

OSCILLATORY SINGULAR INTEGRALS AND COMMUTATORS WITH ROUGH KERNELS

SHANZHEN LU AND HUOXIONG WU

RÉSUMÉ. Par la méthode de la décomposition en blocs pour une fonction noyau et moyennant des inégalités précises pour les polynômes, on obtient une condition nécessaire et suffisante, et aisément vérifiable, pour que soit bornées sur $L^p(\mathbb{R}^n)$ une classe d'intégrales singulières oscillatoires avec noyau. De plus, on obtient un critère pour qu'un commutateur d'ordre supérieur formé d'une intégrale singulière oscillatoire et d'une fonction BMO soit borné dans l'espace L^p avec poids.

ABSTRACT. By the method of block decomposition for kernel function and some precise inequalities concerning polynomials, the authors give a verifiable necessary and sufficient condition so that a class of oscillatory singular integrals with rough kernels are bounded on $L^p(\mathbb{R}^n)$ and establish the L^p -boundedness for these oscillatory singular integral operators. In addition, the authors also discuss the weighted L^p -boundedness for higher order commutators formed by the oscillatory singular integral and a radial BMO function and obtain a criterion on the weighted L^p -boundedness for these commutators.

1. Introduction and statement of results. Let $\mathbb{R}^n$, $n \geq 2$, be the n -dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. And let Ω be homogeneous of degree zero on $\mathbb{R}^n$.

We consider the oscillatory singular integral operator defined by

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy$$

and the corresponding maximal operator defined by

$$T^*f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} e^{iP(x,y)} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|,$$

where $P(x, y)$ is a real polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, and $h \in \text{BV}(\mathbb{R}_+)$ (i.e. $h(r)$ is a bounded variation function on $[0, \infty)$). In 1987, Ricci and Stein [RS] proved that if $h \equiv 1$ and $K(x) = \Omega(x/|x|)|x|^{-n}h(|x|) \in C^1(\mathbb{R}^n \setminus \{0\})$, then T is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Afterwards, Lu and Zhang [LZ] showed that T is bounded on $L^p(\mathbb{R}^n)$,

Reçu le 12 novembre 2002 et, sous forme définitive, le 16 décembre 2002.

$1 < p < \infty$, provided that $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$, and $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. And they also gave a criterion on the L^p -boundedness of T . In 1995, Jiang and Lu [JL1] improved the above results to the case, $\Omega \in L \log^+ L(S^{n-1})$ (see [FP, HP] for other improvements). In addition, for the maximal operator T^* , Lu and Zhang [LZ] proved that T^* is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if $\Omega \in L^q(S^{n-1})$ for $q > 1$.

In this paper, we will focus on the case, $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$. Here, $B_q^{0,0}(S^{n-1})$ denotes certain block spaces, which was introduced by Jiang and Lu (see [LTW] for the details of block spaces). In [KS], Keitoku and Sato showed that for any $q > 1$,

$$\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,0}(S^{n-1}),$$

which is a proper inclusion. And the relationship between $B_q^{0,0}(S^{n-1})$ and $L \log^+ L(S^{n-1})$ remains open. Therefore, it is a meaningful work to study the boundedness of singular integral operators with kernel $\Omega \in B_q^{0,0}(S^{n-1})$. Recently, Al-Hasan and Fan [AF], Hussain and Pan [HP] established some delicate results for certain singular integrals with kernels belonging to $B_q^{0,0}(S^{n-1})$. Meanwhile, Hussain and Pan [HP] also considered the following oscillatory singular integral operator with convolution polynomial phase,

$$\tilde{T}f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x-y)} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

where the phase P is a polynomial. Using the Fourier transform estimates and Littlewood-Paley theory, they proved that $\tilde{T}$ is bounded on $L^p(\mathbb{R}^n)$, if $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$. However, the method used in [HP] fails for the operator T with general polynomial phase $P(x, y)$, since T is not a convolution operator. We shall employ the method of rotation and some delicate techniques, which are very different from that used in [HP], to improve all of results mentioned above to the case, $B_q^{0,0}(S^{n-1})$ for $q > 1$.

On the other hand, for $\Omega \in L \log^+ L(S^{n-1})$, Ding and Lu [DL] studied the weighted L^p -boundedness on the higher order commutator T_b^m formed by T and a radial BMO function b , and gave a criterion on weighted L^p -boundedness for T_b^m . Another purpose of this paper is to study the weighted L^p -boundedness on T_b^m mentioned above, for $\Omega \in B_q^{0,0}(S^{n-1})$. Before stating our results, let us introduce some relevant definitions (see [DL] and [LTW]).

Definition 1. (i) A real-valued polynomial $P(x, y)$ is called non-trivial, if $P(x, y)$ does not take the form of $P_0(x) + P_1(y)$, where P_0 and P_1 are polynomials defined on $\mathbb{R}^n$.

(ii) We will say that the non-trivial polynomial $P(x, y)$ has property $\mathcal{P}$, if $P(x, y)$ satisfies

$$P(x, y) = P(x - t, y - t) + R_1(x, t) + R_2(y, t), \quad t \in \mathbb{R}^n, \quad (1.1)$$

where R_1 and R_2 are real polynomials.

Definition 2. Let $b(r) \in L_{\text{loc}}(\mathbb{R}_+)$. If

$$\|b\|_{*,+} = \sup_{I \subset \mathbb{R}_+} \frac{1}{|I|} \int_I |b(r) - b_I| dr < \infty,$$

then we say $b(r) \in \text{BMO}(\mathbb{R}_+)$, where $b_I = \int_I b(r) dr / |I|$.

Definition 3. Suppose that $\omega \geq 0$ and $\omega \in L_{\text{loc}}(\mathbb{R}_+)$. For $1 < p < \infty$, we say $\omega \in A_p(\mathbb{R}_+)$, if there is a constant $C > 0$ such that for any $I \subset \mathbb{R}_+$,

$$\left(\frac{1}{|I|} \int_I \omega(r) dr \right) \left(\frac{1}{|I|} \int_I \omega(r)^{-1/(p-1)} dr \right)^{p-1} \leq C < \infty.$$

Moreover, if there is a constant $C > 0$ such that $\omega^*(r) \leq C\omega(r)$ a.e. $r \in \mathbb{R}_+$, then we say $\omega \in A_1(\mathbb{R}_+)$, where ω^* denotes the Hardy-Littlewood maximal function of ω defined by

$$\omega^*(t) = \sup_{t \in I \subset \mathbb{R}_+} \frac{1}{|I|} \int_I \omega(r) dr.$$

For $1 < p < \infty$, we denote

$$\tilde{A}_p(\mathbb{R}_+) := \{ \omega : \omega \geq 0, \omega \in L_{\text{loc}}(\mathbb{R}_+) \text{ and } \omega^2 \in A_p(\mathbb{R}_+) \}.$$

Definition 4. A q -block on S^{n-1} is an $L^q(1 < q \leq \infty)$ function $b(x)$ that satisfies

- (i) $\text{supp } b \subset Q$;
- (ii) $\|b\|_q \leq |Q|^{(1/q)-1}$.

where $Q = S^{n-1} \cap \{y \in \mathbb{R}^n : |y - \xi| < \rho, \text{ for some } \xi \in S^{n-1} \text{ and } \rho \in (0, 1]\}$.

Definition 5. The block spaces $B_q^{0,0}$ on S^{n-1} are defined by

$$B_q^{0,0}(S^{n-1}) = \left\{ \Omega \in L^1(S^{n-1}) : \Omega(y') = \sum_s C_s b_s(y'), M_q^{0,0}(\{C_s\}) < \infty \right\},$$

where each C_s is a complex number, each b_s is a q -block supported in Q_s , and

$$M_q^{0,0}(\{C_s\}) = \sum_s |C_s| \left\{ 1 + \log^+ \frac{1}{|Q_s|} \right\}.$$

It should be pointed out that the method of block decomposition for functions was originated by Taibleson and Weiss [TW] in the study of the convergence of the Fourier series. Later on, many application of the block decomposition to harmonic analysis were discovered (see [MTW], [So], [Lo], [Lu]). For further background and information on the theory of spaces generated by blocks and its applications to harmonic analysis one can consult the book [LTW]. We remark that in this paper, we are very much motivated by the work of [LZ] and [HP], some ideas are from [RS], [JL1] and [DL]. Now let us formulate our main results as follows.

Theorem 1. Let $\bar{T}$ be the singular integral operator corresponding to T , i.e.

$$\bar{T}f(x) = p.v. \int_{\mathbb{R}^n} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy.$$

If $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$ and $h \in \text{BV}(\mathbb{R}_+)$, then for $1 < p < \infty$, the following two facts are equivalent:

- (i) If $P(x, y)$ is a non-trivial polynomial, then T is bounded on $L^p(\mathbb{R}^n)$.
- (ii) The truncated operator of $\bar{T}$,

$$\bar{T}_0 f(x) = p.v. \int_{|x-y|<1} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$.

Theorem 2. Let $1 < p < \infty$, $h \in \text{BV}(\mathbb{R}_+)$ and $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$. If the singular integral operator $\bar{T}$ is bounded on $L^p(\mathbb{R}^n)$, then so is T for any real-valued polynomial $P(x, y)$.

Theorem 3. Suppose that $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ and $h \in \text{BV}(\mathbb{R}_+)$. Then T is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, for any real-valued polynomial $P(x, y)$.

Theorem 4. Suppose that $P(x, y)$ is a real valued polynomial. If $h \in \text{BV}(\mathbb{R}_+)$, $\Omega \in B_q^{0,0}(S^{n-1})$ for $q > 1$ and $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, then the maximal operator T^* is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

Remark 1. Since $\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,0}(S^{n-1})$ for any $q > 1$, all of the above results is an essential improvement on the corresponding results in [LZ]. And Theorem 3 greatly improves the corresponding result of [HP].

Theorem 5. Let $1 < p < \infty$, Ω be homogeneous of degree zero, $h(|x|) \in \text{BV}(\mathbb{R}_+)$, $b(x) = b(|x|) \in \text{BMO}(\mathbb{R}_+)$. For any $m \in \mathbb{N} \cup \{0\}$ and any real-valued polynomial $P(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, define the higher order commutator T_b^m formed by T and b ,

$$T_b^m f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy.$$

If $\Omega(x) \in B_q^{0,0}(S^{n-1})$ for $q > 1$, and $\omega(x) = \omega(|x|) \in \tilde{A}_p(\mathbb{R}_+)$, then the following two statements are equivalent:

- (i) If $P(x, y)$ is a nontrivial polynomial, then T_b^m is bounded on $L^p(\omega)$.
- (ii) The truncated operator

$$\bar{T}_{b,0}^m f(x) = p.v. \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$. Here $L^p(\omega)$ denotes the weighted Lebesgue space $L^p(\mathbb{R}^n, \omega(x) dx)$.

Theorem 6. Let $1 < p < \infty$, Ω, h, b, m, ω and T_b^m as in Theorem 5. If $\bar{T}$ is bounded on $L^p(\omega)$, then for any $m \in \mathbb{N} \cup \{0\}$ and any real polynomial $P(x, y)$ on $\mathbb{R}^n \times \mathbb{R}^n$, T_b^m is also bounded on $L^p(\omega)$.

Theorem 7. Let $1 < p < \infty$, $\Omega(x) \in B_q^{0,0}(S^{n-1})$ for $q > 1$, homogeneous of degree zero and $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, $h(|x|) \in \text{BV}(\mathbb{R}_+)$, $b(x) = b(|x|) \in \text{BMO}(\mathbb{R}_+)$. Then for any $m \in \mathbb{N} \cup \{0\}$ and any real-valued polynomial $P(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, the higher order commutator T_b^m is bounded on $L^p(\mathbb{R}^n)$.

Throughout the rest of this paper, we always use the letter C to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Some inequalities for polynomials. As in [RS], our treatment will be based in part on some simple but useful inequalities concerning polynomials. Here, by a slight but crucial modification, we shall give some more precise estimates, which will play a crucial role in proving our main theorems, on polynomials than ones in [RS].

Proposition 1. Suppose $p(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ is a polynomial of degree d and let $\varepsilon < 1/2d$, then

$$\int_{|x| \leq 1} |p(x)|^{-\varepsilon} dx \leq CA^\varepsilon \left(\sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\varepsilon},$$

where C, A depend only on the dimension n , and, in the latter, the total degree of $p(x)$, but not on the coefficients $\{a_\alpha\}$.

Proof. Using the idea of Ricci and Stein in [RS], we verify the lemma as follows:

Case 1 When $n = 1$, write $p(x) = \sum_{j=0}^d a_j x^j$. Without loss of generality, we assume $\sum_{j=0}^d |a_j| = 1$. Since $\int_{-1}^1 |p(x)| dx$ and $\sum_{j=0}^d |a_j|$ give equivalent norms on the polynomials of degree d . Thus $\int_{-1}^1 |p(x)| dx \geq c > 0$, and there is an $x_0 \in [-1, 1]$ so that $|p(x_0)| \geq 2c$. Now write $p(x) = (p(x)/p(x_0))p(x_0)$ and split the roots $\{\alpha_j\}$ into two sets: those of the form $\beta_{j'}$ with $|\beta_{j'}| < 2$, and the remaining roots $\{\gamma_{j''}\}$ with $|\gamma_{j''}| \geq 2$. Hence

$$|p(x)|^{-\varepsilon} = \left| \prod_{j'} (x - \beta_{j'}) / \prod_{j'} (x_0 - \beta_{j'}) \right|^{-\varepsilon} \left| \prod_{j''} (x - \gamma_{j''}) / \prod_{j''} (x_0 - \gamma_{j''}) \right|^{-\varepsilon} |p(x_0)|^{-\varepsilon}.$$

Now, the term $\left| \prod_{j'} (x - \beta_{j'}) \right|^\varepsilon$ is bounded by $3^{d\varepsilon}$, since $|x_0| \leq 1$ and $|\beta_{j'}| \leq 2$. Next the term $\left| \prod_{j''} (x - \gamma_{j''}) / \prod_{j''} (x_0 - \gamma_{j''}) \right|^{-\varepsilon}$ is bounded $\prod_{j''} ((|\gamma_{j''}| + 1)/(|\gamma_{j''}| - 1))^\varepsilon \leq 3^{d\varepsilon}$, since $x, x_0 \in [-1, 1]$ and $|\gamma_{j''}| \geq 2$. Finally, $|p(x_0)|^{-\varepsilon} \leq c^{-\varepsilon}$. Altogether then $|p(x)|^{-\varepsilon} \leq A^\varepsilon \left| \prod_{j'} (x - \beta_{j'}) \right|^{-\varepsilon}$. Notice that $\varepsilon d < 1/2$ and by a d -fold use of Hölder's inequality, we get

$$\begin{aligned} \int_{-1}^1 |p(x)|^{-\varepsilon} dx &\leq \int_{-1}^1 A^\varepsilon \prod_{j'} |x - \beta_{j'}|^{-\varepsilon} dx \\ &\leq A^\varepsilon \prod_{j'} \left(\int_{-1}^1 |x - \beta_{j'}|^{-d\varepsilon} dx \right)^{1/d} \leq CA^\varepsilon. \end{aligned}$$

Case 2 When $n > 1$, by induction, it suffices to illustrate this in $n = 2$. Write $p(x, y) = \sum a_{k\ell} x^k y^\ell$, and assume again that $\sum |a_{k\ell}| = 1$. Choose the pair k_0, ℓ_0 so that $|a_{k_0\ell_0}|$ is maximal, then of course $|a_{k_0\ell_0}| > 0$. Now, by the case 1,

$$\int_{|x| \leq 1} |p(x, y)|^{-\varepsilon} dx \leq C A_1^\varepsilon \left(\sum_k \left| \sum_\ell a_{k,\ell} y^\ell \right| \right)^{-\varepsilon} \leq C A_1^\varepsilon \left| \sum_\ell a_{k_0,\ell} y^\ell \right|^{-\varepsilon}.$$

Integrate the resulting inequality in y to get

$$\int \int_{\substack{|x| \leq 1 \\ |y| \leq 1}} |p(x, y)|^{-\varepsilon} dx dy \leq C A_1^\varepsilon A_2^\varepsilon \left(\sum_\ell |a_{k_0,\ell}| \right)^{-\varepsilon} \leq C A_1^\varepsilon A_2^\varepsilon |a_{k_0,\ell_0}|^{-\varepsilon} \leq C A^\varepsilon,$$

which proves the proposition. $\square$

By Proposition 1, similar to Corollary 1 and 2 in [RS] (see [RS], p. 183), we have the following two propositions:

Proposition 2. Suppose $p(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ is a homogeneous polynomial of degree d in $\mathbb{R}^n$, with $\varepsilon < 1/2d$. Then

$$\int_{S^{n-1}} |p(x)|^{-\varepsilon} d\sigma(x') \leq C A^\varepsilon \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{-\varepsilon},$$

where C and A are as in Proposition 1.

Proposition 3. Suppose $p(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ is a (not necessarily homogeneous) polynomial of degree d in $\mathbb{R}^n$, with $\varepsilon < 1/2d$. Then

$$\sup_{y \in \mathbb{R}^n} \int_{|x| \leq 1} |p(x - y)|^{-\varepsilon} d\sigma(x') \leq C A^\varepsilon \left(\sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\varepsilon},$$

where C and A are as in Proposition 1.

3. Boundedness on oscillatory singular integrals. Let us begin with several lemmas. Set

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |\Omega(y')| |f(x - y)| dy.$$

Lemma 1. Let $1 < p < \infty$, $\Omega \in L^1(S^{n-1})$ and $h \in L^\infty(\mathbb{R}_+)$. If $\overline{T_0}$ and M_Ω are bounded on $L^p(\mathbb{R}^n)$, then so is T_0 for any real polynomial $P(x, y)$, where T_0 and $\overline{T_0}$ is the truncated operator of T and $\overline{T}$, respectively.

Decompose $P(x, y)$ as

$$\begin{aligned} P(x, y) &= \sum_{\substack{0 < |\alpha| \leq k' \\ 0 < |\beta| \leq \ell'}} a_{\alpha\beta} x^\alpha y^\beta + \sum a_{\alpha 0} x^\alpha + \sum a_{0\beta} y^\beta \\ &:= Q(x, y) + P_1(x) + P_2(y). \end{aligned}$$

Clearly, if $P(x, y)$ is non-trivial, then $0 < k' \leq k$ and $0 < \ell' \leq \ell$, where k and ℓ are the highest degree of x and y in $P(x, y)$ respectively.

Lemma 2. Let $1 < p < \infty$, $\Omega \in L^1(S^{n-1})$, $h \in L^\infty(\mathbb{R}_+)$ and $T_\infty = T - T_0$. If $\bar{T}$ and M_Ω are bounded on $L^p(\mathbb{R}^n)$, and T_∞ is bounded on $L^p(\mathbb{R}^n)$ for $Q(x, y)$, then T is bounded on $L^p(\mathbb{R}^n)$ for $P(x, y)$.

Lemma 3. Let $1 < p < \infty$, $\bar{T}_0$ be a truncated operator defined by

$$\bar{T}_0 f(x) = p.v. \int_{|x-y|<1} K(x, y) f(y) dy.$$

If the real polynomial $P(x, y)$ satisfies (1.1) in Definition 1 and the oscillatory integral T_0 defined by

$$T_0 f(x) = p.v. \int_{|x-y|<1} e^{iP(x, y)} K(x, y) f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$, then $\bar{T}_0$ is also bounded on $L^p(\mathbb{R}^n)$.

Lemmas 1-3 are the unweighted cases of Lemmas 3-5 in [JL1], respectively.

Lemma 4. Let $1 < p < \infty$, $h \in \text{BV}(\mathbb{R}_+)$ and $\Omega \in B_q^{0,0}(S^{n-1})$ with $q > 1$. Then $T_\infty = T - T_0$ is bounded on $L^p(\mathbb{R}^n)$, for any real non-trivial polynomial $P(x, y)$.

Proof. Since $\Omega \in B_q^{0,0}(S^{n-1})$, we have

$$\Omega(x') = \sum_s C_s b_s(x'),$$

which each b_s is a q -block, supported in Q_s and

$$\sum_s |C_s| \left(1 + \log^+ \frac{1}{|Q_s|} \right) < \infty.$$

We decompose T_∞ as follows:

$$\begin{aligned} T_\infty f(x) &= \int_{|x-y|\geq 1} e^{iP(x, y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \\ &= \sum_{j=1}^{\infty} \sum_s C_s \int_{2^{j-1} \leq |x-y| < 2^j} e^{iP(x, y)} \frac{b_s(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \\ &:= \sum_{j=1}^{\infty} \sum_s C_s T_{\infty, j}^s f(x). \end{aligned}$$

Thus

$$\|T_\infty f\|_p \leq \sum_s |C_s| \sum_{j=1}^{\infty} \|T_{\infty, j}^s f\|_p.$$

We shall show that

$$\sum_{j=1}^{\infty} \|T_{\infty, j}^s f\|_p \leq C \left(1 + \log^+ \frac{1}{|Q_s|} \right) \|f\|_p.$$

Write

$$P(x, y) = \sum_{\substack{|\alpha| \leq k \\ |\beta| \leq \ell}} a_{\alpha\beta} x^\alpha y^\beta,$$

where $k > 0$ and $\ell > 0$. Since $h \in \text{BV}(\mathbb{R}_+)$, using Proposition 2 and Proposition 3, by a method similar to that used in [LZ], we can prove that, for any $1 < \lambda \leq \infty$ and $0 < \theta < 1$,

$$\|T_{\infty,j}^s f\|_p \leq C A^\delta 2^{-j\theta\delta} \|b_s\|_{L^\lambda(S^{n-1})} \|f\|_p \quad (3.1)$$

holds uniformly in $\delta \in (0, 1]$ such that $\delta < \min\{k/2\ell, k/(k+\ell)\lambda'\}$, where $(1/\lambda) + (1/\lambda') = 1$, C, A depend only on n, p and, in the latter case, $\deg(P)$, but not on the coefficients of $P(x, y)$.

Now, we estimate $\|T_{\infty,j}^s f\|_p$ for each b_s . Notice that for each b_s , $\text{supp } b_s \subset Q_s$ and $\|b_s\|_{L^q(S^{n-1})} \leq |Q_s|^{(1-q)/q}$. If $|Q_s| \geq e^{q/(1-q)}$, we take $\lambda = q$, then

$$\|T_{\infty,j}^s f\|_p \leq C 2^{-j\theta\delta} \|b_s\|_{L^q(S^{n-1})} \|f\|_p \leq C 2^{-j\theta\delta} |Q_s|^{(1-q)/q} \|f\|_p \leq C 2^{-j\theta\delta} \|f\|_p;$$

if $|Q_s| < e^{q/(1-q)}$, we take $\lambda = \log |Q_s| / (1 + \log |Q_s|)$ and we can choose $\delta = \sigma/\lambda' < \min\{k/2\ell, k/(k+\ell)\lambda'\}$, where σ is a positive constant only depending on k and ℓ . Thus, by Hölder's inequality, we obtain

$$\begin{aligned} \|T_{\infty,j}^s f\|_p &\leq C A^{\sigma/\lambda'} 2^{-\theta\sigma j/\lambda'} \|b_s\|_{L^\lambda(S^{n-1})} \|f\|_p \\ &\leq C A^{\sigma/\log(|Q_s|^{-1})} 2^{-\theta\sigma j/\log(|Q_s|^{-1})} \|b_s\|_{L^q(S^{n-1})} |Q_s|^{(1/\lambda)-(1/q)} \|f\|_p \\ &\leq C A^{-\sigma/\log|Q_s|} 2^{\theta\sigma j/\log|Q_s|} |Q_s|^{(1/\lambda)-1} \|f\|_p \\ &\leq C A^{-\sigma/\log|Q_s|} 2^{\theta\sigma j/\log|Q_s|} |Q_s|^{1/\log|Q_s|} \|f\|_p \\ &\leq C A^{-\sigma/\log|Q_s|} 2^{\theta\sigma j/\log|Q_s|} \|f\|_p. \end{aligned}$$

Since $|Q_s| < e^{q/(1-q)}$, we have $-\log |Q_s| > q/(q-1) > 1$. Consequently, if $A \leq 1$, then $A^{-\sigma/\log|Q_s|} \leq 1$; if $A > 1$, then $A^{-\sigma/\log|Q_s|} < A^{\sigma(q-1)/q} = C$. Hence

$$\|T_{\infty,j}^s f\|_p \leq C 2^{\theta\sigma j/\log|Q_s|} \|f\|_p.$$

Therefore, for each b_s , if $|Q_s| \geq e^{q/(1-q)}$, then

$$\sum_{j=1}^{\infty} \|T_{\infty,j}^s f\|_p \leq C \sum_{j=1}^{\infty} 2^{-j\theta\delta} \|f\|_p \leq C \|f\|_p;$$

and if $|Q_s| < e^{q/(1-q)}$, then

$$\sum_{j=1}^{\infty} \|T_{\infty,j}^s f\|_p \leq C \sum_{j=1}^{\infty} 2^{\theta\sigma j/\log|Q_s|} \|f\|_p \leq C \log \frac{1}{|Q_s|} \|f\|_p.$$

This shows that

$$\|T_\infty f\|_p \leq \sum_s |C_s| \sum_{j=1}^{\infty} \|T_{\infty,j}^m f\|_p \leq C \sum_s |C_s| \left(1 + \log^+ \frac{1}{|Q_s|}\right) \|f\|_p \leq C \|f\|_p,$$

which proves the lemma. $\square$

Proof of Theorem 1. (i) $\Rightarrow$ (ii): To do this, we choose $P(x, y)$ such that $P(x, y)$ has property $\mathcal{P}$. Since T is bounded on $L^p(\mathbb{R}^n)$ and using Lemma 4, we know that T_∞ is bounded on $L^p(\mathbb{R}^n)$. So $T_0 = T - T_\infty$ is also bounded on $L^p(\mathbb{R}^n)$. Therefore, by Lemma 3 we obtain that $\overline{T}_0$ is bounded on $L^p(\mathbb{R}^n)$.

(ii) $\Rightarrow$ (i): Since $\overline{T}_0$ is bounded on $L^p(\mathbb{R}^n)$ and the well-known fact that M_Ω is bounded on $L^p(\mathbb{R}^n)$ for $\Omega \in L^1(S^{n-1})$. By Lemma 1, we get that T_0 is bounded on $L^p(\mathbb{R}^n)$. And by Lemma 4, we know that T_∞ is bounded on $L^p(\mathbb{R}^n)$. So $T = T_0 + T_\infty$ is bounded on $L^p(\mathbb{R}^n)$. This proves Theorem 1. $\square$

Proof of Theorem 2. If $P(x, y)$ is trivial, the conclusion is obvious. If $P(x, y)$ is non-trivial, then Theorem 2 is directly deduced from Lemma 2 and Lemma 4, since $B_q^{0,0}(S^{n-1}) \subset L^1(S^{n-1})$ and M_Ω is bounded on $L^p(\mathbb{R}^n)$ for $\Omega \in L^1(S^{n-1})$ (see [Du]). $\square$

Proof of Theorem 3. From [AF], we know that under the assumption of Theorem 3, the singular integral $\overline{T}$ is bounded on $L^p(\mathbb{R}^n)$. Thus Theorem 3 is an immediate consequence of Theorem 2. $\square$

Proof of Theorem 4. We shall carry out the argument by a double induction on the degree in x and y of the polynomial $P(x, y)$ as in the proof of Theorem 2 in [LZ]. Let k and ℓ be positive integers. Note that (see [HP])

$$\overline{T}^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| > \varepsilon} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right|$$

is bounded on $L^p(\mathbb{R}^n)$ under the assumption of Theorem 4. So T is also bounded on $L^p(\mathbb{R}^n)$ for real polynomials with the form $P_1(x) + P_2(y)$. Thus, we may assume that the conclusion of Theorem 4 holds for all real polynomials which are sums of monomials of degree $< k$ in x times monomials of any degree in y , together with monomials which are of degree k in x times monomials which are of degree $< \ell$ in y . Write

$$P(x, y) = \sum_{\substack{|\alpha|=k \\ |\beta|=\ell}} a_{\alpha\beta} x^\alpha y^\beta + R(x, y).$$

Since our conclusion is clearly invariant under dilation, we may assume that

$$\sum_{\substack{|\alpha|=k \\ |\beta|=\ell}} |a_{\alpha\beta}| = 1.$$

If $k = 0$, then $P(x, y)$ is a trivial polynomial. Thus Theorem 4 holds. For general $P(x, y)$, we have

$$\begin{aligned} T^* f(x) &\leq \sup_{0 < \varepsilon < 1} \left| \int_{|x-y| > \varepsilon} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right| \\ &\quad + \sup_{\varepsilon \geq 1} \left| \int_{|x-y| > \varepsilon} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right| \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{0 < \varepsilon < 1} \left| \int_{1 > |x-y| > \varepsilon} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right| \\
&\quad + \left| \int_{|x-y| \geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right| \\
&\quad + \sup_{\varepsilon > 1} \left| \int_{|x-y| > \varepsilon} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right| \\
&:= T_0^* f(x) + \left| \int_{|x-y| \geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy \right| + T_\infty^* f(x).
\end{aligned}$$

By Lemma 4, it suffices to prove that T_0^* and T_∞^* are (L^p, L^p) type operators. Notice that Lemma 1 of [LZ] for $\Omega \in L^1(S^{n-1})$ is valid (see [JL2], p. 399-400), by the method similar to proving (2.4) in [LZ] (see [LZ], p. 208-209), we can easily prove that T_0^* is a (L^p, L^p) type operator.

Now we estimate $T_\infty^* f$. For each fixed $\varepsilon > 1$, we have unique $J \in \mathbb{Z}^+$ such that $2^{J-1} \leq \varepsilon < 2^J$. Hence

$$\begin{aligned}
T_\infty^* f(x) &= \sup_{J \in \mathbb{Z}^+} \int_{2^{J-1} \leq |x-y| < 2^J} |h(|x-y|)| \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)| dy \\
&\quad + \sup_{J \in \mathbb{Z}^+} \sum_{j=J+1}^{\infty} \left| \int_{2^{j-1} \leq |x-y| < 2^j} e^{iP(x,y)} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right| \\
&\leq CM_\Omega f(x) + \sum_{j=1}^{\infty} \left| \int_{2^{j-1} \leq |x-y| < 2^j} e^{iP(x,y)} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|.
\end{aligned}$$

By the L^p -boundedness of M_Ω for $\Omega \in L^1(S^{n-1})$, and the method similar to proving Lemma 4, we get

$$\|T_\infty^* f\|_p \leq C \|f\|_p,$$

which proves Theorem 4. $\square$

4. Weighted boundedness on commutators. To begin with, we give some preliminary lemmas.

Lemma 5 [DL]. *If $1 < p < \infty$, then the weights in $\tilde{A}_p(\mathbb{R}_+)$ have the following properties:*

- (i) $\tilde{A}_p(\mathbb{R}_+) \subset A_p(\mathbb{R}_+)$.
- (ii) For any $\omega(r) \in \tilde{A}_p(\mathbb{R}_+)$, there are weights v_1, v_2 such that $v_i^2 \in A_1(\mathbb{R}_+)$ $i = 1, 2$ and $\omega = v_1 v_2^{1-p}$.
- (iii) For any $\omega(r) \in \tilde{A}_p(\mathbb{R}_+)$, there exists $\varepsilon > 0$ so that $\omega^{1+\varepsilon} \in \tilde{A}_p(\mathbb{R}_+)$.
- (iv) For any $\omega(r) \in \tilde{A}_p(\mathbb{R}_+)$, there exists $\varepsilon > 0$ so that $p-\varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}(\mathbb{R}_+)$.

Lemma 6 [Du]. Let $1 < p < \infty$, $\Omega(x) \in L^1(S^{n-1})$, homogeneous of degree zero. If $\omega \in \tilde{A}_p(\mathbb{R}_+)$, then

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |f(y)| dy$$

is bounded on $L^p(\omega)$ and $\|M_\Omega f\|_{p,\omega} \leq C \|\Omega\|_{L^1(S^{n-1})} \|f\|_{p,\omega}$.

Set

$$Gf(x) = p.v. \int_{\mathbb{R}^n} K(x,y) f(y) dy,$$

$$G_0 f(x) = p.v. \int_{|x-y|<1} K(x,y) f(y) dy.$$

Lemma 7 [JL1]. Let $1 \leq p < \infty$, $\omega(x) \geq 0$. If G_0 is bounded on $L^p(\omega)$, then the inequality

$$\left(\int_{|x-y|<\varepsilon} |G_0 f(x)|^p \omega(x) dx \right)^{1/p} \leq C_\omega \left(\int_{|y-t|<1+\varepsilon} |f(y)|^p \omega(y) dy \right)^{1/p} \quad (4.1)$$

holds for any $\varepsilon > 0$, where C_ε is independent of t and f .

Conversely, if (4.1) holds for certain $\varepsilon > 0$, then G_0 is bounded on $L^p(\omega)$.

Lemma 8 [DL]. Let $1 < p < \infty$, $h(r) \in L^\infty(\mathbb{R}_+)$, $\omega(x) = \omega(|x|) \in \tilde{A}_p(\mathbb{R}_+)$. If $\Omega \in L^1(S^{n-1})$, homogeneous of degree zero, and $\bar{T}$ (see Theorem 1 for the definition) is bounded on $L^p(\omega)$, then for any $b(x) = b(|x|) \in \text{BMO}(\mathbb{R}_+)$, $m \in \mathbb{N} \cup \{0\}$, and any real-valued polynomial $P(x,y)$, the operator

$$T_{b,0}^m f(x) = p.v. \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

Lemma 9. Let $1 < p < \infty$, Ω, h, b, m and ω be as in Theorem 5. Then for any real polynomial $P(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n$, the operator

$$T_{b,\infty}^m f(x) = \int_{|x-y|\geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

Proof. Since $\Omega(x') \in B_q^{0,0}(S^{n-1})$, we have $\Omega(x') = \sum_s C_s b_s(x')$, where each b_s is a q -block supported in Q_s , and $\sum_s |C_s| (1 + \log^+(1/|Q_s|)) < \infty$.

Decompose $T_{b,\infty}^m f$ as follows:

$$\begin{aligned} T_{b,\infty}^m f(x) &= \int_{|x-y|\geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\ &= \sum_{j=1}^{\infty} \sum_s C_s \int_{2^{j-1} \leq |x-y| < 2^j} e^{iP(x,y)} \frac{b_s(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\ &:= \sum_{j=1}^{\infty} \sum_s C_s T_{b,j,s}^m f(x). \end{aligned}$$

Thus

$$\|T_{b,\infty}^m f\|_{p,\omega} \leq \sum_s |C_s| \sum_{j=1}^{\infty} \|T_{b,j,s}^m f\|_{p,\omega}, \quad (4.2)$$

In what follows, we shall prove

$$\sum_{j=1}^{\infty} \|T_{b,j,s}^m f\|_{p,\omega} \leq C \left(1 + \log^+ \frac{1}{|Q_s|} \right) \|f\|_{p,\omega}. \quad (4.3)$$

To prove (4.3), we only need to prove that for each $j \in \mathbb{N}$, there exists $\eta > 0$ such that

$$\|T_{b,j,s}^m f\|_{p,\omega} \leq \begin{cases} C 2^{-\eta j} \|f\|_{p,\omega}, & \text{if } |Q_s| \geq e^{q/(1-q)}; \\ C 2^{\eta j / \log |Q_s|} \|f\|_{p,\omega}, & \text{if } |Q_s| < e^{q/(1-q)}; \end{cases} \quad (4.4)$$

where C, η are independent of j, s, f and Ω .

We shall carry out the proof of (4.4) by induction on m . Since $h \in \text{BV}(\mathbb{R}_+)$ and $P(x, y)$ is nontrivial real-valued polynomial, by a method similar to that in the proof of Lemma 4, we can prove that there exists an $\delta > 0$ such that

$$\|T_{b,j,s}^0 f\|_{p,\omega} \leq \begin{cases} C 2^{-\delta j} \|f\|_{p,\omega}, & \text{if } |Q_s| \geq e^{q/(1-q)}; \\ C 2^{\delta j / \log |Q_s|} \|f\|_{p,\omega}, & \text{if } |Q_s| < e^{q/(1-q)}. \end{cases} \quad (4.5)$$

Without loss of generality, we consider only the case, $|Q_s| < e^{q/(1-q)}$. On the other hand, we have

$$|T_{b,j,s}^0 f(x)| \leq \int_{2^{j-1} \leq |x-y| < 2^j} \frac{|b_s(x-y)|}{|x-y|^n} |h(|x-y|)| |f(y)| dy \leq C M_{b_s} f(x).$$

From Lemma 5 and Lemma 6, it follows that

$$\|T_{b,j,s}^0 f\|_{p,\omega^{1+\varepsilon}} \leq C \|M_{b_s} f\|_{p,\omega^{1+\varepsilon}} \leq C \|f\|_{p,\omega^{1+\varepsilon}}. \quad (4.6)$$

Combining (4.5) with (4.6), and using the Stein-Weiss theorem of interpolation with change of measure [BL], we get

$$\|T_{b,j,s}^0 f\|_{p,\omega} \leq C 2^{\eta_1 j / \log |Q_s|} \|f\|_{p,\omega},$$

where η_1 does not depend on j, s, f and Ω . This shows that (4.4) holds for $m = 0$.

For $m \geq 1$ we assume that (4.4) holds for $m-1$: for any $\varphi \in L^p(\omega)$ with $\omega \in \tilde{A}_p(\mathbb{R}_+)$, we have

$$\|T_{b,j,s}^{m-1} f\|_{p,\omega} \leq C 2^{\eta_{m-1} j / \log |Q_s|} \|f\|_{p,\omega}.$$

By $\omega \in \tilde{A}_p(\mathbb{R}_+)$ and Lemma 5, there exists $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}_p(\mathbb{R}_+)$. Therefore, for any $\varphi \in L^p(\omega^{1+\varepsilon})$,

$$\|T_{b,j,s}^{m-1} \varphi\|_{p,\omega^{1+\varepsilon}} \leq C 2^{\eta'_{m-1} j / \log |Q_s|} \|\varphi\|_{p,\omega^{1+\varepsilon}}. \quad (4.7)$$

Repeating the proof of Theorem 3 in [DL], we can obtain the following results: for any $\theta \in [0, 2\pi]$, $\varphi \in L^p(e^{pb(1+\varepsilon)\cos\theta/\varepsilon})$,

$$\|T_{b,j,s}^{m-1} \varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}} \leq C 2^{\eta''_{m-1} j / \log |Q_s|} \|\varphi\|_{p,e^{pb(1+\varepsilon)\cos\theta/\varepsilon}}, \quad (4.8)$$

where C and η''_{m-1} depend only on p, b and ω , but not on j, s and θ . By interpolating with change measure between (4.7) and (4.8), we see that for any $\theta \in [0, 2\pi]$, $\varphi \in L^p(\omega e^{pb \cos \theta})$,

$$\|T_{b,j,s}^{m-1} \varphi\|_{p, e^{pb \cos \theta}} \leq C 2^{\sigma j / \log |Q_s|} \|\varphi\|_{p, e^{pb \cos \theta}}, \quad (4.9)$$

where C and σ are independent of j, k and θ . Moreover, if we let $g_\theta(x) := f(x) e^{-b(x) e^{i\theta}}$, then it is easy check that for any $\theta \in [0, 2\pi]$,

$$g_\theta \in L^p(\omega e^{pb \cos \theta}) \text{ and } \|g_\theta\|_{p, \omega e^{pb \cos \theta}} = \|f\|_{p, \omega}. \quad (4.10)$$

For simplicity, we denote

$$K_{m-1}(x, y) = e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^{m-1} \chi_{2^{j-1} \leq |x-y| < 2^j}(x-y).$$

So, by the above notation and the following equality (see (3.5) in [DL])

$$b(x) - b(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}(b(x)-b(y))} e^{-i\theta} d\theta,$$

we have

$$\begin{aligned} T_{b,j,s}^m f(x) &= \int_{\mathbb{R}^n} K_{m-1}(x-y) [b(x) - b(y)] f(y) dy \\ &= \int_{\mathbb{R}^n} K_{m-1}(x-y) \left(\frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}(b(x)-b(y))} e^{-i\theta} d\theta \right) f(y) dy \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^n} K_{m-1}(x-y) f(y) e^{-b(y) e^{i\theta}} dy e^{b(x) e^{i\theta}} e^{-i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} T_{b,j,s}^{m-1}(g_\theta)(x) e^{b(x) e^{i\theta}} e^{-i\theta} d\theta. \end{aligned}$$

Hence, by Minkowski's inequality, we get

$$\begin{aligned} \|T_{b,j,s}^m f\|_{p, \omega} &\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{R}^n} \left| T_{b,j,s}^{m-1}(g_\theta)(x) e^{b(x) e^{i\theta}} \right|^p \omega(x) dx \right)^{1/p} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \|T_{b,j,s}^{m-1}(g_\theta)\|_{p, \omega e^{pb \cos \theta}} d\theta. \end{aligned}$$

From (4.9) and (4.10), it follows that

$$\|T_{b,j,s}^m f\|_{p, \omega} \leq \frac{1}{2\pi} \int_0^{2\pi} C 2^{\sigma j / \log |Q_s|} \|g_\theta\|_{p, \omega e^{pb \cos \theta}} d\theta = C 2^{\sigma j / \log |Q_s|} \|f\|_{p, \omega}.$$

This shows that (4.4) also holds for m .

Therefore, for any $m \in \mathbb{N} \cup \{0\}$, if $|Q_s| \geq e^{q/(1-q)}$, then

$$\sum_{j=1}^{\infty} \|T_{b,j,s}^m f\|_{p, \omega} \leq C \sum_{j=1}^{\infty} 2^{-\eta j} \|f\|_{p, \omega} \leq C \|f\|_{p, \omega};$$

if $|Q_s| < e^{q/(1-q)}$, then

$$\sum_{j=1}^{\infty} \|T_{b,j,s}^m f\|_{p,\omega} \leq C \sum_{j=1}^{\infty} 2^{\eta j / \log |Q_s|} \|f\|_{p,\omega} \leq C \log \frac{1}{|Q_s|} \|f\|_{p,\omega}.$$

Summing the above results, we get (4.3). Thus

$$\begin{aligned} \|T_{b,\infty}^m f\|_{p,\omega} &\leq \sum_s |C_s| \sum_{j=1}^{\infty} \|T_{b,j,s}^m f\|_{p,\omega} \\ &\leq C \sum_s |C_s| \left(1 + \log^+ \frac{1}{|Q_s|}\right) \|f\|_{p,\omega} \leq C \|f\|_{p,\omega} \end{aligned}$$

which proves Lemma 9. $\square$

Lemma 10 [DL]. *Let $1 < p < \infty$, $h(r) \in L^\infty(S^{n-1})$, $\omega(x) = \omega(|x|) \in \tilde{A}_p(\mathbb{R}_+)$, $b(x) = b(|x|) \in \text{BMO}(\mathbb{R}_+)$, $m \in \mathbb{N} \cup \{0\}$. If $\Omega \in L^1(S^{n-1})$ and the operator*

$$\bar{T}_{b,0}^m f(x) = p.v. \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$, then for any real-valued polynomial $P(x, y)$, the operator

$$T_{b,0}^m f(x) = p.v. \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

is bounded on $L^p(\omega)$.

Proof of Theorem 5. (ii) $\Rightarrow$ (i): By Lemma 9, it is known that $T_{b,\infty}^m$ is bounded on $L^p(\omega)$. And observe that $B_q^{0,0}(S^{n-1}) \subset L^1(S^{n-1})$, by the $L^p(\omega)$ boundedness of $\bar{T}_{b,0}^m$ and Lemma 10, we see that $T_{b,0}^m$ is also bounded on $L^p(\omega)$. Thus, T_b^m is bounded on $L^p(\omega)$.

(i) $\Rightarrow$ (ii): To do this, as in the proof of Theorem 1, we choose $P(x, y)$ such that $P(x, y)$ has property $\mathcal{P}$, and decompose

$$\begin{aligned} T_b^m f(x) &= p.v. \int_{|x-y|<1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\ &\quad + \int_{|x-y|\geq 1} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy \\ &:= T_{b,0}^m f(x) + T_{b,\infty}^m f(y). \end{aligned}$$

From Lemma 9, it follows that $T_{b,\infty}^m$ is bounded on $L^p(\omega)$. Thus, $T_{b,0}^m$ is also bounded on $L^p(\omega)$.

Now we take $t \in \mathbb{R}^n$. For $|x - t| < 1$, we have

$$T_{b,0}^m f(x) = T_{b,0}^m [f(\cdot) \chi_{B(t,2)}(\cdot)](x).$$

Thus, by Lemma 7, we get

$$\left(\int_{|x-t|<1} |T_{b,0}^m f(x)|^p \omega(x) dx \right)^{1/p} \leq C \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p} \quad (4.11)$$

where C is independent of t and f . By (1.1), we write

$$\bar{T}_{b,0}^m f(x) = e^{-iR_1(x,t)} \int_{|x-y|<1} e^{iP(x,y)} J_m(x,y) f(y) e^{-iP(x-t,y-t)} e^{-iR_2(y,t)} dy,$$

where $t \in \mathbb{R}^n$,

$$J_m(x,y) = \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m.$$

Express $e^{-iP(x-t,y-t)}$ into the Taylor series

$$\begin{aligned} e^{-iP(x-t,y-t)} &= \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} [P(x-t, y-t)]^j \\ &= \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} \left\{ \sum_{\alpha, \beta} a_{\alpha, \beta} (x-t)^\alpha (y-t)^\beta \right\}^j \\ &= \sum_{j=0}^{\infty} \frac{(-i)^j}{j!} \sum_{\mu, \nu} b_{\mu, \nu} (x-t)^\mu (y-t)^\nu. \end{aligned}$$

Hence

$$\begin{aligned} &\left(\int_{|x-t|<1} |\bar{T}_{b,0}^m f(x)|^p \omega(x) dx \right)^{1/p} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu, \nu} |b_{\mu, \nu}| \left(\int_{|x-t|<1} \left| (x-t)^\mu \int_{|x-y|<1} e^{iP(x,y)} J_m(x,y) f(y) \right. \right. \\ &\quad \left. \left. \times e^{-iR_2(y,t)} (y-t)^\nu dy \right|^p \omega(x) dx \right)^{1/p} \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\mu, \nu} |b_{\mu, \nu}| |\xi^\mu| \left(\int_{|x-t|<1} \left| T_{b,0}^m \left[f(\cdot) e^{iR_2(\cdot, t)} (\cdot - t)^\nu \right] (x) \right|^p \omega(x) dx \right)^{1/p}, \end{aligned}$$

where $\xi = (1, 1, \dots, 1)$. By (4.1), we obtain

$$\begin{aligned} &\left(\int_{|x-t|<1} |\bar{T}_{b,0}^m f(x)|^p \omega(x) dx \right)^{1/p} \\ &\leq C \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\mu, \nu} |b_{\mu, \nu}| |\xi^\mu| \left(\int_{|y-t|<2} |f(y) (y-t)^\nu|^p \omega(y) dy \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\mu, \nu} |b_{\mu, \nu}| |\xi^{\mu}| |\eta^{\nu}| \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p} \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{j!} \left(\sum_{\alpha, \beta} |a_{\alpha, \beta}| |\xi^{\alpha} \eta^{\beta}| \right)^j \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p} \\
&\leq C e^{\sum_{\alpha, \beta} |a_{\alpha, \beta}| |\xi^{\alpha} \eta^{\beta}|} \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p} \\
&\leq C \left(\int_{|y-t|<2} |f(y)|^p \omega(y) dy \right)^{1/p},
\end{aligned}$$

where $\eta = (2, 2, \dots, 2)$. By Lemma 7, this implies (ii) and proves Theorem 5. $\square$

Proofs of Theorem 6 and Theorem 7. Since $B_q^{0,0}(S^{n-1}) \subset L^1(S^{n-1})$, Theorem 6 can be directly deduced from Lemma 8 and Lemma 9. And Theorem 7 immediately follows from Theorem 6 and the known result that $\bar{T}$ is bounded on $L^p(\mathbb{R}^n)$ (see [AF]). $\square$

Acknowledgments. This research was supported by the National 973 Project and SEDF of China.

Résumé substantiel en français. Soient S^{n-1} la sphère de rayon 1 dans $\mathbb{R}^n$ ($n \geq 2$) muni de la mesure normalisée de Lebesgue, notée $d\sigma = d\sigma(x')$ et Ω homogène de degré zéro sur $\mathbb{R}^n$. On considère l'opérateur défini par l'intégrale singulière oscillatoire

$$Tf(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) f(y) dy$$

et l'opérateur maximal associé

$$T^*f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} e^{iP(x,y)} h(|x-y|) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|,$$

où $P(x, y)$ est un polynôme réel sur $\mathbb{R}^n \times \mathbb{R}^n$, et $h \in BV(\mathbb{R}_+)$ (c'est-à-dire, $h(r)$ est une fonction à variation bornée sur $[0, \infty)$). L'opérateur T a été étudié pour la première fois dans [RS]. Les résultats qui s'y trouvent ont ensuite été généralisés. (Voir par exemple, [LZ], [JL1], [FP] et [HP].) De plus, il a été démontré par Lu et Zhang [LZ] que T^* est borné sur $L^p(\mathbb{R}^n)$ ($0 < p < \infty$) lorsque $\Omega \in L^q(S^{n-1})$ pour $q > 1$.

Nous considérons ici le cas $\Omega \in B_q^{0,0}(S^{n-1})$ pour $q > 1$, où $B_q^{0,0}(S^{n-1})$ désigne une classe d'espaces blocs introduite pour la première fois par Jiang et Lu sans l'étude des intégrales singulières bornées. (Voir [LTW].) Cet espace est défini comme suit. Un q -bloc sur S^{n-1} est une fonction $b(x)$ dans L^q ($1 < q \leq \infty$) telle que

- (i) $\text{supp } b \subset Q$;
- (ii) $\|b\|_q \leq |Q|^{(1-q)/q}$;

où

$$Q = S^{n-1} \cap \{y \in \mathbb{R}^n : |y - \xi| < \rho \text{ pour un } \xi \in S^{n-1} \text{ et } \rho \in (0, 1]\}.$$

Les espaces blocs $B_q^{0,0}$ sur S^{n-1} sont alors définis par

$$B_q^{0,0}(S^{n-1}) = \left\{ \Omega \in L^1(S^{n-1}) : \Omega(y') = \sum_s C_s b_s(y'), M_q^{0,0}(\{C_s\}) < \infty \right\},$$

où chaque C_s est un nombre complexe, chaque b_s est un q -bloc avec support dans Q_s , et

$$M_q^{0,0}(\{C_s\}) = \sum_s |C_s| \left\{ 1 + \log^+ \frac{1}{|Q_s|} \right\}.$$

Il a été démontré par Keitoku et Sato [KS] que, pour $q > 1$ quelconque,

$$\bigcup_{r>1} L^r(S^{n-1}) \subset B_q^{0,0}(S^{n-1})$$

où l'inclusion est stricte. La relation entre $B_q^{0,0}(S^{n-1})$ et $L \log^+ L(S^{n-1})$ demeure inconnue. Récemment, Hussain et Pan [HP] ont étudié l'opérateur intégral singulier oscillatoire avec polynôme de convolution de phase, $\tilde{T}$. Utilisant des estimés propres aux transformées de Fourier et la théorie de Littlewood-Paley, ils ont montré que $\tilde{T}$ est borné sur $L^p(\mathbb{R}^n)$ lorsque $\Omega \in B_q^{0,0}(S^{n-1})$ pour $q > 1$. Cependant, la méthode utilisée dans [HP] échoue dans le cas d'un opérateur T avec polynôme de phase $P(x, y)$ général, puisque T n'est alors pas nécessairement un opérateur de convolution. Nous employons plutôt la méthode de rotation avec des techniques assez délicates qui diffèrent de celles utilisées dans [HP], afin d'étendre les résultats qui s'y trouvent au cas $B_q^{0,0}(S^{n-1})$ pour $q > 1$.

Lorsque $\Omega \in B_q^{0,0}(S^{n-1})$, on étudie les commutateurs d'ordre supérieur T_b^m formés de T et d'une fonction b qui est à la fois BMO et radiale. On donne, dans ce contexte, un critère pour que T_b^m soit borné dans L^p avec poids. Nos résultats principaux sont les suivants.

Théorème 1. Soit $\bar{T}$ l'intégrale singulière associée à T et définie par

$$\bar{T}f(x) = p.v. \int_{\mathbb{R}^n} h(|x - y|) \frac{\Omega(x - y)}{|x - y|^n} f(y) dy.$$

Si on a $\Omega \in B_q^{0,0}(S^{n-1})$ pour $q > 1$, $h \in \text{BV}(\mathbb{R}_+)$ et $1 < p < \infty$ alors les deux énoncés suivants sont équivalents :

- (i) Si $P(x, y)$ est un polynôme non-trivial, alors T est borné sur $L^p(\mathbb{R}^n)$.
- (ii) L'opérateur $\bar{T}_0$ défini à partir de $\bar{T}$ par

$$\bar{T}_0 f(x) = p.v. \int_{|x-y|<1} h(|x - y|) \frac{\Omega(x - y)}{|x - y|^n} f(y) dy$$

est borné sur $L^p(\mathbb{R}^n)$.

Théorème 2. Soient $1 < p < \infty$, $h \in \text{BV}(\mathbb{R}_+)$ et $\Omega \in B_q^{0,0}(S^{n-1})$ pour $q > 1$. Si $\bar{T}$ est borné sur $L^p(\mathbb{R}^n)$, alors T

Théorème 3. Soient $\Omega \in B_q^{0,0}(S^{n-1})$ pour $q > 1$, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ et $h \in \text{BV}(\mathbb{R}_+)$. Alors T est borné sur $L^p(\mathbb{R}^n)$, $1 < p < \infty$, pour tout polynôme réel $P(x, y)$.

Théorème 4. Soient $\Omega \in B_q^{0,0}(S^{n-1})$ pour $q > 1$, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ et $h \in \text{BV}(\mathbb{R}_+)$. Alors, pour tout polynôme réel $P(x, y)$, l'opérateur maximal T^* est borné sur $L^p(\mathbb{R}^n)$ pour $1 < p < \infty$.

Théorème 5. Soient $1 < p < \infty$, Ω homogène de degré zéro, $h(|x|) \in \text{BV}(\mathbb{R}_+)$ et $b(x) = b(|x|) \in \text{BMO}(\mathbb{R}_+)$. Pour $m \in \mathbb{N} \cup \{0\}$ quelconque et tout polynôme réel $P(x, y)$ défini sur $\mathbb{R}^n \times \mathbb{R}^n$, définissons le commutateur T_b^m formé de T et de b :

$$T_b^m f(x) = p.v. \int_{\mathbb{R}^n} e^{iP(x,y)} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy.$$

Si $\Omega(x) \in B_q^{0,0}(S^{n-1})$ pour $q > 1$ et $\omega(x) = \omega(|x|) \in \tilde{A}_p(\mathbb{R}_+)$ alors les deux énoncés suivants sont équivalents :

- (i) Si $P(x, y)$ est un polynôme non trivial, alors T_b^m est borné sur l'espace de Lebesgue $L^p(\omega) = L^p(\mathbb{R}^n, \omega(x)dx)$ avec poids $\omega(x)$.
- (ii) L'opérateur tronqué

$$\bar{T}_{b,0}^m f(x) = p.v. \int_{|x-y|<1} \frac{\Omega(x-y)}{|x-y|^n} h(|x-y|) [b(x) - b(y)]^m f(y) dy$$

est borné sur $L^p(\omega)$.

Théorème 6. Soient $1 < p < \infty$, Ω, h, b, m, ω et T_b^m comme au théorème 5. Si $\bar{T}$ est borné sur $L^p(\omega)$ alors, pour $m \in \mathbb{N} \cup \{0\}$ quelconque et tout polynôme réel $P(x, y)$ sur $\mathbb{R}^n \times \mathbb{R}^n$, on a que T_b^m est aussi borné sur $L^p(\omega)$.

Théorème 7. Soient $1 < p < \infty$, $\Omega(x) \in B_q^{0,0}(S^{n-1})$ pour $q > 1$ homogène de degré zéro, $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$, $h(|x|) \in \text{BV}(\mathbb{R}_+)$ et $b(x) = b(|x|) \in \text{BMO}(\mathbb{R}_+)$. Alors, pour $m \in \mathbb{N} \cup \{0\}$ quelconque et tout polynôme réel $P(x, y)$ sur $\mathbb{R}^n \times \mathbb{R}^n$, on a que le commutateur T_b^m d'ordre supérieur est borné sur $L^p(\mathbb{R}^n)$.

On obtient ces résultats à partir d'inégalités sur les polynômes, que nous énonçons maintenant.

Proposition 1. Soit $p(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ un polynôme de degré d et soit $\varepsilon < 1/2d$. Alors on a

$$\int_{|x| \leq 1} |p(x)|^{-\varepsilon} dx \leq CA^\varepsilon \left(\sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\varepsilon}$$

où C et A ne dépendent que de la dimension n et A dépend aussi du degré de $p(x)$ mais non des coefficients $\{a_\alpha\}$.

Proposition 2. Soit $p(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ un polynôme homogène de degré d dans $\mathbb{R}^n$ et soit $\varepsilon < 1/2d$. Alors on a

$$\int_{S^{n-1}} |p(x)|^{-\varepsilon} d\sigma(x) \leq CA^\varepsilon \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{-\varepsilon}$$

où C et A sont comme à la proposition 1.

Proposition 3. Soit $p(x) = \sum_{|\alpha| \leq d} a_\alpha x^\alpha$ un polynôme (pas nécessairement homogène) de degré d dans $\mathbb{R}^n$ et soit $\varepsilon < 1/2d$. Alors on a

$$\sup_{y \in \mathbb{R}^n} \int_{|x| \leq 1} |p(x-y)|^{-\varepsilon} d\sigma(x) \leq CA^\varepsilon \left(\sum_{|\alpha| \leq d} |a_\alpha| \right)^{-\varepsilon}$$

où C et A sont comme à la proposition 1.

Nous montrons ensuite les lemmes 1 à 4, dont découlent les théorèmes 1 à 4, et les lemmes 5 à 10 qui entraînent les théorèmes 5 à 7.

REFERENCES

- [AF] A. J. AL-Hasan and D. Fan, *A singular integral operator related to block spaces*, Hokkaido Math. J. **28** (1999), 285–299.
- [BL] J. Bergh and J. Löfström, *Interpolation Spaces, An introduction*, Springer-Verlag, Berlin-New York, 1976.
- [DL] Y. Ding and S. Z. Lu, *Weighted L^p -boundedness for higher order commutators of oscillatory singular integrals*, Tohoku Math. J. (2) **48** (1996), 437–449.
- [Du] J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. **336** (1993), 869–880.
- [HP] A. Q. Hussain and Y. Pan, *L^p estimates for singular integrals with kernels belonging to certain block spaces*; Preprint.
- [JL1] Y. S. Jiang and S. Z. Lu, *Oscillatory singular integrals with rough kernel*, Harmonic analysis in China, 135–145; Math. Appl., vol. 327, Kluwer Acad. Publ., Dordrecht, 1995.
- [JL2] Y. S. Jiang and S. Z. Lu, *A note on oscillatory singular integrals with rough kernel*, Beijing Shifan Daxue Xuebao **27** (1991), 398–402.
- [KS] M. Keitoku and E. Sato, *Block spaces on the unit sphere in $\mathbb{R}^n$* , Proc. Amer. Math. Soc. **119** (1993), 453–455.
- [Lo] R. L. Long, *The spaces generated by blocks*, Sci. Sinica Ser. A **27** (1984), 16–26.
- [Lu] S. Z. Lu, *On block decomposition of functions*, Sci. Sinica Ser. A **27** (1984), 585–596.
- [LTW] S. Z. Lu, M. H. Taibleson and G. Weiss, *Spaces Generated by Blocks*, Beijing Normal University Press, 1989.
- [LZ] S. Z. Lu and Y. Zhang, *Criterion on L^p -boundedness for a class of oscillatory singular integrals with rough kernels*, Rev. Mat. Iberoamericana **8** (1992), 201–219.
- [MTW] Y. Meyer, M. Taibleson and G. Weiss, *Some functional analytic properties of the space B_q generated by blocks*, Indiana Univ. Math. J. **34** (1985), 493–515.
- [RS] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I Oscillatory integrals*, J. Func. Anal. **73** (1987), 179–194.

- [So] F. Soria, *Characterizations of class of functions generated by blocks and associated Hardy Spaces*, Ind. Univ. Math. J. **34** (1985), 463–492.
- [TW] M. H. Taibleson and G. Weiss, *Certain function spaces connected with almost everywhere convergence of Fourier series*, Conference on harmonic analysis in honor of Antoni Zygmund, Vol.I (Chicago, Ill., 1981), 95–113; Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983.

S. LU

DEPARTMENT OF MATHEMATICS

BEIJING NORMAL UNIVERSITY

BEIJING 100875

PEOPLE'S REPUBLIC OF CHINA

E-MAIL: lusz@bnu.edu.cn

H. WU

DEPARTMENT OF MATHEMATICS

XIAMEN UNIVERSITY

XIAMEN 361005

PEOPLE'S REPUBLIC OF CHINA

E-MAIL: huoxwu@163.net