du(t)}{dt} \right),$$

acting on space $H_\ell = L^2(0,1)^\ell$ with Dirichlet-type boundary conditions. Here $h(t) \in C^1(0,1)$, $q(t) \in C^2([0,1], \text{End } \mathbb{C}^\ell)$. Assume that for each $t \in [0,1]$ the matrix function $q(t)$ has ℓ -simple non-zero eigenvalues $\mu_1(t), \dots, \mu_\ell(t)$ in the complex plane. Let $\mu_j(t) \in C^2([0,1], j = 1, \dots, \ell)$. (Note that in Section 4, the latter assumptions are needed to ensure that we are able to diagonalize the matrix function $q(t)$ for each $t \in [0,1]$.) Moreover, let $\Phi = \{z \in \mathbb{C} : |\arg z| \leq \varphi\}$, $\varphi \in (0, \pi)$ be some closed sector with vertex at zero. We now consider different locations for the eigenvalues $\mu_j(t)$ in view of $\Phi \subset \mathbb{C}$. We also suppose that $h(t) \in C^1(0,1)$ be a positive function that satisfies the following inequalities:

$$c_1 t^\alpha (1-t)^\beta \leq |h(t)| \leq M, \quad (1.2)$$

$$|h'(t)| \leq M t^{\alpha/2-1+\varepsilon_1} (1-t)^{\beta/2-1+\varepsilon_2}, \quad (1.2')$$

where $\alpha, \beta \geq 0$, and $\varepsilon_1 = 0$ if $\alpha \neq 1$ and $\varepsilon_1 > 0$ if $\alpha = 1$; $\varepsilon_2 = 0$ if $\beta \neq 1$ and $\varepsilon_2 > 0$ if $\beta = 1$.

For a closed extension of the operator P according to space $\mathcal{H}_\ell = W_{2,h}^2(0,1)^\ell$ above, we need to extend its domain to the closed domain

$$D(P) \subset \left\{ u \in \widehat{\mathcal{H}}_\ell \cap W_{2,\text{loc}}^2(0,1)^\ell : h^{1/2}u' \in H_\ell, (hqu')' \in H_\ell \right\}$$

(see [7]). Here $W_{2,\text{loc}}^2(0,1)^\ell = W_{2,\text{loc}}^2(0,1) \times \cdots \times W_{2,\text{loc}}^2(0,1)$ (ℓ times) where $W_{2,\text{loc}}^2(0,1)$ is the space of the functions $u(t)$ ($0 < t < 1$) in the form

$$\sum_{i=0}^2 \int_\varepsilon^{1-\varepsilon} |u^{(i)}(t)|^2 dt < \infty. \quad \forall \varepsilon \in (0, \frac{1}{2}).$$

In this article, we investigate the spectral properties, and the asymptotic distribution of eigenvalues of the non-selfadjoint differential operator P , which is defined on space $H_\ell = L^2(0, 1)^\ell$ for two boundary conditions. Here, and in the sequel, the value of the function $\arg z \in (-\pi, \pi]$ and $\|T\|$ denotes the norm of the bounded arbitrary operator T acting on H or H_ℓ .

To get a feeling for the history of the subject under study, refer to the papers [2-4]. In [2,3], the asymptotic spectral of the operator P was investigated by assuming that the function $h(t) = t^\gamma(1-t)^\gamma$. In [4], the differential operators of arbitrary binary sequences were studied. In this paper, the conditions we make on the function $h(t)$ are sufficiently more general than [4].

The paper has seven sections. Section 1 is devoted to introduction. In Section 2, we have Theorem 2.1 on the resolvent estimate of the differential operator A , acting on H in a certain case (*i.e.*, in this case, we will study Theorem 2.1 under assumption (2.2)). In Section 3, we have Theorem 3.1 on the estimate resolvent of the differential operator A , acting on H in the general case (*i.e.*, in this case, we will study Theorem 3.1 which, in contrast to Theorem 2.1, does not include assumption (2.2) of Theorem 2.1). It is necessary to make a further remark regarding Theorem 2.1 and Theorem 3.1. We will see that Theorem 2.1, under the assumption (2.2), leads to an assertion including two estimates (2.3) and (2.4). Meanwhile Theorem 3.1, without including assumption (2.2) of Theorem 2.1, leads to a similar assertion, but asserts only statement (2.3) of Theorem 2.1, which becomes (3.2). In Section 4, we assume that for each $t \in [0, 1]$ the matrix function $q(t)$ has ℓ -simple non-zero eigenvalues $\mu_j(t) \in C^2[0, 1]$, $j = 1, \dots, \ell$, arranged in the complex plane in the following way:

$$\mu_1(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi.$$

Then, we estimate the resolvent of differential operator P , acting on space $H_\ell = L^2(0, 1)^\ell$. In Section 5, completeness of the system of root vector functions of the operator P , by the Abelian method with brackets is studied. In Section 6 we let some eigenvalues $\mu_j(t)$ of the matrix function $q(t)$ take another location in the complex plane in the form

$$\mu_1(t), \dots, \mu_\nu(t) \in \mathbb{R}_+, \mu_{\nu+1}(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi,$$

and then, under these conditions, we determine the asymptotic distribution formula for the eigenvalues function $N(t) = \text{card}\{j : |\lambda_j| \leq t\}$, as $t \rightarrow +\infty$. In Section 7, we prove Theorem 7.1, as Theorem 6.1 is proven in Section 6, but under general boundary conditions (*i.e.*, in contrast to Dirichelet-type boundary conditions).

2. On the resolvent estimate of the differential operator A in $H = L_2(0, 1)$.

Theorem 2.1. *Let $\Phi \subset \mathbb{C}$ be some closed sector with vertex at 0, and set $P = A$, $q(t) = \mu(t)$ in the definition of the operator P , as in Section 1. Then, we obtain*

$$(Av)(t) = -\frac{d}{dt} \left(h(t)\mu(t) \frac{dv(t)}{dt} \right),$$

acting on $H = L_2(0, 1)$. Assume that

$$\mu(t) \in C^2[0, 1], \quad \mu(t) \in \mathbb{C} \setminus \Phi, \quad \forall t \in [0, 1], \quad (2.1)$$

$$|\arg\{\mu(t_1)\mu^{-1}(t_2)\}| \leq \frac{\pi}{8}, \quad (\forall t_1, t_2 \in [0, 1]). \quad (2.2)$$

Then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in the space $H = L^2(0, 1)$, and the following estimates hold:

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1} \quad (\lambda \in \Phi, |\lambda| > C_\Phi), \quad (2.3)$$

$$\|h^{1/2} \frac{d}{dt} (A - \lambda I)^{-1}\| \leq M'_\Phi |\lambda|^{-1/2} \quad (\lambda \in \Phi, |\lambda| > C_\Phi), \quad (2.4)$$

where the numbers M_Φ , M'_Φ and $C_\Phi > 0$ are sufficiently large numbers depending on Φ and $\Phi = \{z \in \mathbb{C} : |\arg z| \leq \varphi\}$, $\varphi \in (0, \pi)$.

Proof. Here, to establish Theorem 2.1, we will first prove the assertion of Theorem 2.1 together with estimate (2.3). As in Section 1, for the closed extension the operator A , (for more explain see chapter 6 of [7]), we need to extend its domain to the closed set

$$D(A) = \left\{ v \in \widehat{\mathcal{H}} \cap W_{2,\text{loc}}^2(0, 1) : h^{1/2}u' \in H, (h\mu v')' \in H \right\}.$$

Let the operator A , now satisfy (2.1), (2.2), then, there exists a real $\gamma \in (-\pi, \pi]$, such that for the complex number $e^{i\gamma}$ we have $|e^{i\gamma}| = 1$, and so

$$c' \leq \operatorname{Re}\{e^{i\gamma}\mu(t)\}, \quad c'|\lambda| \leq -\operatorname{Re}\{e^{i\gamma}\lambda\}, \quad c' > 0, \quad \forall t \in [0, 1], \quad \lambda \in \Phi. \quad (2.5)$$

By integrating from both sides of $c' \leq \operatorname{Re}\{e^{i\gamma}\mu(t)\}$ for $v \in D(A)$ we will have

$$c' \int_0^1 h(t)|v'(t)|^2 dt \leq \operatorname{Re} \int_0^1 e^{i\gamma} h(t)\mu|v'(t)|^2 dt = \operatorname{Re}\{e^{i\gamma}(Av, v)\}. \quad (2.6)$$

Here the symbol $(\cdot, \cdot)$ denotes the inner product in H . Notice that the above equality in (2.6) is obtained by the well-known theorem of the m -sectorial operators, which are closed by extending their domain to the closed domain in $\mathcal{H}$. These operators are associated with the closed sectorial bilinear forms that are densely defined in $\mathcal{H}$ (for further explanation see the well-known Theorem 2.1, chapter 6 of [7]). This is why, we extend the domain of the operator A to the closed domain in space $\mathcal{H}$, above.

By (2.5), we have: $c'|\lambda| \leq -\operatorname{Re}\{e^{i\gamma}\lambda\}$, $c' > 0$, $\forall \lambda \in \Phi$. Multiplying the latter inequality by $\int_0^1 |v(t)|^2 dt = (v, v) = \|v\|^2 > 0$, then

$$c'|\lambda| \int_0^1 |v(t)|^2 dt \leq -\operatorname{Re}\{e^{i\gamma}\lambda\}(v, v).$$

By the latter inequality, and (2.6), and by considering $c' = 1/M$, it follows that

$$\begin{aligned} \int_0^1 h(t)|v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2 dt &\leq M \operatorname{Re}\{e^{i\gamma}(Av, v) - e^{i\gamma}\lambda(v, v)\} \\ &= M \operatorname{Re}\{e^{i\gamma}((A - \lambda I)v, v)\} \\ &\leq M \|e^{i\gamma}\| \|v\| \|(A - \lambda I)v\| \\ &= M \|v\| \|(A - \lambda I)v\| \end{aligned} \quad (2.7)$$

or

$$\int_0^1 h(t)|v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2 dt \leq M \|v\| \|(A - \lambda I)v\|.$$

Since $\int_0^1 h(t)|v'(t)|^2 dt$ is positive, we will have also

$$|\lambda| \|v\|^2 = |\lambda| \int_0^1 |v(t)|^2 dt \leq M \|v\| \|(A - \lambda I)v\|, \quad (2.8)$$

i.e.

$$|\lambda| \|v\| \leq M \|(A - \lambda I)v\|.$$

The above relation ensures that the operator $(A - \lambda I)$ is one-to-one, which implies that $\ker(A - \lambda I) = 0$. Therefore, the inverse operator $(A - \lambda I)^{-1}$ exists and its continuity follows from the proof of the estimate (2.3) of Theorem 2.1. To prove (2.3), we set $v = (A - \lambda I)^{-1}f$, $f \in H$ in (2.8), so that

$$|\lambda| \int_0^1 |(A - \lambda I)^{-1}f(t)|^2 dt \leq M \|(A - \lambda I)^{-1}f\| \|(A - \lambda I)(A - \lambda I)^{-1}f\|.$$

Since $(A - \lambda I)(A - \lambda I)^{-1}f = \text{Id}(f) = f$ it follows that

$$|\lambda| \int_0^1 |(A - \lambda I)^{-1}f(t)|^2 dt \leq M \|(A - \lambda I)^{-1}f\| \|f\|.$$

Therefore,

$$|\lambda| \|(A - \lambda I)^{-1}f\|^2 \leq M \|(A - \lambda I)^{-1}f\| \|f\|.$$

By canceling the positive term $\|(A - \lambda I)^{-1}f\|$ from both sides of the latter inequality we find

$$|\lambda| \|(A - \lambda I)^{-1}f\| \leq M \|f\|,$$

and since $\lambda \neq 0$, this implies that $\|(A - \lambda I)^{-1}f\| \leq M |\lambda|^{-1} \|f\|$. The end result is

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}.$$

This estimate completes the proof of the assertion Theorem 2.1, together with the estimate (2.3).

To complete the proof of Theorem 2.1, here, we must prove estimate (2.4), therefore, we begin to prove the estimate (2.4) of Theorem 2.1. As in the first arguments to prove estimate (2.3) above, here, we drop the positive term $|\lambda| \int_0^1 |v(t)|^2 dt$ from

$$\int_0^1 h(t)|v'(t)|^2 dt + |\lambda| \int_0^1 |v(t)|^2 dt \leq M \|v\| \|(A - \lambda I)v\|.$$

It follows that

$$\int_0^1 h(t)|v'(t)|^2 dt \leq M \|v\| \|(A - \lambda I)v\|.$$

Set $v = (A - \lambda I)^{-1}f$, $f \in H$ in the latter inequality, and as above, by proceeding with similar calculations, we then obtain:

$$\int_0^1 h(t) \left| \frac{d}{dt} (A - \lambda I)^{-1} f(t) \right|^2 dt \leq M \|(A - \lambda I)^{-1} f\| \|(A - \lambda I)(A - \lambda I)^{-1} f\|.$$

Since, $(A - \lambda I)(A - \lambda I)^{-1}f = f$, and

$$\int_0^1 h(t) \left| \frac{d}{dt} (A - \lambda I)^{-1} f(t) \right|^2 dt \leq M \|(A - \lambda I)^{-1} f\| \|f\|,$$

consequently by (2.3) we have $\|(A - \lambda I)^{-1} f\| \leq M \|f\| |\lambda|^{-1}$. Then,

$$\int_0^1 h(t) \left| \frac{d}{dt} (A - \lambda I)^{-1} f(t) \right|^2 dt \leq M \|(A - \lambda I)^{-1} f\| \|f\| \leq M^2 |\lambda|^{-1} \|f\|^2.$$

Therefore,

$$\int_0^1 h(t) \left| \frac{d}{dt} (A - \lambda I)^{-1} f(t) \right|^2 dt \leq M'_\Phi |\lambda|^{-1} \|f\|^2$$

i.e.,

$$\left\| h^{1/2} \frac{d}{dt} (A - \lambda I)^{-1} f \right\|^2 \leq M'_\Phi |\lambda|^{-1} \|f\|^2$$

or

$$\left\| h^{1/2} \frac{d}{dt} (A - \lambda I)^{-1} f \right\| \leq M'_\Phi |\lambda|^{-1/2} \|f\|.$$

Consequently,

$$\left\| h^{1/2} \frac{d}{dt} (A - \lambda I)^{-1} \right\| \leq M'_\Phi |\lambda|^{-1/2}.$$

This estimate completes the proof of (2.4). Theorem 2.1 is thereby proved. $\square$

3. On the resolvent estimate of the differential operator A in $H = L_2(0, 1)$ in the general case. In this section, we will derive a new general theorem by dropping the assumption (2.2) from Theorem 2.1 in Section 2.

Theorem 3.1. *As in Section 1, let Φ be some closed sector with vertex at 0 in the complex plane. If we set $P = A$, $q(t) = \mu(t)$ in the definition of the operator P in Section 1, then, the differential operator below follows:*

$$(Av)(t) = -\frac{d}{dt} \left(h(t) \mu(t) \frac{dv(t)}{dt} \right),$$

acting on $H = L_2(0, 1)$. Assume that

$$\mu(t) \in C^2[0, 1], \quad \mu(t) \in \mathbb{C} \setminus \Phi, \quad \forall t \in [0, 1], \quad (3.1)$$

then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(0, 1)$, and the following estimate holds:

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1} \quad (3.2)$$

where $M_\Phi, C_\Phi > 0$ are sufficiently large number depending on Φ and $|\lambda| > C_\Phi$.

Proof. To prove the assertion of theorem 3.1 together with (3.2), we construct the functions $\varphi_{(1)}(t), \dots, \varphi_{(\rho)}(t)$, $\mu_{(1)}(t), \dots, \mu_{(\rho)}(t)$ so that each one of the functions $\mu_{(j)}(t)$, $j = 1, \dots, \rho$ ($t \in \text{supp } \varphi_{(j)}$), as the function $\mu(t)$ in Theorem 2.1 satisfies (2.2). Therefore, let

$$\mu_{(1)}(t), \dots, \mu_{(\rho)}(t), \quad \varphi_{(1)}(t), \dots, \varphi_{(\rho)}(t) \in C^\infty[0, 1],$$

satisfy

$$0 \leq \varphi_{(j)}(t), \quad j = 1, \dots, \rho, \quad \varphi_{(1)}^2(t) + \dots + \varphi_{(\rho)}^2(t) \equiv 1 \quad (0 \leq t \leq 1)$$

$$\frac{d}{dt} \varphi_{(j)}(t) \in C_0^\infty(0, 1), \quad \mu_{(j)}(t) = \mu(t), \quad \forall t \in \text{supp } \varphi_{(j)}$$

$$\mu_{(j)}(t) \in \mathbb{C} \setminus S, \quad \forall t \in [0, 1], \quad j = 1, \dots, \rho.$$

$$|\arg\{\mu_{(j)}(t_1)\mu_{(j)}^{-1}(t_2)\}| \leq \frac{\pi}{8}, \quad (\forall t_1, t_2 \in \text{supp } \varphi_{(j)}), \quad j = 1, \dots, \rho.$$

In view of Theorem 2.1, and by (2.3), and (2.4) set $(A_{(j)}v)(t) = A(t)v(t)$, we will have the differential operator

$$(A_{(j)}v)(t) = -\frac{d}{dt} \left(h(t)\mu_{(j)}(t) \frac{dv(t)}{dt} \right),$$

acting on $H = L_2(0, 1)$ where

$$D(A_{(j)}) = \left\{ v \in \widehat{\mathcal{H}} \cap W_{2,\text{loc}}^2(0, 1) : h^{1/2}u' \in H, (h\mu_{(j)}v')' \in H \right\}.$$

Due to the assertion of Theorem 2.1, for $0 \neq \lambda \in \Phi$ the inverse operator $(A - \lambda I)^{-1}$ exists and is continuous in space $H = L^2(0, 1)$, and satisfies

$$\|(A_{(j)} - \lambda I)^{-1}\| \leq M_1|\lambda|^{-1}, \quad \|h^{1/2} \frac{d}{dt} (A_{(j)} - \lambda I)^{-1}\| \leq M_1|\lambda|^{-1/2}. \quad (3.3)$$

Let us introduce

$$T(\lambda) = \sum_{j=1}^{\rho} \varphi_{(j)}(A_{(j)} - \lambda I)^{-1} \varphi_{(j)}. \quad (3.4)$$

Here $\varphi_{(j)}$ is the multiplication operator in H by the function $\varphi_{(j)}(t)$.

Consequently,

$$(A - \lambda I)T(\lambda)v = I_1 + I_2 + I_3 + I_4$$

where

$$I_1 = - \sum_{j=1}^{\rho} (h(t)\mu(\varphi_{(j)})'_t)'_t (A_{(j)} - \lambda I)^{-1} \varphi_{(j)}v,$$

$$I_2 = - \sum_{j=1}^{\rho} h\mu(\varphi_{(j)})'_t \frac{d}{dt} (A_{(j)} - \lambda I)^{-1} \varphi_{(j)}v,$$

$$I_3 = - \sum_{j=1}^{\rho} \varphi_{(j)} \left(h \mu \frac{d}{dt} (A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v \right)'_t,$$

$$I_4 = -\lambda \sum_{j=1}^{\rho} \varphi_{(j)} (A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v.$$

As $\mu_{(j)}(t) = \mu(t)$ ($\forall t \in \text{supp } \varphi_{(j)}$), replace $\mu(t)$ by $\mu_{(j)}(t)$ in the sum I_3 . Then, in view of $\sum_{j=1}^{\rho} \varphi_{(j)}^2(t) \equiv 1$, we will have

$$\begin{aligned} I_3 + I_4 &= - \sum_{j=1}^{\rho} \varphi_{(j)} \left[\left(h \mu_{(j)} \frac{d}{dt} (A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v \right)'_t + \lambda (A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v \right] \\ &= \sum_{j=1}^{\rho} \varphi_{(j)} (A_{(j)} - \lambda I) A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v = \sum_{j=1}^{\rho} \varphi_{(j)}^2 v = v \end{aligned}$$

i.e. $I_3 + I_4 = v$. When considering $I_1 + I_2 = G(\lambda)v$, then $(A - \lambda I)T(\lambda)v = v + G(\lambda)v$. Or equivalently

$$(A - \lambda I)T(\lambda) = I + G(\lambda). \quad (3.5)$$

Using the fact that $\varphi_{(j)}'_t \in C^\infty(0, 1)$, and by (3.3), we can estimate I_1, I_2 as follows

$$\begin{aligned} \|I_1\| &\leq M \sum_{j=1}^{\rho} \|(A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v\| \leq M |\lambda|^{-1} \|v\|, \\ \|I_2\| &\leq M \sum_{j=1}^{\rho} \left\| \frac{d}{dt} (A_{(j)} - \lambda I)^{-1} \varphi_{(j)} v \right\| \leq M'' |\lambda|^{-1/2} \|v\|. \end{aligned}$$

Using these estimates, and in view of $G(\lambda)$ above, we will have

$$\|G(\lambda)(v)\| \leq \|I_1\| + \|I_2\| \leq M |\lambda|^{-1} \|v\| + M'' |\lambda|^{-1/2} \|v\|,$$

being that λ is sufficiently large, implies that $|\lambda|^{-1} \leq |\lambda|^{-1/2}$ then

$$\|G(\lambda)\| \leq M'_\Phi |\lambda|^{-1/2}. \quad (3.6)$$

By this λ we can also have $\|G(\lambda)\| \leq 1/2 < 1$ where $\lambda \in \Phi$. Now by this, and using the well-known theorem in the operator theory, we conclude that $I + G(\lambda)$ and hence, by (3.5), $(A - \lambda I)T(\lambda)$ are invertible. Therefore $((A - \lambda I)T(\lambda))^{-1}$ exists, and by (3.5)

$$(T(\lambda))^{-1}(A - \lambda I)^{-1} = (I + G(\lambda))^{-1}. \quad (3.7)$$

Adding $+I$ and $-I$ to the right side of (3.7) it follows that

$$(T(\lambda))^{-1}(A - \lambda I)^{-1} = (I + G(\lambda))^{-1} - I + I.$$

Setting $F(\lambda) = (I + G(\lambda))^{-1} - I$ implies that

$$(T(\lambda))^{-1}(A - \lambda I)^{-1} = I + F(\lambda).$$

In view of $\|G(\lambda)\| \leq 1/2 < 1$, and (3.6), by applying the geometric series for $F(\lambda)$

$$\begin{aligned} \|F(\lambda)\| &\leq \sum_{i=2}^{+\infty} \|G^i(\lambda)\| \leq \|G(\lambda)\|^2 (1 + \|G(\lambda)\| + \|G(\lambda)\|^2 + \dots) \\ &\leq \|G(\lambda)\|^2 (1 + 1/2 + 1/4 + \dots) \\ &\leq 2(M'_\Phi |\lambda|^{-1/2})^2 \end{aligned}$$

i.e.,

$$\|F(\lambda)\| \leq K_\Phi |\lambda|^{-1}.$$

Now by (3.3) and (3.4)

$$\begin{aligned} \|T(\lambda)\| &= \left\| \sum_{j=1}^p \varphi_{(j)} (A_{(j)} - \lambda I)^{-1} \varphi_{(j)} \right\| \\ &\leq M''_\Phi \|(A_{(j)} - \lambda I)^{-1}\| \\ &\leq M''_\Phi M_\Phi |\lambda|^{-1} \end{aligned}$$

i.e.,

$$\|T(\lambda)\| \leq K'_\Phi |\lambda|^{-1}.$$

By this and (3.7), it follows that

$$\begin{aligned} \|(A - \lambda I)^{-1}\| &= \|T(\lambda)\| \|I + F(\lambda)\| \\ &\leq K'_\Phi |\lambda|^{-1} (1 + K_\Phi |\lambda|^{-1}) \\ &\leq K'_\Phi |\lambda|^{-1} + K_\Phi K'_\Phi |\lambda|^{-2}. \end{aligned}$$

Since $|\lambda|^{-2} \leq |\lambda|^{-1}$, then

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1} \quad (|\lambda| \geq C_\Phi, \lambda \in \Phi).$$

This estimate completes the proof of (3.2); therefore Theorem 3.1 is proved. $\square$

4. On the resolvent estimate of the differential operator in H_ℓ .

Theorem 4.1. *Let P , $h(t)$, $q(t)$ and Φ be defined as in Section 1. Suppose that for each $t \in [0, 1]$ the matrix function $q(t)$ has ℓ distinct non-zero simple eigenvalues $\mu_j(t) \in C^2[0, 1]$ ($1 \leq j \leq \ell$) located in the complex plane $\mathbb{C}$, in the form*

$$\mu_1(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi. \quad (4.1)$$

As in Theorem 2.1, but here we let the operator P acting on the space $H_\ell = L^2(0, 1)^\ell$ with Dirichelet-type boundary conditions. Then, for sufficiently large in modulus $\lambda \in \Phi$, the inverse operator $(P - \lambda I)^{-1}$ exists and is continuous in the space $H_\ell = L^2(0, 1)^\ell$ and the following estimate holds:

$$\|(P - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1} \quad (4.2)$$

where $M_\Phi, C_\Phi > 0$ are sufficiently large number depending on Φ and $|\lambda| > C_\Phi$.

Proof. The conditions which we consider on the eigenvalues $\mu_j(t)$ of the matrix function $q(t)$ guarantee that one can convert the matrix $q(t)$ to the diagonal form

$$q(t) = U(t)\Lambda(t)U^{-1}(t), \text{ where } U(t), U^{-1}(t) \in C^2([0, 1], \text{End } \mathbb{C}^\ell)$$

and $\Lambda(t) = \text{diag}\{\mu_1(t), \dots, \mu_\ell(t)\}$. Consider space $H_\ell = H \oplus \dots \oplus H$ (ℓ times). Let us introduce the operator

$$B(\lambda) = \text{diag}\{(P_1 - \lambda I)^{-1}, \dots, (P_\ell - \lambda I)^{-1}\}, \quad (4.3)$$

acting on the space H_ℓ . Due to results of Theorem 2.1, if we set $P_j = A$ for $j = 1, \dots, \ell$, then, here as previously, we will have

$$(P_j v)(t) = -\frac{d}{dt} \left(h(t) \mu_j(t) \frac{dv(t)}{dt} \right),$$

$$D(P_j) = \left\{ v \in \widehat{\mathcal{H}} \cap W_{2,\text{loc}}^2(0, 1) : h^{1/2}u' \in H, h^{1/2}u' \in H, (h\mu_j u')' \in H \right\}.$$

According to the results that obtained in Section 2, it follows that the operator $B(\lambda)$, defined in (4.3), exists and is continuous for sufficiently large absolute values of $\lambda \in \Phi$. We consider the operator $\Gamma(\lambda) = UB(\lambda)U^{-1}$ where $(Uu)(t) = U(t)u(t)$, ($u \in H_\ell$). Consequently, it follows that

$$(P - \lambda I)\Gamma(\lambda)u = -\frac{d}{dt} \left(h(t)q(t) \frac{d}{dt} (U(t)B(\lambda)U^{-1}(t)u(t)) \right) = T_1 + T_2 + T_3$$

where

$$\begin{aligned} T_1 &= -\frac{d}{dt} \left(h(t)q(t)U(t) \frac{d}{dt} B(\lambda)U^{-1}(t)u(t) \right) \\ &= -\frac{d}{dt} \left(h^{1/2}(t)U(t)\Lambda(t) \frac{d}{dt} B(\lambda)U^{-1}u(t) \right) \\ &= -U \frac{d}{dt} \left(h^{1/2}(t)\Lambda(t) \frac{d}{dt} B(\lambda)U^{-1}u \right) - U'(t)h(t)\Lambda \frac{d}{dt} B(\lambda)U^{-1}u. \end{aligned}$$

Using the equality

$$-\frac{d}{dt} h(t) \left(\Lambda \frac{d}{dt} B(\lambda) V \right) = V + \lambda B(\lambda) V, \quad V = U^{-1}u.$$

We can then write

$$-U \frac{d}{dt} \left(h(t)\Lambda(t) \frac{d}{dt} B(\lambda)U^{-1}u \right) = -U(V + \lambda B(\lambda)V), \quad V = U^{-1}u.$$

Thus, we have

$$T_1 = \lambda UB(\lambda)U^{-1}u - U'(t)h(t)\Lambda \frac{d}{dt} B(\lambda)U^{-1}u + UU^{-1}u.$$

$$T_2 = -\frac{d}{dt} \left(h^{1/2} q U' B(\lambda) U^{-1} u \right), \quad T_3 = -\lambda U(t) B(\lambda) U^{-1} u.$$

Using (2.3), (2.4), and from the above relations, we have

$$(P - \lambda I) \Gamma(\lambda) = I + T_1^0 + T_2^0$$

where $T_2^0 = (h)' R U' B(\lambda) U^{-1}$, $\|T_1^0\| \leq M |\lambda|^{-1/2}$ ($\lambda \in \Phi$, $|\lambda| \geq C_\Phi$).

By the Hardy-type inequality (see Chapter 6 of [4]), we have

$$\begin{aligned} & \int_0^1 t^{-1+\varepsilon'_1} (1-t)^{-1+\varepsilon'_2} |y(t)|^2 dt \\ & \leq M(\varepsilon'_1, \varepsilon'_2) \int_0^1 |y(t)|^2 dt + M(\varepsilon'_1, \varepsilon'_2) \int_0^1 t^{1+\varepsilon'_1} (1-t)^{1+\varepsilon'_2} |y'(t)|^2 dt, \\ & \quad \forall y \in \widehat{\mathcal{H}}, \quad \varepsilon'_1, \varepsilon'_2 \neq 0. \end{aligned}$$

Since $|q(t)U'(t)|_{\mathbb{C}^\ell \rightarrow \mathbb{C}^\ell} \leq M$, and by (1.2'), we will estimate the operator T_2^0 as follows: using the Hardy-type inequality above, and taking $W = \beta(\lambda)u$, then we have

$$\begin{aligned} & \int_0^1 |h'(t)|^2 |(\beta(\lambda)u)(t)|_{\mathbb{C}^\ell}^2 dt = \int_0^1 |h'(t)|^2 |W(t)|_{\mathbb{C}^\ell}^2 dt \\ & \leq M_2 \int_0^1 t^{\alpha-2+2\varepsilon'_1} (1-t)^{\beta-2+2\varepsilon'_2} |W(t)|_{\mathbb{C}^\ell}^2 dt \\ & \leq M_3 \int_0^1 t^\alpha (1-t)^\beta t^{2\varepsilon'_1} (1-t)^{2\varepsilon'_2} |W'(t)|_{\mathbb{C}^\ell}^2 dt \\ & \quad + M |(\beta(\lambda)u)|_{H_\ell}^2. \end{aligned}$$

i.e.

$$\begin{aligned} & \int_0^1 |h'(t)|^2 |(\beta(\lambda)u)(t)|_{\mathbb{C}^\ell}^2 dt \leq M_3 \int_0^1 t^\alpha (1-t)^\beta t^{2\varepsilon'_1} (1-t)^{2\varepsilon'_2} |W'(t)|_{\mathbb{C}^\ell}^2 dt \\ & \quad + M |(\beta(\lambda)u)|_{H_\ell}^2 \end{aligned}$$

Since $t^{2\varepsilon'_1} (1-t)^{2\varepsilon'_2} \leq M'$, and by (1.2) and the estimates (2.3) and (2.4), the last inequality is equivalent to

$$\begin{aligned} & \int_0^1 |h'(t)|^2 |(\beta(\lambda)u)(t)|_{\mathbb{C}^\ell}^2 dt \leq M_3 M' \int_0^1 t^\alpha (1-t)^\beta |W'(t)|_{\mathbb{C}^\ell}^2 dt + M |(\beta(\lambda)u)|_{H_\ell}^2 \\ & \leq M_3 M' \int_0^1 h(t) |W'(t)|_{\mathbb{C}^\ell}^2 dt + M |(\beta(\lambda)u)|_{H_\ell}^2 \end{aligned}$$

As in Section 2, for finding the estimates (2.3), (2.4) we have

$$\begin{aligned} & \int_0^1 |h'(t)|^2 |(\beta(\lambda)u)(t)|_{\mathbb{C}^\ell}^2 dt \leq M_3' \int_0^1 h(t) |W'(t)|_{\mathbb{C}^\ell}^2 dt + M |(\beta(\lambda)u)|_{H_\ell}^2 \\ & \leq M_3' \int_0^1 h(t) |W'(t)|_{\mathbb{C}^\ell}^2 dt + M |(\beta(\lambda)u)|_{H_\ell}^2 \\ & \leq M_3' |\lambda|^{-1/2} + M |\lambda|^{-1} \\ & \leq M_\Phi |\lambda|^{-1/2} \end{aligned}$$

Then $\|T_2^0\| \leq M_\Phi |\lambda|^{-1/2}$ for sufficiently large absolute-value of $\lambda \in \Phi$. Consequently,

$$(P - \lambda I)\Gamma(\lambda) = I + \mathcal{F}(\lambda), \quad \|\mathcal{F}(\lambda)\| \leq M |\lambda|^{-1/2} \quad (\lambda \in \Phi, \quad |\lambda| > C'). \quad (4.4)$$

Proceeding as in Section 3 (see the calculations (3.6) and (3.7) in Section 3) since $\|\mathcal{F}(\lambda)\| \leq M |\lambda|^{-1/2}$, it easily follows that $I + \mathcal{F}(\lambda)$ and so, by (4.4) $(A - \lambda I)\Gamma(\lambda)$ is invertible. We then have

$$((P - \lambda I)\Gamma(\lambda))^{-1} = (I + \mathcal{F}(\lambda))^{-1}.$$

As in Section 3, recall that by adding $+I$ and $-I$ to the right side of the relation above we will have

$$(I + \mathcal{F}(\lambda))^{-1} = (I + \mathcal{F}(\lambda))^{-1} - I + I$$

Again as in Section 3, setting $\mathcal{Y}(\lambda) = (I + \mathcal{F}(\lambda))^{-1} - I$, since $\|\mathcal{F}(\lambda)\| \leq M |\lambda|^{-1/2}$, this implies

$$\|\mathcal{Y}(\lambda)\| \leq K_\Phi |\lambda|^{-1} \quad (\lambda \in \Phi, \quad |\lambda| > C). \quad (4.5)$$

Consequently,

$$(P - \lambda I)^{-1} = \Gamma(\lambda)(I + \mathcal{Y}(\lambda)) \quad (4.6)$$

since

$$\Gamma(\lambda) = UB(\lambda)U^{-1}, \quad B(\lambda) = \text{diag}\{(P_1 - \lambda I)^{-1}, \dots, (P_\ell - \lambda I)^{-1}\}. \quad (4.7)$$

Putting $A_j = P_j$, $j = 1, \dots, \ell$, in (3.3), one gets $\|(P_j - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}$, for $j = 1, \dots, \ell$. It follows that $\|\Gamma(\lambda)\| \leq K'_\Phi |\lambda|^{-1}$ and therefore

$$\begin{aligned} \|(P - \lambda I)^{-1}\| &\leq \|\Gamma(\lambda)\| \|(I + \mathcal{Y}(\lambda))\| \\ &\leq K'_\Phi |\lambda|^{-1} (1 + K_\Phi |\lambda|^{-1}) \\ &\leq M_\Phi |\lambda|^{-1} \end{aligned}$$

i.e.

$$\|(P - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}$$

where M_Φ is a sufficiently large number depending on Φ . This completes the proof of Theorem 4.1. $\square$

5. Completeness of the system of root vector functions of the operator P in H_ℓ and summability of the Fourier series of elements $f \in H_\ell$.

Theorem 5.1. *Let $h(t) \geq ct^\gamma(1-t)^\gamma$ ($\gamma < 2$), and let $S \subset \bar{\Phi} \setminus \mathbb{R}_+$ be some closed sector with origin at zero. Then for sufficiently large modulus $\lambda \in S$, the operator $(A - \lambda I)^{-1}$ is a compact operator of order not larger than $1/2$; the spectrum of the operator A consists of discrete countable eigenvalues counted according to multiplicity with perhaps an isolated point at infinity; the Fourier series of an arbitrary vector function $f \in H_\ell$ of the system of the root of vector functions of the operator A , converges to f by the Abelian method with brackets of order $\gamma' = 1/2 + \varepsilon$ (where $\varepsilon > 0$ is a sufficiently small number).*

For the definition of summability by Abelian methods (introduced by Lidskij) see [1]. The symbol $N_0(t)$ denotes the number of eigenvalues of the operator A , does not exceed t (counted according to multiplicity), and satisfies the following inequality $N_0(t) \leq M(1+t)^{1/2}$.

Proof. Let $e^{i\gamma}$, $\lambda \in \mathbb{C}$ be the same numbers as in Section 2, and let the operator A have the same conditions as in (2.2), (2.3), such that $\lambda \geq 0$. Since, as in Theorem 2.1, we showed that the operator $A - \lambda I$ has a continuous inverse and by (2.3), it follows that the operator $L(\lambda) = e^{i\gamma}(A - \lambda I)$ is an m -sectorial operator in space $\mathcal{H}$ (see [7]), and so has the bilinear form

$$L_\lambda[v_1, v_2] = e^{i\gamma} \int_0^1 h(t)\mu(t)v_1'(t)\overline{v_2'(t)} dt - \lambda e^{i\gamma} \int_0^1 v_1(t)\overline{v_2(t)} dt$$

where its domain is $D[L_\lambda] = \widehat{\mathcal{H}}$. Recall that as in Section 2, by closed extension its domain implies that this bilinear form is densely define in $\mathcal{H}$, and so it is closed and sectorial in $\mathcal{H}$. Now by Theorem 3.2 [7, volume VI, 8.3.1] the following relation is true

$$L(\lambda) = L_0^{1/2}(\lambda)K(\lambda)L_0^{1/2}(\lambda) \quad (5.1)$$

in which $K(\lambda)$ is a bounded operator having continuous inverse:

$$\|K(\lambda)\| \leq M, \quad \|K^{-1}(\lambda)\| \leq M \quad (0 \neq \lambda \in \Phi) \quad (5.2)$$

and $L_0(\lambda)$ is a selfadjoint operator in $\mathcal{H}$ associated with the bilinear form

$$\mathcal{L}_{0,\lambda}[v_1, v_2] = \int_0^1 \operatorname{Re}[e^{i\gamma}h(t)\mu(t)]v_1'(t)\overline{v_2'(t)} dt - \operatorname{Re}(e^{i\gamma}\lambda) \int_0^1 v_1(t)\overline{v_2(t)} dt,$$

$$D[\mathcal{L}_{0,\lambda}] = \widehat{\mathcal{H}}.$$

By (2.2) we have

$$\|(L' + |\lambda|I)^{1/2}L_0^{-1/2}(\lambda)\| \leq M \quad (|\lambda| > 1, \lambda \in S), \quad (5.3)$$

in which L' is a positive selfadjoint operator in $\mathcal{H}$ associated with the bilinear form

$$\mathcal{L}'[v_1, v_2] = \int_0^1 h(t)\mu(t)v_1'(t)\overline{v_2'(t)} dt, \quad D[\mathcal{L}'] = \widehat{\mathcal{H}}.$$

Now suppose that $h(t) \geq ct^\gamma(1-t)^\gamma$ ($\gamma < 2$), then

$$\|(L'' + |\lambda|I)^{1/2}(L' + |\lambda|I)^{-1/2}\| \leq M \quad (|\lambda| > 1, \lambda \in S) \quad (5.4)$$

in which L'' is a positive selfadjoint operator in $\mathcal{H}$ associated with the bilinear form

$$\mathcal{L}''[v_1, v_2] = \int_0^1 t^\gamma(1-t)^\gamma\mu(t)v_1'(t)\overline{v_2'(t)} dt, \quad D[\mathcal{L}''] = \widehat{\mathcal{H}}.$$

Since $\gamma < 2$, $(L'' + T)^{-1} \in \sigma_\infty(H)$. Here $\sigma_\infty(H)$ denotes the space of compact linear operators on H , and for the symbol $N(\tau, L'')$ the corresponding eigenvalues of the operator L'' does not exceed τ , we have (see [7] Chapter 7):

$$N(\tau, L'') \sim \text{const} \times \tau^{1/2}, \quad \tau \rightarrow +\infty. \quad (5.5)$$

Since L'' is a positive selfadjoint operator, we have

$$|(L'' + |\lambda|I)|_2^{-1/2} \leq M|\lambda|^{-1/4} \quad (0 \neq \lambda \in S) \quad (5.6)$$

and

$$(L'' + |\lambda|I)^{-1/2} \in \sigma_p(H), \quad \forall p > 1. \quad (5.7)$$

For the definition of $\sigma_p(H)$ and the norm $|\cdot|_p$ refer to [7]. According to (5.1)-(5.4), we have

$$L^{-1}(\lambda) = Z^{-1}(A - \lambda I)^{-1} = (L'' + |\lambda|I)^{1/2} K_1(\lambda) (L'' + |\lambda|I)^{-1/2} K_2(\lambda),$$

in which the functions $K_1(\lambda), K_2(\lambda)$ satisfy the following inequalities

$$\|K_1(\lambda)\| \leq M, \quad \|K_2(\lambda)\| \leq M, \quad (|\lambda| \geq 1, \lambda \in S).$$

From these inequalities, and from (5.5), (5.6) we have $(A - \lambda I)^{-1} \in \sigma_q(H)$ for all $q > 1/2$ and

$$|(A - \lambda I)^{-1}|_1 \leq M_1 |(L'' + |\lambda|I)^{-1/2}|_2^2 \leq M_2 |\lambda|^{-1/2} \quad (|\lambda| \geq 1, \lambda \in S).$$

These estimates are fulfilled when the operator A , satisfies the condition (2.3) in Theorem 2.1 (as we saw in the proof of the first step of Theorem 2.1). Analogously, when the operator A does not satisfy (2.3), these estimates are again satisfied for the operator A (this was shown in the proof of Theorem 3.1). Therefore, the order of the resolvent of the operator does not exceed $1/2$. From the closed extension of the domain of the operator P to the closed set, we note that $D(P)$ is a closed set in the compact space $\mathcal{H}_\ell$, which this implies that the imbedding $\hat{\mathcal{H}}_\ell \subset \mathcal{H}_\ell$ is compact (for the definition see [1]). Consequently, the operator A has a discrete countable spectrum (because, compact operators have a discrete countable spectrum). Now, according to Theorems 6.4.1 and 6.4.2 of [1], we obtain the proof of Theorem 5.1, by helping of the system of root vector functions of the operator A in $\mathcal{H}_\ell$, and for the summability of the Fourier series of elements $f \in \mathcal{H}_\ell$, by applying the Abelian methods with brackets. Indeed, from (5.7), (5.8), and by applying Theorem 6.4.1 of [1], we can establish that the system of root vector functions of the operator A is completed in $\mathcal{H}_\ell$. Analogously, by applying Theorem 6.4.2 of [1], we can prove that the Fourier series of any vector function $f \in \mathcal{H}_\ell$, by the the system of root vector functions of the operator A , is summable to f by the Abelian methods with brackets of order $\gamma' = 1/2 + \varepsilon$ (for sufficiently small $\varepsilon > 0$). Moreover, we now get the estimate $N_0(t) \leq M(1+t)^{1/2}$, $t > 0$, in which $N_0(t) = \text{card}\{j : |\lambda_j| \leq t\}$, and $\lambda_1, \lambda_2, \dots$ is the sequence of eigenvalues of the operator A . Since for sufficiently large absolute values of $\lambda \in S$, the operator $A - \lambda I$

has continuous inverse, then for sufficiently large $j \geq j_0$, we have $\lambda_j \notin S$. Without loss of generality we can supposed that

$$\{z \in \mathbb{C} : |\arg z| \leq \theta\} \subset \{0\} \cup \text{Int}(S), \quad 0 < \theta < \frac{\pi}{8}$$

where

$$|\lambda_j + t| \geq C_\theta(t + |\lambda_j|), \quad \forall t > 1, j \geq j_0. \quad (5.8)$$

Since the sum of the eigenvalues of the kernel operator does not exceed its kernel norm (see [7]), therefore (see (5.7)), now, by (5.8), we have

$$\begin{aligned} N_0(t) &\leq \int_0^t dN_0(\tau) \leq 2t \int_0^t \frac{dN_0(\tau)}{\tau + t} \leq 2t \int_0^{+\infty} \frac{dN_0(\tau)}{\tau + t} \\ &= 2t \sum_{j=1}^{\infty} \frac{1}{|\lambda_j| + t} \leq 2t C_\theta^{-1} \sum_{j=t}^{\infty} \left| \frac{1}{\lambda_j + t} \right| \leq 2t C_\theta^{-1} |(A + tI)^{-1}|_1 \\ &\leq 2t C_\theta^{-1} (1 + t)^{-1/2} \leq M(1 + t)^{1/2}. \end{aligned}$$

This completes the proof of Theorem 5.1. $\square$

6. On the asymptotic distribution of eigenvalues of the differential operator P in H_ℓ . Let $P, h(t), q(t)$ and Φ be defined as in Section 1. Moreover, suppose that for each $t \in [0, 1]$ the matrix function $q(t)$ has ℓ -simple non-zero eigenvalues $\mu_1(t), \dots, \mu_\ell(t)$ in the complex plane. Let $\mu_j(t) \in C^2[0, 1], j = 1, \dots, \ell$. In view of the above $\Phi \subset \mathbb{C}$, we assume that the eigenvalues $\mu_j(t)$ ($1 \leq j \leq \ell$) are arranged in the complex plane in the form

$$\mu_1(t), \dots, \mu_\nu(t) \in \mathbb{R}_+, \mu_{\nu+1}(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi.$$

(For a more indepth explanation, we can say that for $j = 1, \dots, \nu$ the eigenvalues $\mu_j(t)$ lie on the positive real line $\mathbb{R}_+$, and for $j = \nu + 1, \dots, \ell$ the eigenvalues $\mu_j(t)$ lie outside of the closed angel Φ in the complex plane $\mathbb{C}$). Under the above conditions, we will have the following theorem.

Theorem 6.1. *Let the above assumptions be satisfied, then the distribution eigenvalues function $N(t) = \text{card}\{j : |\lambda_j| \leq t\}$ satisfies the following asymptotic formula:*

$$N(t) \sim \Lambda t^{1/2} \text{ as } t \rightarrow \infty, \quad \Lambda = \frac{1}{\pi} \sum_{j=1}^{\nu} \int_0^1 h^{-1/2}(t) \mu_j^{-1/2}(t) dt.$$

Proof. We know that for an arbitrary bounded operator T_2 and an arbitrary kernel operator T_1

$$\text{tr}(T_1 T_2) = \text{tr}(T_2 T_1), \quad |T_1 T_2|_1 \leq |T_1|_1 \|T_2\|$$

(e.g., see [7]). Using the above relations, and by the results (4.5)–(4.7), we get

$$\text{tr}((P - \lambda I)^{-1}) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda} = \text{tr}(\Gamma(\lambda)) + O(1) |\Gamma(\lambda)|_1 \|\mathcal{Y}(\lambda)\|,$$

$$\operatorname{tr}(\Gamma(\lambda)) = \operatorname{tr}(UB(\lambda)U^{-1}) = \operatorname{tr}(B(\lambda)) = \sum_{j=1}^{\ell} \operatorname{tr}((P_j - \lambda I)^{-1}),$$

$$|\Gamma(\lambda)|_1 \leq \|U\| \|U^{-1}\| |B(\lambda)|_1 \leq M |\lambda|^{-1} \quad (\lambda \in \Phi, |\lambda| > C_0).$$

From these last relations we see that

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda} = \sum_{j=1}^{\ell} \operatorname{tr}((P_j - \lambda I)^{-1}) + O(|\lambda|^{-1}), \quad \lambda \rightarrow +\infty, \quad \lambda \in \Phi. \quad (6.1)$$

Let $\lambda_1, \lambda_2, \dots$, and $\lambda_{j,1}, \lambda_{j,2}, \dots$ be the sequences of the eigenvalues of the operators P and P_j ($j = \nu + 1, \dots, \ell$), respectively. By (6.1),

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j - \lambda} = \sum_{k=1}^{\ell} \sum_{j=1}^{\infty} \frac{1}{\lambda_{j,k} - \lambda} + O(|\lambda|^{-1}), \quad \lambda \rightarrow +\infty, \quad \lambda \in \Phi. \quad (6.2)$$

We take $\psi = \{z \in \mathbb{C} : |\arg z| \leq \psi\}$, where $\psi \in (0, \varphi)$ (we know from Section 1, that $\varphi \in (0, \pi)$) is a fixed number). We can also take the index j_0 , such that for $j \geq j_0$, the following inequality is true:

$$|\arg \lambda_{j,k}| > \varphi, \quad k = \nu + 1, \dots, \ell, \quad |\arg \lambda_j| < \psi. \quad (6.3)$$

Moreover, also let the properties of the eigenvalues $\mu_j(t)$ above, satisfy the following conditions for the eigenvalues $\lambda_{j,k}$:

$$\lambda_{j,k} \in \mathbb{R}_+, \quad k = 1, \dots, \nu, \quad \lambda_{j,k} \in \mathbb{C} \setminus \Phi, \quad k = \nu + 1, \dots, \ell.$$

From these conditions we conclude that $\lambda_{j,k} > 0$ (for $j = 1, \dots, \nu$). With these considerations in mind, we can change the sum $\sum_{k=1}^{\ell}$ to the $\sum_{k=1}^{\nu}$ in (6.2). We then have

$$\sum_{j=j_0}^{\infty} \frac{1}{\lambda_j - \lambda} = \sum_{k=1}^{\nu} \sum_{j=j_0}^{\infty} \frac{1}{\lambda_{j,k} - \lambda} + O(|\lambda|^{-1}), \quad \lambda \rightarrow +\infty, \quad \lambda \in \partial\Phi.$$

Multiplying both sides of the above equation by $(\lambda + t)^{-1}$, $t > 1$, using the contour integral method (see [5], chapter 4), and by (6.3), integrate with respect to $\lambda \in \partial\psi$, it follows that

$$\sum_{j=j_0}^{\infty} \frac{1}{\lambda + \lambda_j} = \sum_{k=1}^{\nu} \sum_{j=j_0}^{\infty} \frac{1}{\lambda + \lambda_{j,k}} + O(|\lambda|^{-1}), \quad \lambda \rightarrow +\infty.$$

(Remark that here, as above, by applying the contour integral method, we can change the negative-sign to the positive-sign in the denominator of the above relation). We now use the countable discreteness of the eigenvalues of the operator P (note that this property of the eigenvalues of P can be easily seen). Then, we can change the above series to the following integral

$$\int_0^{\infty} \frac{dN(\tau)}{t + \tau} = \sum_{k=1}^{\nu} \int_0^{\infty} \frac{dN_k(\tau)}{t + \tau} + O(t^{-1}), \quad t \rightarrow +\infty, \quad (6.4)$$

where

$$N(\tau) = \text{card}\{j : |\lambda_j| \leq \tau\}, \quad N_k(\tau) = \text{card}\{j : \lambda_{j,k} \leq \tau\}, \quad k = 1, \dots, \nu.$$

In view of the previous conditions concerning the functions $h(t)$, $\mu_k(t)$, and by applying [7, Chapter 7], we can derive the following formula for the functions $N_k(t)$, $k = 1, \dots, \nu$ as follows

$$N_k(t) \sim c_k t^{1/2}, \quad t \rightarrow +\infty$$

where

$$c_k = \frac{1}{\pi} \int_0^1 h^{-1/2}(t) \mu_k^{-1/2}(t) dt, \quad k = 1, \dots, \nu.$$

By (6.4), we have that

$$\int_0^\infty \frac{dN(\tau)}{t + \tau} \sim \sum_{k=1}^\nu \int_0^\infty \frac{dN_k(\tau)}{t + \tau}, \quad t \rightarrow +\infty,$$

Now, if for this last relation we apply the Shkalikov multi-ray Tauberian theorem (see [11]), that implies

$$N(t) \sim \sum_{k=1}^\nu N_k(t), \quad t \rightarrow +\infty$$

where $N_k(t) \sim c_k t^{1/2}$ and c_k as above. Consequently, the following asymptotic formula is valid:

$$N(t) \sim \Lambda t^{1/2}, \quad t \rightarrow +\infty$$

where

$$\Lambda = \sum_{j=1}^\nu c_k = \frac{1}{\pi} \sum_{j=1}^\nu \int_0^1 h^{-1/2}(t) \mu_k^{-1/2}(t) dt.$$

This completes the proof of Theorem 6.1. $\square$

7. Spectral asymptotic of differential operators for general boundary conditions.

Let the differential operator

$$P_0 u = -\frac{d}{dt}(h(t)R(t)\frac{du}{dt}), \quad D(P_0) = C_0^\infty(0, 1)^\ell$$

acting on space H_ℓ , suppose that $h(t)$, $R(t)$ be defined as in Section 1, but, in contrast to Section 1, here we consider another extended operator P_0 which, however, cannot be given by the help of Dirichlet-type boundary conditions. Let Φ be the same angle as in Theorem 6.1, such that the eigenvalues $\mu_j(t)$ of the matrix $R(t)$, satisfy $\mu_1(t), \dots, \mu_\nu(t) \in \mathbb{R}_+$, $\mu_{\nu+1}(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi$ where $1 \leq \nu \leq \ell$. Let $\mu_j(t) \in C^2[0, 1]$, $\forall t \in [0, 1]$, $j = 1, \dots, \ell$, also let $\mathcal{P}$ be a closed extension of the operator P_0 such that:

- 1) $S \subset \Phi$ be an arbitrary sector with origin at zero, such that $S \cap \mathbb{R}_+ = \emptyset$, and S contains a finite number of the eigenvalues of the operator $\mathcal{P}$.

- 2) There exists a number $\varepsilon > 1/2$, $\psi \in (0, \varphi)$, $c > 0$, such that the inverse operator $(\mathcal{P} - \lambda I)^{-1}$ exists and is continuous for $|\lambda| \geq c$, $\arg \lambda = \pm\psi$ and the following estimate is true

$$\|(\mathcal{P} - \lambda I)^{-1}\| \leq M|\lambda|^{-\varepsilon} \quad (|\lambda| \geq c, \arg \lambda = \pm\psi).$$

Here, as before, let the symbol $N(t)$ denotes the number of eigenvalues of the operator $\mathcal{P}$ located inside the angle Φ and not exceeding the number t . From the above conditions, we have the following theorem (similar to Theorem 6.1)

Theorem 7.1. *Let the above assumption be satisfied, then the distribution eigenvalues function $N(t) = \text{card}\{j : |\lambda_j| \leq t\}$ satisfies the following asymptotic formula:*

$$N(t) \sim ct^{1/2} \text{ as } t \rightarrow \infty, \quad c = \frac{1}{\pi} \sum_{j=1}^{\nu} \int_0^1 h^{-1/2}(t) \mu_j^{-1/2}(t) dt$$

Proof. Let $\mathcal{P}$ be the same operator as in Section 6. For sufficiently large absolute values $\lambda \in S$, let $\arg \lambda = \pm\psi$ and consider

$$\theta(\lambda) = (\mathcal{P} - \lambda I)^{-1}u - (P - \lambda I)^{-1}u, \quad u \in H_\ell, \quad |u|_{H_\ell} = 1.$$

Choose the orthonormal basis $W_1^{(\lambda)}, \dots, W_k^{(\lambda)}$ in the space of solutions of the equation

$$-\frac{d}{dt} \left(h(t) R(t) \frac{du(t)}{dt} \right) = \lambda u(t), \quad 0 < t < 1.$$

We know that $k \leq 2$ (see [8]), so that we have

$$\theta(\lambda) = \sum_{j=1}^k q_j(\lambda) W_j(\lambda), \quad q_j(\lambda) = (\theta(\lambda), W_j(\lambda)).$$

Therefore

$$\begin{aligned} |\text{tr}((\mathcal{P} - \lambda I)^{-1}) - \text{tr}((P - \lambda I)^{-1})| &\leq \sum_{j=1}^k |q_j(\lambda)| \leq 2|\theta(\lambda)|_{H_\ell} \\ &\leq 2(\|(\mathcal{P} - \lambda I)^{-1}\| + \|(P - \lambda I)^{-1}\|) \\ &\leq M(|\lambda|^{-1} + |\lambda|^{-\varepsilon}) \quad \lambda \rightarrow \infty, \quad \arg \lambda = \pm\psi. \end{aligned}$$

Hence, by applying the asymptotic formula for the function $\text{tr}((\mathcal{P} - \lambda I)^{-1})$, as in Section 6, the proof of Theorem 7.1 is completed. $\square$

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Résumé substantiel en français. Dans cet article, on étudie les propriétés spectrales d'un opérateur différentiel elliptique non-adjoint

$$(Pu)(t) = -\frac{d}{dt} \left(h(t)q(t) \frac{du(t)}{dt} \right),$$

défini sur l'espace $H_\ell = L^2(0, 1)^\ell$ sous des conditions frontières de type Dirichlet. Ici $h(t) \in C^1(0, 1)$, $q(t) \in C^2([0, 1], \text{End } \mathbb{C}^\ell)$.

On y aborde différentes questions telles la complétude des fonctions propres généralisées de l'opérateur P dans H_ℓ ; la sommabilité au sens des méthodes d'Abel avec crochets de la série de Fourier des éléments $f \in H_\ell$ représentés comme combinaison linéaire des fonctions propres généralisées de P ; un estimé de la norme de la résolvante de P dans différents cas ; la distribution asymptotique des valeurs propres de P dans le cas des conditions de type Dirichlet ainsi que pour d'autres types de conditions aux frontières.

Soit $\mathcal{H}_\ell$ l'espace de Sobolev $W_{2,h}^2(0, 1)^\ell$ pondéré par une fonction positive $h(t)$ des fonctions vectorielles $u(t) = (u_1(t), \dots, u_\ell(t))$ définies sur l'intervalle $(0, 1)$, muni de la norme (finie)

$$|u|_+ = \left(\int_0^1 h(t) \left| \frac{du(t)}{dt} \right|_{\mathbb{C}^\ell}^2 dt + \int_0^1 |u(t)|_{\mathbb{C}^\ell}^2 dt \right)^{1/2}.$$

La fermeture de l'espace $C_0^\infty(0, 1)^\ell$ par rapport à la norme $|\cdot|_+$ est notée $\widehat{\mathcal{H}}_\ell$. Soit

$$(Pu)(t) = -\frac{d}{dt} \left(h(t)q(t) \frac{du(t)}{dt} \right),$$

un opérateur différentiel elliptique non auto-adjoint agissant sur l'espace $\mathcal{H}_\ell = L^2(0, 1)^\ell$ avec des conditions frontières de type Dirichlet. Les conditions que nous imposons sur la fonction $h(t)$ sont plus générales que celles que l'on retrouve dans [4], i.e., $h(t) \in C^1(0, 1)$ une fonction positive satisfaisant les inégalités :

$$c_1 t^\alpha (1-t)^\beta \leq |h(t)| \leq M, \quad (1.2)$$

$$|h'(t)| \leq M t^{\alpha/2-1+\varepsilon_1} (1-t)^{\beta/2-1+\varepsilon_2}, \quad (1.2')$$

où $\alpha \geq 0$, $\beta \geq 0$, avec $\varepsilon_1 = 0$ si $\alpha \neq 1$ et $\varepsilon_1 > 0$ si $\alpha = 1$; $\varepsilon_2 = 0$ si $\beta \neq 1$ et $\varepsilon_2 > 0$ si $\beta = 1$. Afin d'obtenir une extension fermée de l'opérateur P relativement à l'espace $\mathcal{H}_\ell$, on étend le domaine de cet opérateur au sous-ensemble fermé

$$D(P) \subset \left\{ u \in \widehat{\mathcal{H}}_\ell \cap W_{2,\text{loc}}^2(0, 1)^\ell : h^{1/2}u' \in H_\ell, (hqu')' \in H_\ell \right\}.$$

L'article est divisé en sept sections. À la section 2, on y présente le théorème 2.1 qui fournit un estimé de la norme de la résolvante de l'opérateur différentiel

$$(Av)(t) = -\frac{d}{dt} \left(h(t)\mu(t) \frac{dv(t)}{dt} \right),$$

agissant sur l'espace $H = L_2(0, 1)$. En supposant que

$$\mu(t) \in C^2[0, 1], \quad \mu(t) \in \mathbb{C} \setminus \Phi, \quad \forall t \in [0, 1], \quad (2.1)$$

$$|\arg\{\mu(t_1)\mu^{-1}(t_2)\}| \leq \frac{\pi}{8}, \quad (\forall t_1, t_2 \in [0, 1]), \quad (2.2)$$

on conclut que pour $\lambda \in \Phi$ tel que $|\lambda|$ est suffisamment grand, l'opérateur inverse $(A - \lambda I)^{-1}$ existe et est continu sur l'espace $H = L^2(0, 1)$, et que les estimés suivants sont valides :

$$\|(A - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1} \quad (\lambda \in \Phi, \quad |\lambda| > C_\Phi), \quad (2.3)$$

$$\|h^{1/2} \frac{d}{dt}(A - \lambda I)^{-1}\| \leq M'_\Phi |\lambda|^{-1/2} \quad (\lambda \in \Phi, \quad |\lambda| > C_\Phi). \quad (2.4)$$

Ici les quantités M_Φ , M'_Φ et $C_\Phi > 0$ sont des réels positifs dépendant de l'ensemble Φ où $\Phi = \{z \in \mathbb{C} : |\arg z| \leq \varphi\}$, $\varphi \in (0, \pi)$.

La section 3 est consacrée à la détermination d'estimés de la résolvante de l'opérateur A agissant sur $L^2(0, 1)$ dans un cas plus général. Dans cette section on obtient un nouveau théorème, le théorème 3.1, en laissant tomber l'hypothèse (2.2).

À la section 4, le théorème 4.1 fournit un estimé de la résolvante de l'opérateur différentiel P agissant sur H_ℓ et soumis à des conditions frontières de type Dirichlet. Dans ce théorème on suppose que pour chaque $t \in [0, 1]$ la fonction matricielle $q(t)$ possède ℓ valeurs propres non nulles distinctes et simples, $\mu_j(t) \in C^2[0, 1]$ ($1 \leq j \leq \ell$), où $\mu_1(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi$. De façon semblable au résultat obtenu au théorème 2.1 dans le cas $\ell = 1$, on a que pour $\lambda \in \Phi$, $|\lambda| > C_\Phi$, l'opérateur inverse $(P - \lambda I)^{-1}$ existe, est continu sur l'espace H_ℓ et satisfait l'estimé $\|(P - \lambda I)^{-1}\| \leq M_\Phi |\lambda|^{-1}$ où M_Φ et C_Φ sont des réels positifs qui dépendent de l'ensemble Φ . On remarque que les conditions imposées sur les valeurs propres $\mu_j(t)$ de la fonction matricielle $q(t)$ peuvent être représentées sous la forme

$$q(t) = U(t)\Lambda(t)U^{-1}(t), \quad \text{où } U(t), U^{-1}(t) \in C^2([0, 1], \text{End } \mathbb{C}^\ell)$$

et où $\Lambda(t) = \text{diag}\{\mu_1(t), \dots, \mu_\ell(t)\}$. En utilisant cette représentation de $q(t)$, le théorème 4.1 est obtenu comme une conséquence du théorème 2.1.

À la section 5, la complétude de l'ensemble des fonctions propres généralisées de l'opérateur P est étudié pour la méthode de sommabilité d'Abel. À la section 6, en supposant cette fois que les valeurs propres $\mu_j(t)$ sont telles que $\mu_1(t), \dots, \mu_\nu(t) \in \mathbb{R}_+$, $\mu_{\nu+1}(t), \dots, \mu_\ell(t) \in \mathbb{C} \setminus \Phi$, on établit au théorème 6.1 une formule sur la distribution asymptotique des valeurs propres $\lambda_1, \lambda_2, \dots$ de P comme suit :

$$N(t) \stackrel{\text{déf.}}{=} \text{card}\{j : |\lambda_j| \leq t\} \sim \Lambda t^{1/2}, \quad t \rightarrow +\infty$$

où

$$\Lambda = \frac{1}{\pi} \sum_{j=1}^{\nu} \int_0^1 h^{-1/2}(t) \mu_j^{-1/2}(t) dt.$$

Des résultats sur la distribution asymptotique des valeurs propres de l'opérateur différentiel P , cette fois soumis à d'autres types de conditions aux frontières, sont également obtenus à la section 7.

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