

ESTIMATES NEAR THE BOUNDARY FOR SOLUTIONS OF PDE AND INTERPOLATION INEQUALITIES

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RÉSUMÉ. On établit les estimations à priori dans les espaces de Besov et de Hölder pour les solutions de certains problèmes aux limites elliptiques singuliers du type Hermite et Tricomi. Une application est donnée à la régularité des solutions de certains problèmes non linéaires. On utilise des inégalités d'interpolation avec poids pour des fonctions L^p vectorielles à valeurs dans les espaces de Besov. Ce travail prolonge les résultats de [8], [9] et [16].

ABSTRACT. We give a priori-estimates in Besov and Hölder spaces for solutions of linear degenerate elliptic boundary value problems of Hermite and Tricomi type. An application of these estimates is given to obtain the regularity of some nonlinear problems solutions. We make use of some interpolation inequalities with weights for functions of vector-valued L^p -spaces built on Besov spaces $B_{p,p}^s$ in the sense of Bochner's integrals. This work extends the results of [8], [9] and [16].

1. Interpolation inequalities with weights.

1.1. Introduction and results. In [7] Caffarelli, Kohn and Nirenberg give a Sobolev interpolation inequality with weight. Such a result was generalized by Lin [17] by giving a sufficient and necessary condition for all parameters such that the following inequality holds

$$\| |t|^\gamma D_t^j u \|_{L^r(\mathbf{R}^n)} \leq \| |t|^\alpha D_t^m u \|_{L^p(\mathbf{R}^n)}^a \| |t|^\beta u \|_{L^q(\mathbf{R}^n)}^{1-a} \quad (1.1)$$

The main aim of this section is to give an interpolation inequality like (1.1) for vector-valued functions u taking their values in Besov-Peetre spaces $B_{p,q}^s$, Campanato spaces $\mathcal{L}^{p,\lambda}$ and John and Nirenberg space BMO. Thus, we get an extension of some interpolation inequalities interfering in a weighted Sobolev space and in the study of some degenerate elliptic boundary value problems (see [4, 10, 11, 12, 15] among others).

Another aim of this section will be to give an application of these inequalities to obtain the regularity of a tempered distribution u in function of its normal and weighted derivatives (Corollary 1.9 and 1.11). These results extend the interpolation inequalities of [10], [11], [12] and [15] to a more general frame.

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First, we give the hypotheses, we define the spaces and we state the results.

To define the spaces, we will use a Littlewood-Paley partition of unity. Denote $x = (t, x') \in \mathbf{R}^n \times \mathbf{R}^N$ and $\xi = (\tau, \xi')$ its dual variable. Let $\varphi \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$, $\varphi \geq 0$ and φ equal to 1 for $|\xi| \leq 1$, 0 for $|\xi| \geq 2$. Let us take $\theta(\xi) = \varphi(\xi) - \varphi(2\xi)$, we have $\text{supp } \theta \subset \{1/2 \leq |\xi| \leq 2\}$. For $j \in \mathbf{Z}$ we set

$$\dot{\Delta}_j u = \theta(2^{-j} D_x) u, \quad \dot{\Delta}'_j u = \theta(2^{-j} D_{x'}) u, \quad \dot{\Delta}''_j u = \theta(2^{-j} D_t) u$$

and denoting $\psi_j(\xi) = \sum_{k \leq j} \theta(2^{-k} \xi)$ ($= \varphi(2^{-j} \xi)$ for $\xi \neq 0$) we set

$$\dot{S}_j u = \psi_j(D_x) u, \quad \dot{S}'_j u = \psi_j(D_{x'}) u, \quad \dot{S}''_j u = \psi_j(D_t) u.$$

If $j \geq 1$ we set also

$$\begin{aligned} \Delta_j u &= \dot{\Delta}_j u, & \Delta'_j u &= \dot{\Delta}'_j u, & \Delta''_j u &= \dot{\Delta}''_j u \\ S_j u &= \varphi(2^{-j} D_x) u, & S'_j u &= \varphi(2^{-j} D_{x'}) u, & S''_j u &= \varphi(2^{-j} D_t) u \end{aligned}$$

and

$$S_0 u = \Delta_0 u = \varphi(D_x) u, \quad S'_0 u = \Delta'_0 u = \varphi(D_{x'}) u, \quad S''_0 u = \Delta''_0 u = \varphi(D_t) u.$$

If $u \in \mathcal{S}'(\mathbf{R}^{n+N})$ then the nonhomogeneous decomposition of u is $u = \sum_{k \geq 0} \Delta_k u = \sum_{k \geq 0} \mathcal{F}^{-1}[\theta_k \mathcal{F} u]$ where $\theta_0 = \varphi$ and $\theta_k(\xi) = \varphi(2^{-k} \xi) - \varphi(2^{-k+1} \xi)$ for $k \geq 1$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote respectively the Fourier and the inverse Fourier transforms. On the other hand, if $u \in \mathcal{S}'(\mathbf{R}^{n+N})$ with $0 \notin \text{supp } u$ then the homogeneous decomposition of u is $u = \sum_{k \in \mathbf{Z}} \dot{\Delta}_k u$.

Following H. Triebel [21] we give the definition of the spaces $B_{p,q}^s$.

Definition 1.1. Let $s \in \mathbf{R}$ and $1 \leq p, q \leq +\infty$. The space $B_{p,q}^s(\mathbf{R}^n \times \mathbf{R}^N)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^N)$ such that

$$\|u\|_{B_{p,q}^s(\mathbf{R}^n \times \mathbf{R}^N)} = \left\{ \sum_{k \geq 0} 2^{kqs} \|\Delta_k u\|_{L^p(\mathbf{R}^n \times \mathbf{R}^N)}^q \right\}^{1/q} < +\infty$$

with the usual modification if $q = +\infty$.

Notation. We denote $L^q(B_{p,q}^s)$ the space $L^q(\mathbf{R}^n; B_{p,q}^s(\mathbf{R}^N))$ of functions which are L^q with respect to the (normal) variable t and of class $B_{p,q}^s$ with respect to the (tangential) variable x' , equipped with the following norm

$$\|u\|_{L^q(B_{p,q}^s)} = \left\{ \sum_{k \geq 0} 2^{kqs} \|\Delta'_k u\|_{L^p(\mathbf{R}^n \times \mathbf{R}^N)}^q \right\}^{1/q}.$$

The spaces $B_{p,q}^s(\mathbf{R}^n \times \mathbf{R}^N)$ and $L^q(B_{p,q}^s)$ are Banach spaces.

Remark 1.2. It is classical that if $s > 0$, then $B_{\infty,\infty}^s$ coincides with the Hölder-Zygmund space $\mathcal{C}^s$, and if further $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$, then $B_{p,q}^s$ coincides with the Besov space $\Lambda_{p,q}^s$ (cf. [21]).

We recall the definition of the scale $\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n \times \mathbf{R}^N)$ given in [15] and [14].

Definition 1.3. Let $s \in \mathbf{R}$, $\lambda \geq 0$, $1 \leq p < +\infty$. The space $\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n \times \mathbf{R}^N)$ denotes the set of all tempered distributions $u \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^N)$ such that

$$\|u\|_{\mathcal{L}^{p,\lambda,s}(\mathbf{R}^n \times \mathbf{R}^N)} = \left\{ \sup_B \frac{1}{|B|^{\lambda/(n+N)}} \sum_{j \geq J^+} 2^{jps} \|\Delta_j u\|_{L^p(B)}^p \right\}^{1/p} < +\infty$$

where the supremum may be taken over all $J \in \mathbf{Z}$ and all balls (or cubes) B of $\mathbf{R}^n \times \mathbf{R}^N$ of radius (or length side) 2^{-J} ; $|B|$ denotes the measure of B and $J^+ = \max(J, 0)$.

Now we introduce the homogeneous version of the spaces $\mathcal{L}^{p,\lambda,s}$ which contain many well known spaces as special cases: John and Nirenberg space BMO and Campanato spaces $\mathcal{L}^{2,\lambda}$ modulo polynomials. For this, we recall the notation of [21, chapter 5]: $Z'(\mathbf{R}^n \times \mathbf{R}^N) \cong \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^N)/\mathcal{P}$ denotes the set of all tempered distributions modulo the set $\mathcal{P}$ of polynomials of $\mathbf{R}^n \times \mathbf{R}^N$ with complex coefficients.

Definition 1.4. Let $s \in \mathbf{R}$, $\lambda \geq 0$, $1 \leq p < +\infty$. The space $\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n \times \mathbf{R}^N)$ denotes the set of all $u \in Z'(\mathbf{R}^n \times \mathbf{R}^N)$ such that

$$\|u\|_{\dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^n \times \mathbf{R}^N)} = \left\{ \sup_B \frac{1}{|B|^{\lambda/(n+N)}} \sum_{j \geq J} 2^{jps} \|\dot{\Delta}_j u\|_{L^p(B)}^p \right\}^{1/p} < +\infty$$

where the supremum may be taken over all $J \in \mathbf{Z}$ and all balls (or cubes) B of $\mathbf{R}^n \times \mathbf{R}^N$ of radius (or length side) 2^{-J} ; $|B|$ denotes the measure of B .

The sets $\mathcal{L}^{p,\lambda,s}$ and $\dot{\mathcal{L}}^{p,\lambda,s}$ are Banach spaces equipped with the above norms. As for $B_{p,q}^s$ we introduce the following.

Notation. We denote $L^p(\mathcal{L}^{p,\lambda,s})$ (resp. $L^p(\dot{\mathcal{L}}^{p,\lambda,s})$) the space $L^p(\mathbf{R}^n; \mathcal{L}^{p,\lambda,s}(\mathbf{R}^N))$ (resp. $L^p(\mathbf{R}^n; \dot{\mathcal{L}}^{p,\lambda,s}(\mathbf{R}^N))$) which is equipped with the following norm

$$\begin{aligned} \|u\|_{L^p(\mathcal{L}^{p,\lambda,s})} &= \left\{ \sup_{B'} \frac{1}{|B'|^{\lambda/N}} \sum_{j \geq J^+} 2^{jps} \|\Delta'_j u\|_{L^p(\mathbf{R}^n \times B')}^p \right\}^{1/p} \\ (\text{resp. } \|u\|_{L^p(\dot{\mathcal{L}}^{p,\lambda,s})} &= \left\{ \sup_{B'} \frac{1}{|B'|^{\lambda/N}} \sum_{j \geq J} 2^{jps} \|\dot{\Delta}'_j u\|_{L^p(\mathbf{R}^n \times B')}^p \right\}^{1/p} \end{aligned}$$

where the supremum may be taken over all $J \in \mathbf{Z}$ and all balls (or cubes) B' of $\mathbf{R}^N$ of radius (or length side) 2^{-J} .

Remark 1.5. We gave in [13], [15] and [14] some properties of these spaces. For instance, the derivation D_x^α is a bounded operator from $\mathcal{L}^{p,\lambda,s}$ to $\mathcal{L}^{p,\lambda,s-|\alpha|}$ and $\dot{\mathcal{L}}^{2,\lambda,s}$ coincides algebraically and topologically with the Riesz-potential space $I^s(\mathcal{L}^{2,\lambda})$ built on Campanato space $\mathcal{L}^{2,\lambda}$ modulo polynomials for any $0 \leq \lambda < n+2$ and $s \in \mathbf{R}$. For $s = 0$ we obtain the Littlewood-Paley characterization of BMO (case $\lambda = n$) and Campanato spaces $\mathcal{L}^{2,\lambda}$ modulo polynomials.

Now we introduce the hypotheses. Let $j \in \mathbf{N}$, m, n and $N \in \mathbf{N} \setminus \{0\}$, further suppose $\alpha, \beta, \gamma, p, q, r$ and a are real numbers satisfying

$$p, q, r \geq 1, \quad \frac{j}{m} \leq a \leq 1 \quad \text{and} \quad m - j - \frac{n}{p} \notin \mathbf{N} \quad (1.2)$$

$$\frac{1}{p} + \frac{\alpha}{n} > 0, \quad \frac{1}{q} + \frac{\beta}{n} > 0 \quad \text{and} \quad \frac{1}{r} + \frac{\gamma}{n} > 0 \quad (1.3)$$

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q} \quad \text{and} \quad \gamma - j = a(\alpha - m) + (1-a)\beta. \quad (1.4)$$

Remark 1.6. From hypotheses (1.2), (1.3), (1.4) follow assumptions (0.1) to (0.9) of [17].

The main result of this section is as follows.

Theorem 1.7. *Under hypotheses (1.2) to (1.4), there is a constant $C > 0$ such that*

$$\begin{aligned} \| |t|^\gamma D_t^j u \|_{L^r(B_{r,r}^{2sa(1-a)+\gamma-j})} &\leq C \| |t|^\alpha D_t^m u \|_{L^p(B_{p,p}^{s(1-a)+\alpha-m})}^a \| |t|^\beta u \|_{L^q(B_{q,q}^{sa+\beta})}^{1-a} \\ \| |t|^\gamma D_t^j u \|_{L^r(\mathcal{L}^{r,\lambda,2sa(1-a)+\gamma-j})} &\leq C \| |t|^\alpha D_t^m u \|_{L^p(\mathcal{L}^{p,\lambda,s(1-a)+\alpha-m})}^a \| |t|^\beta u \|_{L^q(\mathcal{L}^{q,\lambda,sa+\beta})}^{1-a} \end{aligned}$$

hold for any $s \in \mathbf{R}$, $\lambda \geq 0$ and $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$.

The statement remains true if we replace $\mathcal{L}$ by the dotted space $\dot{\mathcal{L}}$.

Inequalities of Theorem 1.7 can be written in a different manner as compactness inequalities.

Theorem 1.8. *Let $s \in \mathbf{R}$ and $0 < a < 1$. Under hypotheses (1.2), (1.3) and (1.4), there exists a constant $C > 0$ such that for any $\varepsilon > 0$ and $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$, the inequalities*

$$\begin{aligned} \| |t|^\gamma D_t^j u \|_{L^r(B_{r,r}^{2sa(1-a)+\gamma-j})}^r &\leq C \left\{ \varepsilon \| |t|^\alpha D_t^m u \|_{L^p(B_{p,p}^{s(1-a)+\alpha-m})}^p \right. \\ &\quad \left. + \varepsilon^{-aq/(1-a)p} \| |t|^\beta u \|_{L^q(B_{q,q}^{sa+\beta})}^q \right\} \\ \| |t|^\gamma D_t^j u \|_{L^r(\mathcal{L}^{r,\lambda,2sa(1-a)+\gamma-j})}^r &\leq C \left\{ \varepsilon \| |t|^\alpha D_t^m u \|_{L^p(\mathcal{L}^{p,\lambda,s(1-a)+\alpha-m})}^p \right. \\ &\quad \left. + \varepsilon^{-aq/(1-a)p} \| |t|^\beta u \|_{L^q(\mathcal{L}^{q,\lambda,sa+\beta})}^q \right\} \end{aligned}$$

hold.

The statement remains true if we replace $\mathcal{L}$ by the dotted space $\dot{\mathcal{L}}$.

An application of the last result is given to the regularity of a tempered distribution u in function of the one of certain normal and weight derivatives of u . We assume that

$$p, q, r \geq 1, \quad 0 < a < 1 \quad \text{and} \quad m - \frac{n}{p} \notin \mathbf{N} \quad (1.5)$$

$$\frac{1}{p} + \frac{\alpha}{n} > 0 \text{ and } \frac{1}{q} + \frac{\beta}{n} > 0 \quad (1.6)$$

$$\frac{1}{r} = \frac{a}{p} + \frac{1-a}{q} \text{ and } a(\alpha - m) + (1-a)\beta = 0. \quad (1.7)$$

We have the following result.

Corollary 1.9. *Let s be a real number $> -m$. Assume that (1.5), (1.6), and (1.7) hold. If $u \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^N)$ is such that*

- (1) $|t|^\beta u \in L^q(B_{q,q}^{(s+m)/2(1-a)+\beta})$,
- (2) $|t|^\alpha D_t^m u \in L^p(B_{p,p}^{(s+m)/2a+\alpha-m})$, and
- (3) $D_t^m u \in B_{r,r}^s(\mathbf{R}^n \times \mathbf{R}^N)$

then $u \in B_{r,r}^{s+m}(\mathbf{R}^n \times \mathbf{R}^N)$ and there exists a constant $C > 0$ independent of u such that

$$\|u\|_{B_{r,r}^{s+m}(\mathbf{R}^n \times \mathbf{R}^N)} \leq C \left\{ \sup_{|\ell|=m} \|D_t^\ell u\|_{B_{r,r}^s(\mathbf{R}^n \times \mathbf{R}^N)} + \left\| |t|^\alpha D_t^m u \right\|_{L^p(B_{p,p}^{(s+m)/2a+\alpha-m})}^{p/r} + \left\| |t|^\beta u \right\|_{L^q(B_{q,q}^{(s+m)/2(1-a)+\beta})}^{q/r} \right\}$$

Example 1.10. If $n = 1$, $s > 0$, $p = q = r$, $\alpha = 0$ and $\beta = -m + \delta m$ we find again some results of [10] and [19]. For instance, in the case $m = \delta = 2$ (thus $\beta = 2$ and $a = 1/2$), if $u \in \mathcal{S}'(\mathbf{R}^{N+1})$ is such that $t^2 u \in L^p(\mathbf{R}; B_{p,p}^{s+4}(\mathbf{R}^N))$ and $D_t^2 u \in B_{p,p}^s(\mathbf{R}^{N+1})$ then $u \in B_{p,p}^{s+2}(\mathbf{R}^{N+1})$.

We can state an analogous result for the spaces $\mathcal{L}^{p,\lambda,s}$ and $\dot{\mathcal{L}}^{p,\lambda,s}$.

Corollary 1.11. *Assume that (1.5), (1.6), and (1.7) hold. If $u \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^N)$ is such that*

- (1) $|t|^\beta u \in L^q(\mathcal{L}^{q,\lambda,(s+m)/2(1-a)+\beta})$,
- (2) $|t|^\alpha D_t^m u \in L^p(\mathcal{L}^{p,\lambda,(s+m)/2a+\alpha-m})$, and
- (3) $D_t^k u \in \mathcal{L}^{r,\lambda+m,s}(\mathbf{R}^n \times \mathbf{R}^N)$ for any $|k| \leq m$

then $u \in \mathcal{L}^{r,\lambda,s+m}(\mathbf{R}^n \times \mathbf{R}^N)$. Moreover, there is a constant $C > 0$ independent of u such that

$$\|u\|_{\mathcal{L}^{r,\lambda,s+m}(\mathbf{R}^n \times \mathbf{R}^N)} \leq C \left\{ \sup_{|k| \leq m} \|D_t^k u\|_{\mathcal{L}^{r,\lambda+m,s}(\mathbf{R}^n \times \mathbf{R}^N)} + \left\| |t|^\alpha D_t^m u \right\|_{L^p(\mathcal{L}^{p,\lambda,(s+m)/2a+\alpha-m})}^{p/r} + \left\| |t|^\beta u \right\|_{L^q(\mathcal{L}^{q,\lambda,(s+m)/2(1-a)+\beta})}^{q/r} \right\}.$$

The statement remains true if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

1.2. Proofs. First of all, we recall the basic result of [17].

Theorem 1.12. *Under hypotheses (1.2), (1.3), and (1.4), there exists a constant $C > 0$ such that for any $u \in \mathcal{S}(\mathbf{R}^n)$ we get*

$$\| |t|^\gamma D_t^j u \|_{L^r(\mathbf{R}^n)} \leq C \| |t|^\alpha D_t^m u \|_{L^p(\mathbf{R}^n)}^a \| |t|^\beta u \|_{L^q(\mathbf{R}^n)}^{1-a}.$$

1.2.1. Proof of Theorem 1.7. Throughout this paper $C, C_M, \dots$ denote positive constants whose values may change from line to line.

Applying Theorem 1.12 to $u(t) = w(\lambda t)$ where λ is a parameter > 0 , we deduce

$$\begin{aligned} \lambda^{(j-\gamma)r} \| |t|^\gamma D_t^j w \|_{L^r(\mathbf{R}^n)}^r \\ \leq C \lambda^{(m-\alpha)ar} \| |t|^\alpha D_t^m w \|_{L^p(\mathbf{R}^n)}^{ar} \lambda^{-\beta(1-a)r} \| |t|^\beta w \|_{L^q(\mathbf{R}^n)}^{(1-a)r}. \end{aligned} \quad (1.8)$$

Let $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$. Replacing w by $\Delta'_k u(\cdot, x')$, $k \in \mathbf{N}$ and $x' \in \mathbf{R}^N$, and integrating over x' it follows

$$\begin{aligned} \lambda^{(j-\gamma)r} \int_{x' \in \mathbf{R}^N} \| |t|^\gamma D_t^j \Delta'_k u(\cdot, x') \|_{L^r(\mathbf{R}^n)}^r dx' \\ \leq C \int_{x' \in \mathbf{R}^N} \lambda^{(m-\alpha)ar} \| |t|^\alpha D_t^m \Delta'_k u(\cdot, x') \|_{L^p(\mathbf{R}^n)}^{ar} \lambda^{-\beta(1-a)r} \| |t|^\beta \Delta'_k u(\cdot, x') \|_{L^q(\mathbf{R}^n)}^{(1-a)r} dx'. \end{aligned}$$

Using Hölder's inequality and assumption (1.4) we obtain

$$\begin{aligned} \lambda^{(j-\gamma)r} \int_{x' \in \mathbf{R}^N} \| |t|^\gamma D_t^j \Delta'_k u(\cdot, x') \|_{L^r(\mathbf{R}^n)}^r dx' \\ \leq C \left\{ \int_{x' \in \mathbf{R}^N} \lambda^{(m-\alpha)p} \| |t|^\alpha D_t^m \Delta'_k u(\cdot, x') \|_{L^p(\mathbf{R}^n)}^p dx' \right\}^{ar/p} \\ \times \left\{ \int_{x' \in \mathbf{R}^N} \lambda^{-\beta q} \| |t|^\beta \Delta'_k u(\cdot, x') \|_{L^q(\mathbf{R}^n)}^q dx' \right\}^{(1-a)r/q}. \end{aligned}$$

Choosing $\lambda = 2^{-k}$, multiplying by $2^{2kr sa(1-a)}$ and noting that the operators Δ'_k and $|t|^\gamma D_t^j$ commute, we deduce

$$\begin{aligned} 2^{k(2sa(1-a)+\gamma-j)r} \| \Delta'_k(|t|^\gamma D_t^j u) \|_{L^r(\mathbf{R}^n \times \mathbf{R}^N)}^r \\ \leq C 2^{k(s(1-a)+\alpha-m)ar} \| \Delta'_k(|t|^\alpha D_t^m u) \|_{L^p(\mathbf{R}^n \times \mathbf{R}^N)}^{ar} \\ \times 2^{k(sa+\beta)(1-a)r} \| \Delta'_k(|t|^\beta u) \|_{L^q(\mathbf{R}^n \times \mathbf{R}^N)}^{(1-a)r}. \end{aligned}$$

Summing over $k \geq 0$ and using again Hölder's inequality

$$\begin{aligned} \sum_{k \geq 0} 2^{k(2sa(1-a)+\gamma-j)r} \| \Delta'_k(|t|^\gamma D_t^j u) \|_{L^r(\mathbf{R}^n \times \mathbf{R}^N)}^r \\ \leq C \left\{ \sum_{k \geq 0} 2^{k(s(1-a)+\alpha-m)p} \| \Delta'_k(|t|^\alpha D_t^m u) \|_{L^p(\mathbf{R}^n \times \mathbf{R}^N)}^p \right\}^{ar/p} \\ \times \left\{ \sum_{k \geq 0} 2^{k(sa+\beta)q} \| \Delta'_k(|t|^\beta u) \|_{L^q(\mathbf{R}^n \times \mathbf{R}^N)}^q \right\}^{(1-a)r/q}. \end{aligned}$$

The proof of the first part of Theorem 1.7 is complete. To prove the second part, we proceed in the same manner. Let $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$, and B' a ball of $\mathbf{R}^N$ centered at x'_0 and of radius 2^{-J} , $J \in \mathbf{Z}$. Replacing in (1.8) w by $\Delta'_k u(\cdot, x')$, $k \in \mathbf{N}$, integrating over $x' \in B'$ and using Hölder's inequality it follows

$$\begin{aligned} & \lambda^{(j-\gamma)r} \int_{x' \in B'} \| |t|^\gamma D_t^j \Delta'_k u(\cdot, x') \|_{L^r(\mathbf{R}^n)}^r dx' \\ & \leq C \left\{ \int_{x' \in B'} \lambda^{(m-\alpha)p} \| |t|^\alpha D_t^m \Delta'_k u(\cdot, x') \|_{L^p(\mathbf{R}^n)}^p dx' \right\}^{ar/p} \\ & \quad \times \left\{ \int_{x' \in B'} \lambda^{-\beta q} \| |t|^\beta \Delta'_k u(\cdot, x') \|_{L^q(\mathbf{R}^n)}^q dx' \right\}^{(1-a)r/q}. \end{aligned}$$

Replace λ by 2^{-k} and multiply by $2^{2kr sa(1-a)} / |B'|^{\lambda/N}$

$$\begin{aligned} & \frac{1}{|B'|^{\lambda/N}} 2^{k(2sa(1-a)+\gamma-j)r} \| \Delta'_k (|t|^\gamma D_t^j u) \|_{L^r(\mathbf{R}^n \times B')}^r \\ & \leq \frac{C}{|B'|^{\lambda/N}} 2^{k(s(1-a)+\alpha-m)ar} \| \Delta'_k (|t|^\alpha D_t^m u) \|_{L^p(\mathbf{R}^n \times B')}^{ar} \\ & \quad \times 2^{k(sa+\beta)(1-a)r} \| \Delta'_k (|t|^\beta u) \|_{L^q(\mathbf{R}^n \times B')}^{(1-a)r}. \end{aligned}$$

Hölder's inequality and assumption (1.4) give

$$\begin{aligned} & \frac{1}{|B'|^{\lambda/N}} \sum_{k \geq J^+} 2^{k(2sa(1-a)+\gamma-j)r} \| \Delta'_k (|t|^\gamma D_t^j u) \|_{L^r(\mathbf{R}^n \times B')}^r \\ & \leq C \left\{ \frac{1}{|B'|^{\lambda/N}} \sum_{k \geq J^+} 2^{k(s(1-a)+\alpha-m)p} \| \Delta'_k (|t|^\alpha D_t^m u) \|_{L^p(\mathbf{R}^n \times B')}^p \right\}^{ar/p} \\ & \quad \times \left\{ \frac{1}{|B'|^{\lambda/N}} \sum_{k \geq J^+} 2^{k(sa+\beta)q} \| \Delta'_k (|t|^\beta u) \|_{L^q(\mathbf{R}^n \times B')}^q \right\}^{(1-a)r/q} \\ & \leq C \| |t|^\alpha D_t^m u \|_{L^p(\mathcal{L}^{p,\lambda,s(1-a)+\alpha-m})}^{ar} \| |t|^\beta u \|_{L^q(\mathcal{L}^{q,\lambda,sa+\beta})}^{(1-a)r}. \end{aligned}$$

The proof of Theorem 1.7 is complete.

Theorem 1.8 is a consequence of Theorem 1.7 and the following classical inequality (see [1] for example):

$$xy \leq \frac{x^{p'}}{p'} + \frac{y^{q'}}{q'}$$

with

$$p' = \frac{p}{ar}, \quad q' = \frac{q}{(1-a)r},$$

$$x = \left(\varepsilon^{1/p} \| |t|^\alpha D_t^m u \|_{L^p(B_{p,p}^{s(1-a)+\alpha-m})} \right)^{ar} \quad \text{and} \quad y = \varepsilon^{-ar/p} \| |t|^\beta u \|_{L^q(B_{q,q}^{sa+\beta})}^{(1-a)r}.$$

1.2.2. Proof of Corollary 1.9. First of all, we recall the following classical lemma [3].

Lemma 1.13. *Let $\lambda > 0$, $1 \leq p \leq q \leq +\infty$. For any $u \in L^p(\mathbf{R}^n)$ with $\text{supp } \mathcal{F}u \subset \{|\xi| \leq \lambda\}$ and for all multi-index α , there exists $C_\alpha > 0$ such that*

$$\|D_x^\alpha u\|_{L^q(\mathbf{R}^n)} \leq C_\alpha \lambda^{|\alpha|+n(1/p-1/q)} \|u\|_{L^p(\mathbf{R}^n)}.$$

The following lemma is proved in [14].

Lemma 1.14. *Let R be a real > 1 ; there exists $C > 0$ such that for all $\lambda > 0$ and $1 \leq p \leq +\infty$, for any $u \in L^p(\mathbf{R}^n)$ with $\text{supp } \mathcal{F}u \subset \{\lambda \leq |\xi| \leq R\lambda\}$ and for any $k \in \mathbf{N}$ we get*

$$\|u\|_{L^p(\mathbf{R}^n)} \leq C^k \lambda^{-k} \sup_{|\alpha|=k} \|D_x^\alpha u\|_{L^p(\mathbf{R}^n)}.$$

Now we prove the following interpolation lemma.

Lemma 1.15. *Let s and δ be two real numbers satisfying $s > -m/\delta$, $\delta \geq 1$, and let $1 \leq r \leq +\infty$. If $u \in L^r(B_{r,r}^{s+m/\delta})$ and $D_t^m u \in B_{r,r}^{s-(\delta m-m)/\delta}(\mathbf{R}^n \times \mathbf{R}^N)$, then $u \in B_{r,r}^{s+m/\delta}(\mathbf{R}^n \times \mathbf{R}^N)$. Furthermore, there exists a constant $C > 0$ independent of u such that*

$$\|u\|_{B_{r,r}^{s+m/\delta}(\mathbf{R}^n \times \mathbf{R}^N)} \leq C \left\{ \sup_{|\ell|=m} \|D_t^\ell u\|_{B_{r,r}^{s-(\delta m-m)/\delta}(\mathbf{R}^n \times \mathbf{R}^N)} + \|u\|_{L^r(B_{r,r}^{s+m/\delta})} \right\}.$$

Proof. Let $k \in \mathbf{N}$; we write

$$\Delta_k u = \Delta_k \left(\sum_{i,j \geq 0} \Delta_i'' \Delta_j' u \right) = \Delta_k \left(\sum_{j=k-2}^{k+1} S_j'' \Delta_j' u + \sum_{j=k-2}^{k+1} \Delta_j'' S_{j-1}' u \right).$$

The operators Δ_k , S_j'' and S_{j-1}' are uniformly bounded on $L^r = L^r(\mathbf{R}^n \times \mathbf{R}^N)$, thus

$$\|\Delta_k u\|_{L^r} \leq C \left\{ \sum_{j=k-2}^{k+1} \|\Delta_j' u\|_{L^r} + \sum_{j=k-2}^{k+1} \|\Delta_k \Delta_j'' u\|_{L^r} \right\}.$$

Now Lemma 1.14 applied to the function $t \mapsto \Delta_k \Delta_j'' u(t, x')$, $x' \in \mathbf{R}^N$, gives for any $k \geq 3$

$$\|\Delta_k \Delta_j'' u\|_{L^r} \leq C 2^{-jm} \sup_{|\ell|=m} \|D_t^\ell \Delta_k \Delta_j'' u\|_{L^r} \leq C 2^{-jm} \sup_{|\ell|=m} \|\Delta_k D_t^\ell u\|_{L^r}.$$

Hence

$$\|\Delta_k u\|_{L^r} \leq C \left\{ \sum_{j=k-2}^{k+1} \|\Delta_j' u\|_{L^r} + \sum_{j=k-2}^{k+1} 2^{-jm} \sup_{|\ell|=m} \|\Delta_k D_t^\ell u\|_{L^r} \right\}$$

and thus for any $k \geq 3$

$$2^{k(s+m/\delta)} \|\Delta_k u\|_{L^r} \leq C \left\{ \sum_{j=k-2}^{k+1} 2^{j(s+m/\delta)} \|\Delta_j' u\|_{L^r} + 2^{k(s-m(\delta-1)/\delta)} \sup_{|\ell|=m} \|\Delta_k D_t^\ell u\|_{L^r} \right\}.$$

Since $s + m/\delta > 0$, we have $\|\Delta_k u\|_{L^r} \leq C\|u\|_{L^r} \leq C\|u\|_{L^r(B_{r,r}^{s+m/\delta})}$, for $0 \leq k \leq 3$. Therefore

$$\|u\|_{B_{r,r}^{s+m/\delta}(\mathbf{R}^n \times \mathbf{R}^N)} \leq C \left\{ \sup_{|\ell|=m} \|D_t^\ell u\|_{B_{r,r}^{s-(\delta m-m)/\delta}(\mathbf{R}^n \times \mathbf{R}^N)} + \|u\|_{L^r(B_{r,r}^{s+m/\delta})} \right\}.$$

The proof of Lemma 1.15 is complete. $\square$

Corollary 1.9 is an immediate consequence of Theorem 1.8 and the case $\delta = 1$ of Lemma 1.15. The following intermediate derivatives inequalities will be useful in the second part of this paper.

Lemma 1.16. *There exists a constant $C > 0$ such that for any $\varepsilon > 0$, $j, k \in \mathbf{N}$, $j + k \leq m$, $j < m$ and $-m + \delta k + j \geq 0$, $1 \leq p \leq +\infty$*

$$\begin{aligned} \|D_t^j u\|_{L^p(\mathbf{R}^+; B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^{N-1}))} \\ \leq C \left\{ \varepsilon \|D_t^m u\|_{L^p(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))} + C_\varepsilon \|u\|_{L^p(\mathbf{R}; B_{p,p}^{s+m/\delta}(\mathbf{R}^{N-1}))} \right\}, \end{aligned}$$

$$\begin{aligned} \|t^{-m+\delta k+j} D_t^j u\|_{L^p(\mathbf{R}; B_{p,p}^{s+k}(\mathbf{R}^{N-1}))} \\ \leq C \left\{ \varepsilon \|D_t^m u\|_{L^p(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))} + C_\varepsilon \|t^{-m+\delta m} u\|_{L^p(\mathbf{R}; B_{p,p}^{s+m}(\mathbf{R}^{N-1}))} \right\} \end{aligned}$$

and

$$\|D_t^j u\|_{B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^N)} \leq C \left\{ \varepsilon \|D_t^m u\|_{B_{p,p}^s(\mathbf{R}^N)} + C_\varepsilon \|u\|_{B_{p,p}^{s+m/\delta}(\mathbf{R}^N)} \right\}$$

hold for any $u \in C_0^\infty(\mathbf{R}^N)$.

Proof. It is classical that if v and $D_t^m v$ belong to $L^p(\mathbf{R})$ then $D_t^j v \in L^p(\mathbf{R})$ for $j = 0, \dots, m-1$ and there exists a constant $C > 0$ independent of v satisfying

$$\|D_t^j v\|_{L^p(\mathbf{R})} \leq C \left\{ \|D_t^m v\|_{L^p(\mathbf{R})} + \|v\|_{L^p(\mathbf{R})} \right\}.$$

Applying this inequality to $v(\varepsilon^{1/(m-j)} \lambda)$, $\lambda > 0$, we deduce

$$\|D_t^j v\|_{L^p(\mathbf{R})} \leq C \left\{ \varepsilon \lambda^{m-j} \|D_t^m v\|_{L^p(\mathbf{R})} + \varepsilon^{-j/(m-j)} \lambda^{-j} \|v\|_{L^p(\mathbf{R})} \right\}.$$

Put $v(t, x') = \Delta'_\ell u(t, x')$ [or $v(t, x') = \Delta_\ell u(t, x')$], $\ell \in \mathbf{N}$, $\lambda = 2^{-\ell/\delta}$, we integrate with respect to x' and next we multiply by $2^{\ell(s+(m-j)/\delta)}$ to get the first and the third inequality. The second inequality is obtained in the same way if we start with the following inequality

$$\|t^{-m+\delta k+j} D_t^j v\|_{L^p(\mathbf{R})} \leq C \left\{ \|D_t^m v\|_{L^p(\mathbf{R})} + \|t^{-m+\delta m} v\|_{L^p(\mathbf{R})} \right\}.$$

Lemma 1.16 is proved. $\square$

The following interpolation lemmas will be the key to estimate the normal derivatives for solutions of the degenerate PDE considered in the second part of this work.

Lemma 1.17. *Let s be a real number $0 < s < m$ and let $1 \leq p \leq +\infty$. If $u \in L^p(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))$ and $D_t^m u \in L^p(\mathbf{R}; B_{p,p}^{s-m}(\mathbf{R}^{N-1}))$, then $u \in B_{p,p}^s(\mathbf{R}^N)$. Further, there exists a constant $C > 0$ independent of u such that*

$$\|u\|_{B_{p,p}^s(\mathbf{R}^N)} \leq C \left\{ \|D_t^m u\|_{L^p(\mathbf{R}; B_{p,p}^{s-m}(\mathbf{R}^{N-1}))} + \|u\|_{L^p(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))} \right\}.$$

Proof. Let $\ell \in \mathbf{N}$; we write

$$\Delta_\ell u = \Delta_\ell \left(\sum_{k,j \geq 0} \Delta_k'' \Delta_j' u \right) = \Delta_\ell \left(\sum_{j=\ell-2}^{\ell+1} S_j'' \Delta_j' u + \sum_{k=\ell-2}^{\ell+1} \sum_{j=0}^{k-1} \Delta_k'' \Delta_j' u \right).$$

Lemma 1.14 applied to the function $t \mapsto \Delta_l \Delta_j' u(t, x')$, $x' \in \mathbf{R}^{N-1}$, gives for any $\ell \geq 3$, $\|\Delta_l \Delta_k'' \Delta_j' u\|_{L^p(\mathbf{R}^N)} \leq C 2^{-km} \|D_t^m \Delta_j' u\|_{L^p(\mathbf{R}^N)}$. Hence,

$$\begin{aligned} \|\Delta_\ell u\|_{L^p(\mathbf{R}^N)} \leq C & \left\{ \sum_{j=\ell-2}^{\ell+1} \|\Delta_j' u\|_{L^p(\mathbf{R}^N)} \right. \\ & \left. + \sum_{k=\ell-2}^{\ell+1} 2^{-ks} \sum_{j=0}^{k-1} 2^{(k-j)(s-m)} 2^{j(s-m)} \|D_t^m \Delta_j' u\|_{L^p(\mathbf{R}^N)} \right\}. \end{aligned}$$

The sequence $\varepsilon_k = \sum_{j=0}^{k-1} 2^{(k-j)(s-m)} 2^{j(s-m)} \|D_t^m \Delta_j' u\|_{L^p(\mathbf{R}^N)}$ is a convolution product of the sequence $(2^{j(s-m)})_j \in \ell^1$ and the sequence $(2^{j(s-m)} \|D_t^m \Delta_j' u\|_{L^p(\mathbf{R}^N)})_j \in \ell^p$. Hence $(\varepsilon_k)_k \in \ell^p$ and $(2^{ls} \|\Delta_\ell u\|_{L^p(\mathbf{R}^N)})_l \in \ell^p$.

The proof of Lemma 1.17 is complete. $\square$

Lemma 1.18. *Let $\delta \geq 1$ be a real number satisfying $-m + \delta m \in \mathbf{N}$ and let $1 \leq p \leq +\infty$. If $u \in \mathcal{S}'(\mathbf{R}^N)$, $t^{-m+\delta m} u \in L^p(\mathbf{R}; B_{p,p}^{s+m}(\mathbf{R}^{N-1}))$ and $D_t^m u \in L^p(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))$ then $u \in L^p(\mathbf{R}; B_{p,p}^{s+m/\delta}(\mathbf{R}^{N-1}))$ and there is a constant $C > 0$ such that*

$$\begin{aligned} \|u\|_{L^p(\mathbf{R}; B_{p,p}^{s+m/\delta}(\mathbf{R}^{N-1}))} \\ \leq C \left\{ \|D_t^m u\|_{L^p(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))} + \|t^{-m+\delta m} u\|_{L^p(\mathbf{R}; B_{p,p}^{s+m}(\mathbf{R}^{N-1}))} \right\}. \end{aligned}$$

Proof. If $v \in \mathcal{S}'(\mathbf{R})$ is such that $t^{-m+\delta m} v \in L^p(\mathbf{R})$ and $D_t^m v \in L^p(\mathbf{R})$ then $v \in L^p(\mathbf{R})$ and there is a constant $C > 0$ satisfying

$$\|v\|_{L^p(\mathbf{R})} \leq C \left\{ \|D_t^m v\|_{L^p(\mathbf{R})} + \|t^{-m+\delta m} v\|_{L^p(\mathbf{R})} \right\}.$$

Next we argue in the same way as for Lemma 1.16. $\square$

Corollary 1.19. *Let s and δ be two real numbers satisfying $s > -m/\delta$, $\delta \geq 1$, $-m + \delta m \in \mathbf{N}$ and let $1 \leq p \leq +\infty$. If $u \in \mathcal{S}'(\mathbf{R}^N)$, $t^{-m+\delta m} u \in L^p(\mathbf{R}; B_{p,p}^{s+m}(\mathbf{R}^{N-1}))$*

and $D_t^m u \in B_{p,p}^s(\mathbf{R}^N)$, then $D_t^j u \in B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^N)$, $j = 0, \dots, m-1$. Moreover, there exists a constant $C > 0$ such that

$$\|D_t^j u\|_{B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^N)} \leq C \left\{ \|D_t^m u\|_{B_{p,p}^s(\mathbf{R}^N)} + \|t^{-m+\delta m} u\|_{L^p(\mathbf{R}; B_{p,p}^{s+m}(\mathbf{R}^{N-1}))} \right\}.$$

Proof. Lemmas 1.18 and 1.15 yield $u \in B_{p,p}^{s+m/\delta}(\mathbf{R}^N)$ and we conclude with Lemma 1.16. $\square$

1.2.3. Proof of Corollary 1.11. We recall the following needed lemmas proved in [14].

Lemma 1.20. *Let $1 \leq p < +\infty$, and A a real < 0 . If $(a_{j\nu})_{j,\nu}$ is a sequence of positive real numbers satisfying $(a_{j\nu})_j \in \ell^p$ for any $\nu \geq 1$, then there exists a constant $C > 0$ such that*

$$\sum_{j \geq 1} \left(\sum_{\nu \geq 1} 2^{\nu A} a_{j\nu} \right)^p \leq C \sup_{\nu \geq 1} \sum_{j \geq 1} a_{j\nu}^p.$$

We denote αB , $\alpha > 0$, the ball with the same center x_0 as B and of radius αr , where r is the radius of B .

Lemma 1.21. *For any integer $M > 0$, there exists a constant $C_M > 0$ such that for any $J \in \mathbf{Z}$, $x_0 \in \mathbf{R}^{n+N}$, for any ball B centered at x_0 with radius 2^{-J} , for any $\ell \in \mathbf{Z}$ and $u \in L^p(\mathbf{R}^{n+N})$, $1 \leq p \leq +\infty$,*

$$\|A_\ell u\|_{L^p(B)} \leq C_M \left\{ \|u\|_{L^p(2B)} + \sum_{\nu \geq -J+1} 2^{-(\ell+\nu)M} \|u\|_{L^p(F_\nu)} \right\}$$

holds for $F_\nu = \{x \in \mathbf{R}^{n+N} : 2^\nu \leq |x - x_0| \leq 2^{\nu+1}\}$ and A_ℓ denotes any product of the operators Δ_ℓ, S_ℓ and the dotted operators.

Lemma 1.22. *Let $1 \leq p < +\infty$ and $s \in \mathbf{R}$. Further, let λ and $\alpha \geq 0$. Then we have the following continuous embedding*

$$\mathcal{L}^{p,\lambda,s+\alpha/p}(\mathbf{R}^N) \hookrightarrow \mathcal{L}^{p,\lambda+\alpha,s}(\mathbf{R}^N).$$

We have the same result for the dotted spaces $\dot{\mathcal{L}}$.

Lemma 1.23. *For any integer $M > 0$, there exists a constant $C_M > 0$ such that for any $J \in \mathbf{Z}$, $x_0 \in \mathbf{R}^{n+N}$, for any ball B centered at x_0 with radius 2^{-J} , for any $\ell \in \mathbf{Z}$, $k \in \mathbf{N}$ and $u \in W_{\text{loc}}^{k,p}(\mathbf{R}^{n+N}) \cap L^p(\mathbf{R}^{n+N})$, $1 \leq p \leq +\infty$,*

$$\|\dot{\Delta}_\ell u\|_{L^p(B)} \leq C_k 2^{k(J-\ell)} \sup_{|\alpha| \leq k} \|D_x^\alpha u\|_{L^p(3B)} + C_M \sum_{\nu \geq -J+1} 2^{-(\ell+\nu)M} \|u\|_{L^p(F_\nu)} \quad (1.9)$$

holds for $F_\nu = \{x \in \mathbf{R}^{n+N} : 2^\nu \leq |x - x_0| \leq 2^{\nu+1}\}$.

Proof. Let $\ell \in \mathbf{Z}$ and $\psi \in C_0^\infty(\mathbf{R}^{n+N})$, $\psi = 1$ on $2B$ and $\text{supp } \psi \subset 3B$. We have

$$\|\dot{\Delta}_\ell u\|_{L^p(B)} = \|\psi \dot{\Delta}_\ell u\|_{L^p(B)} \leq \|\dot{\Delta}_\ell \psi u\|_{L^p(B)} + \|[\dot{\Delta}_\ell, \psi] u\|_{L^p(B)}.$$

From Lemma 1.14 we deduce

$$\begin{aligned}
\|\dot{\Delta}_\ell \psi u\|_{L^p(B)} &\leq \|\dot{\Delta}_\ell \psi u\|_{L^p(\mathbf{R}^{n+N})} \\
&\leq C^k 2^{-k\ell} \sup_{|\alpha|=k} \|D_x^\alpha \dot{\Delta}_\ell \psi u\|_{L^p(\mathbf{R}^{n+N})} \\
&\leq C' C^k 2^{-k\ell} \sup_{|\alpha|=k} \|D_x^\alpha \psi u\|_{L^p(\mathbf{R}^{n+N})} \\
&\leq C_k 2^{-k\ell} \sup_{|\alpha|=k} \sum_{\beta \leq \alpha} C_{\alpha\beta} \|D_x^{\alpha-\beta} \psi\|_{L^\infty(\mathbf{R}^{n+N})} \|D_x^\beta u\|_{L^p(3B)}.
\end{aligned}$$

The classical estimations of the cutoff functions ψ yield

$$\|D_x^{\alpha-\beta} \psi\|_{L^\infty(\mathbf{R}^{n+N})} \leq C 2^{J|\alpha-\beta|}.$$

Hence

$$\|\dot{\Delta}_\ell \psi u\|_{L^p(B)} \leq C_k 2^{-k\ell} 2^{kJ} \sup_{|\alpha| \leq k} \|D_x^\alpha u\|_{L^p(3B)}.$$

To estimate $\|[\dot{\Delta}_\ell, \psi] u\|_{L^p(B)}$ we write for $x \in B$,

$$\begin{aligned}
[\dot{\Delta}_\ell, \psi] u(x) &= \dot{\Delta}_\ell \psi u(x) - \psi(x) \dot{\Delta}_\ell u(x) \\
&= -2^{(n+N)\ell} \int_{\mathbf{R}^{n+N} \setminus 2B} u(y) \mathcal{F}^{-1} \theta(2^\ell(x-y)) dy \\
&\quad + 2^{(n+N)\ell} \int_{3B \setminus 2B} \psi(y) u(y) \mathcal{F}^{-1} \theta(2^\ell(x-y)) dy.
\end{aligned}$$

Since $\mathcal{F}^{-1} \theta \in \mathcal{S}(\mathbf{R}^{n+N})$ we obtain in the same way as in the proof of Lemma 1.21, for any integer $L > n$,

$$\begin{aligned}
&\left| 2^{(n+N)\ell} \int_{\mathbf{R}^{n+N} \setminus 2B} u(y) \mathcal{F}^{-1} \theta(2^\ell(x-y)) dy \right| \\
&= 2^{(n+N)\ell} \left| \sum_{\nu \geq -J+1} \int_{F_\nu} u(y) \mathcal{F}^{-1} \theta(2^\ell(x-y)) dy \right| \\
&\leq C_L 2^{(n+N)\ell} \sum_{\nu \geq -J+1} 2^{-(\ell+\nu)L} \|u\|_{L^p(F_\nu)} |F_\nu|^{1-1/p}.
\end{aligned}$$

Using $|F_\nu| \sim 2^{(n+N)\nu}$ and $|B| \sim 2^{-J(n+N)}$ it follows that

$$\left\| 2^{(n+N)\ell} \int_{\mathbf{R}^{n+N} \setminus 2B} u(y) \mathcal{F}^{-1} \theta(2^\ell(x-y)) dy \right\|_{L^p(B)} \leq C_L \sum_{\nu \geq -J+1} 2^{(\ell+\nu)(n+N-L)} \|u\|_{L^p(F_\nu)}. \quad (1.10)$$

Noting that $3B \setminus 2B \subset F_{-J+1}$, we prove in the same way as above

$$\left\| 2^{(n+N)\ell} \int_{3B \setminus 2B} \psi(y) u(y) \mathcal{F}^{-1} \theta(2^\ell(x-y)) dy \right\|_{L^p(B)} \leq C_L 2^{(\ell-J)(n+N-L)} \|u\|_{L^p(F_{-J+1})}. \quad (1.11)$$

Adding (1.10) and (1.11) we obtain

$$\| [\dot{\Delta}_\ell, \psi] u \|_{L^p(B)} \leq C_L \sum_{\nu \geq -J+1} 2^{(\ell+\nu)(n+N-L)} \|u\|_{L^p(F_\nu)}.$$

The proof of Lemma 1.23 is complete. $\square$

To prove Corollary 1.11 we will need the following interpolation lemma.

Lemma 1.24. *If $u \in L^p(\mathcal{L}^{p,\lambda,s+m})$ and $D_t^k u \in \mathcal{L}^{p,\lambda+m,s}(\mathbf{R}^n \times \mathbf{R}^N)$ for any $|k| \leq m$, then $u \in \mathcal{L}^{p,\lambda,s+m}(\mathbf{R}^n \times \mathbf{R}^N)$ and there is a constant $C > 0$ independent of u such that*

$$\|u\|_{\mathcal{L}^{p,\lambda,s+m}} \leq C \left\{ \sup_{|k| \leq m} \|D_t^k u\|_{\mathcal{L}^{p,\lambda+m,s}} + \|u\|_{L^p(\mathcal{L}^{p,\lambda,s+m})} \right\}.$$

The statement remains true if we replace $\mathcal{L}$ by $\dot{\mathcal{L}}$.

Proof. For any integer $\ell \geq 3$ we have

$$\Delta_\ell u = \Delta_\ell \left(\sum_{k,j \geq 0} \Delta_k'' \Delta_j' u \right) = \Delta_\ell \left(\sum_{j=\ell-2}^{\ell+1} S_j'' \Delta_j' u + \sum_{k=\ell-2}^{\ell+1} \Delta_k'' S_{k-1}' u \right).$$

Let $B = Q \times B'$ be a cube of $\mathbf{R}^n \times \mathbf{R}^N$ centered at $x_0 = (t_0, x_0')$ and of length side 2^{-J} , $J \in \mathbf{Z}$. We have

$$\begin{aligned} 2^{\ell(s+m)} \|\Delta_\ell u\|_{L^p(B)} &\leq C \left\{ 2^{\ell(s+m)} \sum_{j \sim \ell} \|\Delta_\ell S_j'' \Delta_j' u\|_{L^p(B)} \right. \\ &\quad \left. + 2^{\ell(s+m)} \sum_{k \sim \ell} \|\Delta_\ell \Delta_k'' S_{k-1}' u\|_{L^p(B)} \right\}. \end{aligned} \quad (1.12)$$

For the first term of the right hand side of the above inequality we use Lemma 1.21 to get for any integer $M > 0$

$$\begin{aligned} &\frac{1}{|B|^{\lambda/(n+N)p}} 2^{\ell(s+m)} \|\Delta_\ell S_\ell'' \Delta_\ell' u\|_{L^p(B)} \\ &\leq \frac{C_M}{|B|^{\lambda/(n+N)p}} 2^{\ell(s+m)} \left\{ \|\Delta_\ell' u\|_{L^p(2B)} + \sum_{\nu \geq -J+1} 2^{-(\ell+\nu)M} \|\Delta_\ell' u\|_{L^p(F_\nu)} \right\}. \end{aligned}$$

Now

$$\frac{C_M}{|B|^{\lambda/(n+N)p}} 2^{\ell(s+m)} \|\Delta_\ell' u\|_{L^p(2B)} \leq \frac{C}{|2B'|^{\lambda/Np}} 2^{\ell(s+m)} \|\Delta_\ell' u\|_{L^p(\mathbf{R}^n \times 2B')}$$

and denoting B_k' the cube of $\mathbf{R}^N$ with the center x_0' and the length side 2^{k+1} , we obtain

for M sufficiently large

$$\begin{aligned}
& \frac{C_M}{|B|^{\lambda/(n+N)p}} 2^{\ell(s+m)} \sum_{\nu \geq -J+1} 2^{-(\ell+\nu)M} \|\Delta'_\ell u\|_{L^p(F_\nu)} \\
& \leq \frac{C_M}{|B|^{\lambda/(n+N)p}} 2^{\ell(s+m)} 2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M} \|\Delta'_\ell u\|_{L^p(F_{\nu-J})} \\
& \leq C 2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{\nu(-M+\lambda/p)} \frac{2^{\ell(s+m)}}{|B'_{\nu-J}|^{\lambda/(Np)}} \|\Delta'_\ell u\|_{L^p(\mathbf{R}^n \times B'_{\nu-J})} \\
& \leq C 2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{\nu(-M+\lambda/p)} \frac{2^{\ell(s+m)}}{|B'_{\nu-J}|^{\lambda/(Np)}} \|\Delta'_\ell u\|_{L^p(\mathbf{R}^n \times B'_{\nu-J})}.
\end{aligned}$$

We deduce

$$\begin{aligned}
& \frac{1}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} \sum_{j \sim \ell} 2^{\ell(s+m)p} \|\Delta_\ell S_j'' \Delta'_j u\|_{L^p(B)}^p \\
& \leq \frac{C}{|B'|^{\lambda/N}} \sum_{\ell \geq J^+} 2^{\ell(s+m)p} \|\Delta'_\ell u\|_{L^p(\mathbf{R}^n \times 2B')}^p \\
& \quad + C \sum_{\ell \geq J^+} \left(\sum_{\nu \geq 1} 2^{\nu(-M+\lambda/p)} \frac{2^{\ell(s+m)}}{|B'_{\nu-J}|^{\lambda/(Np)}} \|\Delta'_\ell u\|_{L^p(\mathbf{R}^n \times B'_{\nu-J})} \right)^p.
\end{aligned}$$

Using Lemma 1.20 for the last term of the right

$$\begin{aligned}
& \sum_{\ell \geq J^+} \left(\sum_{\nu \geq 1} 2^{\nu(-M+\lambda/p)} \frac{2^{\ell(s+m)}}{|B'_{\nu-J}|^{\lambda/(Np)}} \|\Delta'_\ell u\|_{L^p(\mathbf{R}^n \times B'_{\nu-J})} \right)^p \\
& \leq C \sup_{\nu \geq 1} \sum_{\ell \geq (J-\nu-1)^+} \frac{2^{\ell(s+m)p}}{|B'_{\nu-J}|^{\lambda/N}} \|\Delta'_\ell u\|_{L^p(\mathbf{R}^n \times B'_{\nu-J})}^p. \quad (1.13)
\end{aligned}$$

Hence

$$\frac{1}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} \sum_{j \sim \ell} 2^{\ell(s+m)p} \|\Delta_\ell S_j'' \Delta'_j u\|_{L^p(B)}^p \leq C \|u\|_{L^p(\mathcal{L}^{p,\lambda,s+m})}^p. \quad (1.14)$$

To bound from above the second term of the right hand side of (1.12), we use Lemma 1.23 with respect to the variable t to get for any $\ell \geq 3$ and any $M > 0$

$$\begin{aligned}
& \|\Delta''_\ell \Delta_\ell S'_{\ell-1} u\|_{L^p(B)} = \left[\int_{B'} \|\Delta''_\ell \Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(Q)}^p dx' \right]^{1/p} \\
& \leq C_p 2^{(J-\ell)m} \left[\int_{B'} \sup_{0 \leq |k| \leq m} \|D_t^k \Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(3Q)}^p dx' \right]^{1/p} \\
& \quad + C_{M',p} \left[\int_{B'} \left(\sum_{\nu \geq -J+1} 2^{-(\ell+\nu)M} \|\Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(F'_\nu)} \right)^p dx' \right]^{1/p}
\end{aligned}$$

$$\begin{aligned} &\leq C 2^{(J-\ell)m} \|D_t^{k_0} \Delta_\ell S'_{\ell-1} u\|_{L^p(3B)} \\ &\quad + C 2^{-(\ell-J)M} \left[\int_{B'} \left(\sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(\tilde{F}_{\nu-J}^t)} \right)^p dx' \right]^{1/p} \end{aligned}$$

where $F_\nu^t = \{t \in \mathbf{R}^n : 2^\nu \leq |t - t_0| \leq 2^{\nu+1}\} \subset \tilde{F}_\nu^t = \{t \in \mathbf{R}^n : |t - t_0| \leq 2^{\nu+1}\}$ and k_0 is a multi-index $|k_0| \leq m$. Hence

$$\begin{aligned} &\frac{1}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} \sum_{k \sim \ell} \|\Delta_k'' S'_{k-1} \Delta_\ell u\|_{L^p(B)}^p \\ &\leq \frac{C}{|B|^{(\lambda+m)/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell s} \|D_t^{k_0} \Delta_\ell S'_{\ell-1} u\|_{L^p(3B)}^p \\ &\quad + \frac{C}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} \left(2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \right. \\ &\quad \left. \times \int_{B'} \left\{ \sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(\tilde{F}_{\nu-J}^t)} \right\}^p dx' \right). \end{aligned} \tag{1.15}$$

For the first term of the right, we use Lemma 1.21 to deduce for any $M' > 0$

$$\begin{aligned} &\frac{C}{|B|^{(\lambda+m)/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell s} \|S'_{\ell-1} D_t^{k_0} \Delta_\ell u\|_{L^p(3B)}^p \\ &\leq \frac{C}{|B|^{(\lambda+m)/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell s} \left\{ \|D_t^{k_0} \Delta_\ell u\|_{L^p(6B)}^p \right. \\ &\quad \left. + \left(\sum_{\nu \geq -J+3} 2^{-(\ell+\nu)M'} \|D_t^{k_0} \Delta_\ell u\|_{L^p(F_\nu)} \right)^p \right\} \\ &\leq C \|D_t^{k_0} u\|_{\mathcal{L}^{p,\lambda+m,s}(\mathbf{R}^{n+N})}^p \\ &\quad + \frac{C}{|B|^{(\lambda+m)/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell s} 2^{-(\ell-J)M'p} \left(\sum_{\nu \geq 3} 2^{-\nu M'} \|D_t^{k_0} \Delta_\ell u\|_{L^p(F_{\nu-J})} \right)^p \\ &\leq C \|D_t^{k_0} u\|_{\mathcal{L}^{p,\lambda,s}(\mathbf{R}^{n+N})}^p \\ &\quad + C \sum_{\ell \geq J^+} \left\{ \sum_{\nu \geq 3} 2^{\nu(-M'+(\lambda+m)/p)} \frac{2^{\ell s}}{|F_{\nu-J}|^{(\lambda+m)/(n+N)p}} \|D_t^{k_0} \Delta_\ell u\|_{L^p(F_{\nu-J})} \right\}^p. \end{aligned}$$

Denote $B_{\nu-J}$ the cube $2^{\nu+1}B \supset F_{\nu-J}$, and use Lemma 1.20 for the last term

$$\begin{aligned} &\frac{C}{|B|^{(\lambda+m)/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell s} \|D_t^{k_0} \Delta_\ell S'_{\ell-1} u\|_{L^p(3B)}^p \\ &\leq C \|D_t^{k_0} u\|_{\mathcal{L}^{p,\lambda+m,s}(\mathbf{R}^{n+N})}^p + C \sup_{\nu \geq 1} \sum_{\ell \geq J^+} 2^{p\ell s} \frac{1}{|B_{\nu-J}|^{(\lambda+m)/(n+N)}} \|D_t^{k_0} \Delta_\ell u\|_{L^p(B_{\nu-J})}^p \\ &\leq C \sup_{|k| \leq m} \|D_t^k u\|_{\mathcal{L}^{p,\lambda+m,s}(\mathbf{R}^{n+N})}^p. \end{aligned} \tag{1.16}$$

For the second term of the right hand side of (1.15) we apply Hölder's inequality to the sum inside the integral, thus

$$\begin{aligned}
& \frac{C}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \int_{B'} \left\{ \sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(\tilde{F}_{\nu-J}^t)} \right\}^p dx' \\
& \leq \frac{C'}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \sum_{\nu \geq 1} 2^{-\nu Mp/2} \|\Delta_\ell S'_{\ell-1} u\|_{L^p(\tilde{F}_{\nu-J}^t \times B')}^p \\
& \leq \frac{C}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \left(\sum_{\nu \geq 1} 2^{-\nu M/2} \|\Delta_\ell S'_{\ell-1} u\|_{L^p(\tilde{F}_{\nu-J}^t \times B')} \right)^p.
\end{aligned}$$

Note that $B_{\nu-J} = 2^{\nu+1}B \supset \tilde{F}_{\nu-J}^t \times B'$ and $S'_{\ell-1}u = \sum_{j=0}^{\ell-1} \Delta'_j u$; we apply Lemma 1.21 to the term $\|\Delta_\ell S'_{\ell-1} u\|_{L^p(\tilde{F}_{\nu-J}^t \times B')}$ to deduce for any $M, M' > 0$

$$\begin{aligned}
& \frac{C}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \int_{B'} \left\{ \sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(\tilde{F}_{\nu-J}^t)} \right\}^p dx' \\
& \leq \frac{C'}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \left(\sum_{\nu \geq 1} 2^{-\nu M/2} C_{M'} \left\{ \|S'_{\ell-1} u\|_{L^p(2B_{\nu-J})} \right. \right. \\
& \quad \left. \left. + \sum_{\mu \geq \nu-J+1} 2^{-(\mu+\ell)M'} \|S'_{\ell-1} u\|_{L^p(F_\mu)} \right\} \right)^p \\
& \leq C \sum_{\ell \geq J^+} \left(2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M/2} \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(2B_{\nu-J})} \right)^p \\
& \quad + C \sum_{\ell \geq J^+} \left(2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M/2} \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} 2^{-\ell M'} \sum_{\mu \geq \nu-J+1} 2^{-\mu M'} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(F_\mu)} \right)^p.
\end{aligned}$$

Now $|B|^{\lambda/(n+N)p} \sim 2^{-J\lambda/p}$, and put $A = -M + (2\lambda/p) + 2R$ where R is any integer greater than $s + m$, we obtain by Lemma 1.20 for M sufficiently large (thus $A < 0$):

$$\begin{aligned}
& \sum_{\ell \geq J^+} \left(2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M/2} \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(2B_{\nu-J})} \right)^p \\
& \leq C \sum_{\ell \geq J^+} \left(2^{(\ell-J)(R-M)} \sum_{\nu \geq 1} 2^{\nu A/2} \sum_{j=0}^{\ell} 2^{(\ell-j)(s+m-R)} \right. \\
& \quad \left. \times \frac{2^{j(s+m-R)}}{|B_{\nu-J}|^{(\lambda+Rp)/(n+N)p}} \|\Delta'_j u\|_{L^p(2B_{\nu-J})} \right)^p \\
& \leq C \sum_{\ell \geq J^+} \left(\sum_{\nu \geq 1} 2^{\nu A/2} \sum_{j=0}^{\ell} 2^{(\ell-j)(s+m-R)} \frac{2^{j(s+m-R)}}{|B_{\nu-J}|^{(\lambda+Rp)/(n+N)p}} \|\Delta'_j u\|_{L^p(2B_{\nu-J})} \right)^p
\end{aligned}$$

$$\begin{aligned}
&\leq C \sup_{\nu \geq 1} \sum_{\ell \geq (J-\nu-1)^+} \left(\sum_{j=0}^{\ell} 2^{(\ell-j)(s+m-R)} \frac{2^{j(s+m-R)}}{|B_{\nu-J}|^{(\lambda+Rp)/(n+N)p}} \|\Delta'_j u\|_{L^p(2B_{\nu-J})} \right)^p \\
&\leq C \sup_{\nu \geq 1} \sum_{\ell \geq (J-\nu-1)^+} \left(\sum_{j=0}^{\ell} 2^{(\ell-j)(s+m-R)} \frac{2^{j(s+m-R)}}{|B'_{\nu-J}|^{(\lambda+Rp)/(Np)}} \|\Delta'_j u\|_{L^p(\mathbf{R}^n \times 2B'_{\nu-J})} \right)^p.
\end{aligned}$$

Since $R > s + m$ the last sum is a convolution product of the sequences

$$(2^{k(s+m-R)})_k \in \ell^1 \quad \text{and} \quad \left(\frac{2^{k(s+m-R)}}{|B'_{\nu-J}|^{(\lambda+Rp)/(Np)}} \|\Delta'_k u\|_{L^p(\mathbf{R}^n \times 2B'_{\nu-J})} \right)_k \in \ell^p.$$

Therefore with the aid of Lemma 1.22,

$$\begin{aligned}
&\sum_{\ell \geq J^+} \left(2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M/2} \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(2B_{\nu-J})} \right)^p \\
&\leq C \|u\|_{L^p(\mathcal{L}^{p,\lambda+Rp,s+m-R})}^p \leq C \|u\|_{L^p(\mathcal{L}^{p,\lambda,s+m})}^p.
\end{aligned} \tag{1.18}$$

To estimate from above the last term of (1.17) we proceed in the same manner as above

$$\begin{aligned}
&\sum_{\ell \geq J^+} \left(2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M/2} \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} 2^{-\ell M'} \sum_{\mu \geq \nu-J+1} 2^{-\mu M'} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(F_{\mu})} \right)^p \\
&\leq \sum_{\ell \geq J^+} 2^{-(\ell-J)Mp} \left(\sum_{\nu \geq 1} 2^{-\nu M/2} \left\{ \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} 2^{-(\ell+\nu-J)M'} \right. \right. \\
&\quad \left. \left. \times \sum_{\mu \geq 1} 2^{-\mu M'} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(B_{\mu+\nu-J})} \right\} \right)^p \\
&\leq C \sum_{\ell \geq J^+} 2^{(\ell-J)(-M-M'+R)p} \left(\sum_{\nu \geq 1} 2^{\frac{\nu}{2}(-M-2M'+(2\lambda/p)+2R)} \left\{ \sum_{\mu \geq 1} 2^{\mu(-M'+(\lambda/p)+R)} \right. \right. \\
&\quad \left. \left. \sum_{j=0}^{\ell} 2^{(\ell-j)(s+m-R)} \frac{2^{j(s+m-R)}}{|B_{\mu+\nu-J}|^{(\lambda+Rp)/(n+N)p}} \|\Delta'_j u\|_{L^p(B_{\mu+\nu-J})} \right\} \right)^p
\end{aligned}$$

where R is any integer $> s + m$.

Choosing larger and larger M and M' , (thus $2^{(\ell-J)(-M-M'+R)p} \leq 1$) we apply Lemma 1.20 twice and we get in the same way as for (1.18),

$$\begin{aligned}
&\sum_{\ell \geq J^+} \left(2^{-(\ell-J)M} \sum_{\nu \geq 1} 2^{-\nu M/2} \frac{2^{\ell(s+m)}}{|B|^{\lambda/(n+N)p}} 2^{-\ell M'} \sum_{\mu \geq \nu-J+1} 2^{-\mu M'} \sum_{j=0}^{\ell} \|\Delta'_j u\|_{L^p(F_{\mu})} \right)^p \\
&\leq C \sup_{B'} \sum_{\ell \geq J^+} \left(\sum_{j=0}^{\ell} 2^{(\ell-j)(s+m-R)} \frac{2^{j(s+m-R)}}{|B'|^{(\lambda+Rp)/(Np)}} \|\Delta'_j u\|_{L^p(\mathbf{R}^n \times B')} \right)^p \\
&\leq C \|u\|_{L^p(\mathcal{L}^{p,\lambda+Rp,s+m-R})}^p \leq C \|u\|_{L^p(\mathcal{L}^{p,\lambda,s+m})}^p.
\end{aligned} \tag{1.19}$$

From (1.17), (1.18) and (1.19) we obtain

$$\begin{aligned} \frac{C}{|B|^{\lambda/(n+N)}} \sum_{l \geq J^+} 2^{p\ell(s+m)} 2^{-(\ell-J)Mp} \int_{B'} \left\{ \sum_{\nu \geq 1} 2^{-\nu M} \|\Delta_\ell S'_{\ell-1} u(\cdot, x')\|_{L^p(\tilde{F}_{\nu-J}^t)} \right\}^p dx' \\ \leq C \|u\|_{L^p(\mathcal{L}^{p,\lambda,s+m})}^p. \end{aligned} \quad (1.20)$$

Finally inequalities (1.15), (1.16) and (1.20) yield

$$\begin{aligned} \frac{1}{|B|^{\lambda/(n+N)}} \sum_{\ell \geq J^+} 2^{p\ell(s+m)} \sum_{k \sim \ell} \|\Delta_k'' S'_{k-1} \Delta_\ell u\|_{L^p(B)}^p \\ \leq C \left\{ \sup_{|k| \leq m} \|D_t^k u\|_{\mathcal{L}^{p,\lambda+m,s}(\mathbf{R}^{n+N})}^p + \|u\|_{L^p(\mathcal{L}^{p,\lambda,s+m})}^p \right\} \end{aligned} \quad (1.21)$$

Combining inequalities (1.12), (1.14) and (1.21), Lemma 1.24 is proved. $\square$

Corollary 1.11 is a consequence of Lemma 1.24 and Theorem 1.8.

2. Estimates near the boundary for solutions of degenerate elliptic problems.

2.1. Introduction and statement of the results. In this section we use the last interpolation inequalities to prove the optimal a priori estimates in Besov spaces for solutions of a class of linear degenerate elliptic boundary value problems. These estimates allow us to deduce the regularity for some nonlinear degenerate partial differential equations. For more precision, we denote $L(x; D_x)$ the operator which can be written in an appropriate local coordinates

$$L(x; D_x) = \sum_{|\alpha'|+j \leq m} a_{\alpha'j}(t, x') t^{[-m+\delta|\alpha'|+j]^+} D_{x'}^{\alpha'} D_t^j \quad (2.1)$$

where $x = (t, x') \in \mathbf{R}_+ \times \mathbf{R}^{N-1}$, $m \in \mathbf{N}$, δ is a real number such that $-m + \delta m \in \mathbf{N}$ and $[-m + \delta|\alpha'| + j]^+$ is the smallest nonnegative integer $\geq -m + \delta|\alpha'| + j$. Here, the coefficients $a_{\alpha'j}(t, x')$ are complex-valued functions, infinitely differentiable in $\overline{\mathbf{R}_+^N}$. This class of operators (2.1) contains some classical differential operators. For instance,

- if $\delta = 1$ then $L(x; D_x)$ is a regular elliptic differential operator,
- if $m = 2$ and $\delta = 3/2$ [resp. $\delta = 2$] then $L(x; D_x)$ is the Tricomi [resp. Hermite] operator whose model is given in $\mathbf{R}_+^2$ by $L = D_t^2 + tD_{x'}^2$ [resp. $L = D_t^2 + t^2D_{x'}^2 + \lambda D_{x'}$; $\lambda \in \mathbf{C}$] where $(t, x') \in \mathbf{R}_+^2$.

This class of operators (2.1) was studied by many authors in the classical spaces of Sobolev H^s and $W^{s,p}$, Hölder C^α and in the space C^∞ . Extensive references could be found in the bibliography of this paper. On the other hand, in BMO and $\mathcal{L}^{p,\lambda}$ spaces we give in [15] the optimal a priori estimates for the linear boundary value problems associated to the operator (2.1).

The proof of these estimates for the general boundary value problems with respect to the operators (2.1) is given in two steps. The first one is based on one Peetre's method which is described in [6, 15, 18] and consists to use the Fourier transform with respect to $x' \in \mathbf{R}^{N-1}$ and to do a tangential harmonic analysis for Besov spaces $B_{p,p}^s(\mathbf{R}_+^N)$,

thereby we reduce the problem to one isomorphic theorem for an ordinary differential equation and we estimate the “almost tangential” derivatives of the solution. The second step is based on the interpolation inequalities given in the last section needed to estimate the normal derivatives of the solution.

We will need the following spaces.

Definition 2.1. Let $1 \leq p \leq +\infty$ and $\delta \geq 1$. Further let $s \in \mathbf{R}$ and $m \in \mathbf{N}$. We set

$$\begin{aligned}
 \text{(i)} \quad & W^{m,p}(\mathbf{R}) = \left\{ u \in L^p(\mathbf{R}) : D_t^j u \in L^p(\mathbf{R}) \text{ for } 0 \leq j \leq m \right\} \\
 & \text{and } W_\delta^{m,p}(\mathbf{R}) = \left\{ u \in W^{m,p}(\mathbf{R}) : t^{-m+\delta k+j} D_t^j u \in L^p(\mathbf{R}) \text{ for } 0 \leq k+j \leq m, \right. \\
 & \quad \left. -m+\delta k+j \in \mathbf{N} \right\} \\
 \text{(ii)} \quad & W_\delta^{m,p}(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1})) \\
 & = \left\{ u \in L^p(\mathbf{R}; B_{p,p}^{s+m/\delta}(\mathbf{R}^{N-1})) : D_t^j u \in L^p(\mathbf{R}; B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^{N-1})) \text{ and } \right. \\
 & \quad \left. t^{-m+\delta k+j} D_t^j u \in L^p(\mathbf{R}; B_{p,p}^{s+k}(\mathbf{R}^{N-1})) \text{ for } 0 \leq k+j \leq m, -m+\delta k+j \in \mathbf{N} \right\} \\
 \text{(iii)} \quad & W_\delta^{m+s,p}(\mathbf{R}^N) = \left\{ u \in B_{p,p}^{s+m/\delta}(\mathbf{R}^N) : D_t^j u \in B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^N) \text{ and } \right. \\
 & \quad \left. t^{[-m+\delta|\alpha'|+j]^+} D_{x'}^{\alpha'} D_t^j u \in B_{p,p}^s(\mathbf{R}^N) \text{ for } 0 \leq |\alpha'|+j \leq m, -m+\delta|\alpha'|+j \geq 0 \right\}.
 \end{aligned}$$

The space $W_\delta^{m,p}(\mathbf{R}_+)$ [resp. $W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$] denotes the set of all restrictions to $\mathbf{R}_+$ of elements of $W_\delta^{m,p}(\mathbf{R})$ [resp. $W_\delta^{m,p}(\mathbf{R}; B_{p,p}^s(\mathbf{R}^{N-1}))$]. In the same way, the space $W_\delta^{m+s,p}(\mathbf{R}_+^N)$ denotes the set of all restrictions to $\mathbf{R}_+^N$ of elements of $W_\delta^{m+s,p}(\mathbf{R}^N)$.

Now we give the assumptions:

(H₁) The operator $L(x, D_x)$ is elliptic for $t > 0$ in $\mathbf{R}_+^N$.

Let L^0 be the “principal part” of the operator $L(x, D_x)$ defined as

$$L^0 = L^0(t, x'; D_{x'}, D_t) = \sum_{\substack{|\alpha'|+j \leq m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} a_{\alpha'j}(0, x') t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j$$

We assume then

(H₂) The operator L^0 is elliptic for $t > 0$ in $\mathbf{R}_+^N$.

This hypothesis (H₂) is equivalent to assume that for any couples $x' \in \mathbf{R}^N$ and $\xi' \in \mathbf{R}^N \setminus \{0\}$, the polynomials in the variable τ

$$P^+(\tau) = \sum_{\substack{|\alpha'|+j \leq m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} a_{\alpha'j}(0, x') \xi'^{\alpha'} \tau^j \text{ and } P^-(\tau) = \sum_{\substack{|\alpha'|+j \leq m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} a_{\alpha'j}(0, x') \xi'^{\alpha'} (-1)^{\delta j} \tau^j$$

have no real roots. Let m_+ be the number of the roots of the polynomial $P^+(\tau)$ having positive imaginary part. Finally, we suppose

(H₃) For any $\omega' \in \mathbf{R}^N$, $|\omega'| = 1$, and for any $x' \in \mathbf{R}^N$ the problem

$$\begin{cases} L^0(t, x'; \omega', D_t) u(t) = 0 \\ D_t^j u(0) = 0 \quad \text{for } j = 0, \dots, m_+ - 1 \end{cases}$$

has only the solution $u = 0$ in $\mathcal{S}(\overline{\mathbf{R}}_+)$.

The main result is as following.

Theorem 2.2. Let s be a real number > 0 and $1 \leq p \leq +\infty$. Under hypotheses (H₁), (H₂) and (H₃), for any compact set K of $\overline{\mathbf{R}}_+^N$, there exists a constant $C_K > 0$ such that for any $u \in W_\delta^{m+s,p}(\mathbf{R}_+^N)$ with $\text{supp } u \subset K$,

$$\begin{aligned} \|u\|_{W_\delta^{m+s,p}(\mathbf{R}_+^N)} \leq C \Bigg\{ & \|Lu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \|u\|_{B_{p,p}^{s+m/\delta-1}(\mathbf{R}_+^N)} \\ & + \sum_{\ell=0}^{m_+-1} \|\gamma_\ell u\|_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})} \Bigg\} \end{aligned}$$

holds.

This theorem extends the results of [8], [9] and [16]. To shed some light on this purpose we give the following examples.

Examples 2.3.

1. If $\delta = 1$ we find again the case of elliptic operators, cf. [2].
2. If $m = \delta = 2$ then we obtain Hermite's operator given in $\mathbf{R}_+^2$ by

$$L = D_t^2 + t^2 D_{x'}^2 + \lambda D_{x'}, \quad (t, x') \in \mathbf{R}_+ \times \mathbf{R} \text{ and } \lambda \in \mathbf{C}.$$

If $\lambda \neq \pm(4k-1)$, $k \in \mathbf{N} \setminus \{0\}$, and if u is a solution of the problem

$$\begin{cases} u \in B_{p,p}^s(\mathbf{R}_+^2) \\ Lu = f \in B_{p,p}^s(\mathbf{R}_+^2) \\ \gamma_0 u = g \in B_{p,p}^{s+1-1/2p}(\mathbf{R}) \end{cases}$$

then u belongs to

$$W_{2,\text{loc}}^{2+s,p}(\mathbf{R}_+^2) = \left\{ u \in B_{p,p,\text{loc}}^{s+1}(\mathbf{R}_+^2) : D_t^2 u, t^2 D_{x'}^2 u, t D_t D_{x'} u \in B_{p,p,\text{loc}}^s(\mathbf{R}_+^2) \right\}.$$

If $p = +\infty$, we find the Hölder regularity given in [8] and if $p < +\infty$ we find the L^p regularity given in [9].

3. If $m = 2$ and $\delta = 3/2$ then we obtain Tricomi's operator given in $\mathbf{R}_+^2$ by

$$L = D_t^2 + t D_{x'}^2, \quad (t, x') \in \mathbf{R}_+ \times \mathbf{R}.$$

If u is a solution of the problem

$$\begin{cases} u \in B_{p,p}^s(\mathbf{R}_+^2) \\ Lu = f \in B_{p,p}^s(\mathbf{R}_+^2) \\ \gamma_0 u = g \in B_{p,p}^{s+4/3-2/3p}(\mathbf{R}) \end{cases}$$

then u belongs to

$$W_{3/2,\text{loc}}^{2+s,p}(\mathbf{R}_+^2) = \{ u \in B_{p,p,\text{loc}}^{s+4/3}(\mathbf{R}_+^2) : D_t^2 u, tD_{x'}^2 u, tD_t D_{x'} u \in B_{p,p,\text{loc}}^s(\mathbf{R}_+^2) \}.$$

If $p = +\infty$, we find the Hölder regularity given in [16] and if $p < +\infty$ we find the L^p regularity given in [9] and [16].

Remark 2.4. The method used in this paper can be adapted to the following class of degenerate operators considered in [20], [22] and [6]:

$$L = \sum_{h=0}^{\min(k,2m)} \varphi^{k-h} P^{2m-h}(x; D_x) \quad (2.2)$$

where $k, m \in \mathbf{N}$, the function φ is C^∞ from $\mathbf{R}^n$ to $\mathbf{R}$ with $\Omega = \{x \in \mathbf{R}^n; \varphi(x) > 0\}$, $\Gamma = \{x \in \mathbf{R}^n; \varphi(x) = 0\}$ and $d\varphi \neq 0$ on Γ ; $P^{2m-h}(x; D)$ is a differential operator with smooth coefficients on $\overline{\Omega}$ and of order $\leq 2m - h$, $P^{2m}(x; D)$ is a properly elliptic differential operator on $\overline{\Omega}$. The model of the class (2.2) is given on $\mathbf{R}_+^2$ by $L = t\Delta_x + \lambda(D_t + D_{x'})$, $x = (t, x') \in \mathbf{R}_+ \times \mathbf{R}$. We can state an analogous result as Theorem 2.2 for the class of operators (2.2).

An application of Theorem 2.2 is given to the regularity for solutions to the following completely nonlinear boundary value problem

$$\begin{cases} u \in W_\delta^{m+\mu,+\infty}(\mathbf{R}_+^N), \quad 0 < \mu < 1 \\ F(t, x'; t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u) = 0 \\ \gamma_\ell u = 0, \quad \ell = 0, \dots, m_+ - 1 \end{cases} \quad (2.3)$$

where $\partial_y = \partial/\partial y$ is the real derivation, u and $F = F(t, x'; p_{\alpha'j})_{0 \leq |\alpha'|+j \leq m}$ are real functions, F smooth in its arguments. Let L be the linearized operator of (2.3):

$$L = \sum_{0 \leq |\alpha'|+j \leq m} a_{\alpha'j}(t, x') t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j$$

where the coefficients $a_{\alpha'j}(t, x') = \partial F(t, x'; t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u) / \partial p_{\alpha'j}$ belong to the Hölder space $C^\mu(\mathbf{R}_+^N)$.

We assume

(H'_1) The operator L is elliptic for $t > 0$ in $\mathbf{R}_+^N$.

(H₂') The following operator L^0 is elliptic for $t > 0$ in $\mathbf{R}_+^N$

$$L^0 = L^0(t, x'; \partial_{x'}, \partial_t) = \sum_{\substack{|\alpha'|+j \leq m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} a_{\alpha'j}(0, x') t^{-m+\delta|\alpha'|+j} \partial_{x'}^{\alpha'} \partial_t^j$$

(H₃') For any $\omega' \in \mathbf{R}^N$, $|\omega'| = 1$, and for any $x' \in \mathbf{R}^N$ the problem

$$\begin{cases} L^0(t, x'; \omega', \partial_t) u(t) = 0 \\ \partial_t^j u(0) = 0 \quad \text{for } j = 0, \dots, m_+ - 1 \end{cases}$$

has only the solution $u = 0$ in $\mathcal{S}(\overline{\mathbf{R}_+})$.

Theorem 2.5. Under hypotheses (H₁'), (H₂') and (H₃'), if u solves the problem (2.3), then $u \in C^\infty(\overline{\mathbf{R}_+^N})$.

2.2. Traces. In this section we characterize the trace for elements of $W_\delta^{m+s,p}(\mathbf{R}_+^N)$ and $W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$.

Theorem 2.6. Let m be an integer ≥ 1 , $\ell \in \{0, 1, \dots, m-1\}$, $s \in \mathbf{R}$ and $1 \leq p \leq +\infty$. For $u \in W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$, the series $\sum_{j \geq 0} D_t^\ell \Delta_j' u(0, \cdot)$ converges in $\mathcal{S}'(\mathbf{R}^{N-1})$ and define an element $\gamma_\ell u$ belonging to the space $B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$ and the map $u \mapsto \gamma_\ell u$ is a continuous operator from $W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$ to $B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$.

Also, if $s > 0$ there exists an extension operator R_ℓ from $B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$ to $W_\delta^{m+s,p}(\mathbf{R}_+^N) \subset W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$ such that

$$\gamma_\ell \circ R_\ell = \text{Id}_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})}.$$

In particular, if $s > 0$ the operator γ_ℓ is bounded and surjective from $W_\delta^{m+s,p}(\mathbf{R}_+^N)$ onto $B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$.

Proof. Since $W_\delta^{m,p}(\mathbf{R}_+) \hookrightarrow W^{m,p}(\mathbf{R}_+) \hookrightarrow C^{m-1}(\mathbf{R}_+)$, for any integer $0 \leq \ell \leq m-1$, there exists a constant $C_\ell > 0$ such that for every $v \in W_\delta^{m,p}(\mathbf{R}_+)$

$$|D_t^\ell v(0)| \leq C_\ell \|v\|_{W_\delta^{m,p}(\mathbf{R}_+)}$$

holds. We deduce for any $\alpha > 0$ (if $p < +\infty$)

$$\begin{aligned} \alpha^{\ell p} |D_t^\ell v(0)|^p &\leq C_\ell^p \sum_{r=0}^m \alpha^{rp-1} \|D_t^r v\|_{L^p(\mathbf{R}_+)}^p \\ &\quad + \sum_{\substack{0 \leq k+r \leq m \\ -m+\delta k+r \in \mathbf{N}}} \alpha^{(m-\delta k)p-1} \|t^{-m+\delta k+r} D_t^r v\|_{L^p(\mathbf{R}_+)}^p. \end{aligned} \quad (2.4)$$

Now let $u \in W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$. For j integer ≥ 0 we set

$$u_j(t, x') = \Delta_j' u(t, x') \in W_\delta^{m,p}(\mathbf{R}_+; C^\infty(\mathbf{R}^{N-1}) \cap L^p(\mathbf{R}^{N-1})).$$

Apply inequality (2.4) to $t \mapsto u_j(t, x')$, take $\alpha = 2^{-j}$, integrate with respect to $x' \in \mathbf{R}^{N-1}$ and multiply by $2^{j(s+m/\delta)p-j/\delta}$ to get

$$\begin{aligned} & 2^{j(s+(m-\ell)/\delta-1/\delta p)p} \|D_t^\ell \Delta'_j u(0, \cdot)\|_{L^p(\mathbf{R}^{N-1})}^p \\ & \leq C_\ell^p \left\{ \sum_{r=0}^m 2^{j(s+(m-r)/\delta)p} \|D_t^r \Delta'_j u\|_{L^p(\mathbf{R}_+^N)}^p \right. \\ & \quad \left. + \sum_{\substack{0 \leq k+r \leq m \\ -m+\delta k+r \in \mathbf{N}}} 2^{jp(s+k)} \|t^{-m+\delta k+r} D_t^r \Delta'_j u\|_{L^p(\mathbf{R}_+^N)}^p \right\}. \end{aligned}$$

We deduce that $u \mapsto \gamma_\ell u$ is a bounded operator from $W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$ to $B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$. Let $u \in B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$, $0 \leq \ell \leq m-1$. Let $\varphi_0 \in C_0^\infty(\mathbf{R})$ be equal to 1 in a neighborhood of 0. If we set $\varphi_\ell(t) = t^\ell \varphi_0(t)/\ell!$ we have $\partial_t^\ell \varphi_\ell(0) = 1$ and $\partial_t^k \varphi_\ell(0) = 0$ for $k \neq \ell$, $0 \leq k \leq m-1$.

For ℓ integer ≥ 0 , we set

$$\tilde{R}_\ell u(t, x') = \sum_{j \geq 0} \tilde{R}_\ell^j u(t, x')$$

where $\tilde{R}_\ell^j u(t, x') = 2^{-j\ell/\delta} \varphi_\ell(2^{j/\delta} t) \Delta'_j u(x')$. Let $0 \leq r \leq m$, K an integer $> s + (m-r)/\delta$ and $k + |\alpha'| \leq K$. We have

$$\begin{aligned} \|D_{x'}^{\alpha'} D_t^k D_t^r \tilde{R}_\ell^j u\|_{L^p(\mathbf{R}^N)} & \leq 2^{-j\ell/\delta} 2^{j(k+r)/\delta} C 2^{-j/\delta p} \varepsilon_j 2^{j(|\alpha'|-s+(m-\ell)/\delta-1/\delta p)} \\ & \leq \varepsilon_j 2^{j(k+|\alpha'|-s+(m-r)/\delta)} \quad (\text{since } \delta \geq 1) \end{aligned}$$

where $(\varepsilon_j)_j \in \ell^p(\mathbf{N})$. The real interpolation of Besov spaces yields that $D_t^r \tilde{R}_\ell u$ belongs to $B_{p,p}^{s+(m-r)/\delta}(\mathbf{R}^N)$ and

$$\|D_t^r \tilde{R}_\ell u\|_{B_{p,p}^{s+(m-r)/\delta}(\mathbf{R}^N)} \leq C \|u\|_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})}.$$

In the same way as above we prove that if $0 \leq r \leq m$, K integer $> s$ and $r + |\alpha'| \leq K$ then $t^{[-m+\delta|\alpha'|+r]^+} D_{x'}^{\alpha'} D_t^r \tilde{R}_\ell u \in B_{p,p}^s(\mathbf{R}^N)$ and

$$\|t^{[-m+\delta|\alpha'|+r]^+} D_{x'}^{\alpha'} D_t^r \tilde{R}_\ell u\|_{B_{p,p}^s(\mathbf{R}^N)} \leq C \|u\|_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})}.$$

Consequently $\tilde{R}_\ell$ is bounded from $B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})$ to $W_\delta^{m+s,p}(\mathbf{R}^N)$. We set $R_\ell u = \tilde{R}_\ell u|_{\mathbf{R}_+^N}$, and it is not hard to see that

$$\gamma_\ell \circ R_\ell = \text{Id}_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})}.$$

The proof of Theorem 2.6 is complete. $\square$

2.3 An ordinary differential equation. We consider a class of ordinary differential operators defined on $\mathbf{R}_+$ by

$$L = \sum_{\substack{0 \leq j+k \leq m \\ -m+\delta k+j \in \mathbf{N}}} a_{kj} t^{-m+\delta k+j} D_t^j$$

where $a_{kj} \in \mathbf{C}$. We assume

$$(H) \text{ For any } \tau \in \mathbf{R} \text{ and } t > 0, \sum_{\substack{0 \leq j+k \leq m \\ -m+\delta k+j \in \mathbf{N}}} a_{kj} t^{-m+\delta k+j} \tau^j \neq 0.$$

This condition (H) is equivalent to say that the polynomial

$$P^+(\tau) = \sum_{\substack{k+j \leq m \\ -m+\delta k+j \in \mathbf{N}}} a_{kj} \tau^j$$

does not vanish on the line $\mathbf{R}$. Let m_+ be the number of the roots of the polynomial $P^+(\tau)$ having positive imaginary part.

Theorem 2.7. *Under the condition (H) the operator L from $W_\delta^{m,p}(\mathbf{R}_+)$ to $L^p(\mathbf{R}_+)$ is a Fredholm operator and its index is equal to m_+ .*

Corollary 2.8. *We have*

$$\text{Ker}(L) \cap W_\delta^{m,p}(\mathbf{R}_+) = \text{Ker}(L) \cap \mathcal{S}(\overline{\mathbf{R}}_+)$$

where $\text{Ker}(L) = \{u \in \mathcal{D}'(\mathbf{R}_+) : Lu = 0\}$.

The proof of the last results is similar to the one of the case $p = 2$ given in [5].

2.4. Proof of Theorem 2.2. In order to prove Theorem 2.2, the first step is the following

Proposition 2.9. *Let $s \in \mathbf{R}$. Under hypotheses (H_1) , (H_2) and (H_3) , for any compact set K of $\overline{\mathbf{R}}_+^N$, there is a constant $C_K > 0$ such that for any $u \in W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$, $\text{supp } u \subset K$, the following inequality holds:*

$$\begin{aligned} \|u\|_{W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} &\leq C_K \left\{ \|Lu\|_{L^p(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} \right. \\ &\quad \left. + \sum_{\ell=0}^{m_+-1} \|\gamma_\ell u\|_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})} + \|u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s+m/\delta-1}(\mathbf{R}^{N-1}))} \right\} \end{aligned}$$

Proof. Using the freezing technique of the coefficients of L at the point 0 we can restrict ourselves to prove Proposition 2.9 for the following homogeneous operator with constant coefficients

$$L^0 \equiv L^0(t; D_{x'}, D_t) = D_t^m + \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} a_{\alpha'j} t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j$$

where $a_{\alpha'j} \in \mathbf{C}$.

Set $\gamma = (\gamma_\ell)_{0 \leq \ell \leq m_+-1}$. With the aid of Theorem 2.7, we prove (as in [6,10]) that under hypotheses (H₁), (H₂) and (H₃), for every $\xi' \in \mathbf{R}^{N-1} \setminus \{0\}$, the operator $(L^0(t; \xi', D_t), \gamma)$ is left invertible from $W_\delta^{m,p}(\mathbf{R}_+)$ onto $L^p(\mathbf{R}_+) \times \mathbf{C}^{m_+}$ and if $K_{\xi'}$ denotes its left inverse, then the mapping $\xi' \mapsto K_{\xi'}$ is C^∞ from $\mathbf{R}^{N-1} \setminus \{0\}$ to $\mathcal{L}(L^p(\mathbf{R}_+) \times \mathbf{C}^{m_+}; W_\delta^{m,p}(\mathbf{R}_+))$ and for any multi-index α' , there exists $C_{\alpha'} > 0$ such that for any ξ' , $1/2 \leq |\xi'| \leq 2$, and for any $(f, g) \in L^p(\mathbf{R}_+) \times \mathbf{C}^{m_+}$,

$$\|D_{\xi'}^{\alpha'} K_{\xi'}(f, g)\|_{W_\delta^{m,p}(\mathbf{R}_+)} \leq C_{\alpha'} \|(f, g)\|_{L^p(\mathbf{R}_+) \times \mathbf{C}^{m_+}} \quad (2.5)$$

holds.

First of all, we will prove that there is a constant $C > 0$ such that

$$\|u\|_{L^p(\mathbf{R}^{N-1}; W_\delta^{m,p}(\mathbf{R}_+))} \leq C \left\{ \|L^0 u\|_{L^p(\mathbf{R}^{N-1}; L^p(\mathbf{R}_+))} + \sum_{\ell=0}^{m-1} \|\gamma_\ell u\|_{L^p(\mathbf{R}^{N-1})} \right\} \quad (2.6)$$

holds for any $u \in \mathcal{S}(\mathbf{R}^{N-1}; W_\delta^{m,p}(\mathbf{R}_+))$ whose tangential spectrum belongs to the annulus $1/2 \leq |\xi'| \leq 2$.

For this, apply the operator $(L^0(t; \xi', D_t), \gamma)$ to the relation

$$\mathcal{F}u(\cdot, \xi') = \int e^{-iy'\xi'} u(\cdot, y') dy'$$

yields the system

$$\begin{cases} L^0(t; \xi', D_t) \mathcal{F}u(\cdot, \xi') &= \mathcal{F}L^0 u(\cdot, \xi') &= \int e^{-iy'\xi'} (L^0 u)(\cdot, y') dy' \\ \gamma_j \mathcal{F}u(\cdot, \xi') &= \mathcal{F}\gamma_j u(\xi') &= \int e^{-iy'\xi'} (\gamma_j u)(y') dy'. \end{cases}$$

Apply $K_{\xi'}$ to that system gives

$$\mathcal{F}u(\cdot, \xi') = \int e^{-iy'\xi'} K_{\xi'}(L^0 u(\cdot, y'), \gamma u(y')) dy'.$$

Since

$$u(\cdot, x') = \int e^{ix'\xi'} \Phi(\xi') \mathcal{F}u(\cdot, \xi') \frac{d\xi'}{(2\pi)^{N-1}}, \quad \Phi \in C_0^\infty(\mathbf{R}^{N-1})$$

Φ is equal to 1 on $1/2 \leq |\xi'| \leq 2$ and its support belongs to an annulus, we integrate by parts with respect to ξ' to obtain

$$u(\cdot, x') = \iint \frac{e^{i(x'-y')\xi'}}{(1 + |x' - y'|^2)^M} (I - \Delta_{\xi'})^M \{\Phi(\xi') K_{\xi'}(L^0 u(\cdot, y'), \gamma u(y'))\} dy' \frac{d\xi'}{(2\pi)^{N-1}}.$$

Inequality (2.5) yields

$$\|u(\cdot, x')\|_{W_\delta^{m,p}(\mathbf{R}_+)} \leq C \int \frac{1}{(1 + |x' - y'|^2)^M} \|(L^0 u(\cdot, y'), \gamma u(y'))\|_{L^p(\mathbf{R}_+) \times \mathbf{C}^{m_+}} dy'.$$

This integral is a convolution product of a function of $L^1(\mathbf{R}^{N-1})$ (for M large) and a function of $L^p(\mathbf{R}^{N-1})$. Inequality (2.6) is proved.

Let $u \in W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))$ with $\text{supp } u \subset K$, K is a compact set of $\overline{\mathbf{R}_+^N}$. For $j \in \mathbf{N}$, we set $u_j(x) = \Delta'_j u(2^{-j/\delta}t, 2^{-j}x')$. If $j \geq 1$, then $u_j \in \mathcal{S}(\mathbf{R}^{N-1}; W_\delta^{m,p}(\mathbf{R}_+))$ and its tangential spectrum belongs to the annulus $\{1/2 \leq |\xi'| \leq 2\}$. Since

$$(L^0 u)_j = 2^{jm/\delta} L^0 u_j \quad \text{and} \quad (\gamma_\ell u)_j = 2^{j\ell/\delta} \gamma_\ell u_j$$

we apply inequality (2.6) to each u_j , $j \geq 1$,

$$\begin{aligned} & 2^{jmp/\delta} \|u_j\|_{L^p(\mathbf{R}^{N-1}; W_\delta^{m,p}(\mathbf{R}_+))}^p \\ & \leq C \left\{ \|(L^0 u)_j\|_{L^p(\mathbf{R}^{N-1}; L^p(\mathbf{R}_+))}^p + \sum_{\ell=0}^{m_+-1} 2^{j(m-\ell)p/\delta} \|(\gamma_\ell u)_j\|_{L^p(\mathbf{R}^{N-1})}^p \right\} \end{aligned}$$

holds for $p < +\infty$.

Develop and multiply by 2^{jsp} ,

$$\begin{aligned} & \sum_{r=0}^m 2^{jp(s+(m-r)/\delta)} \|\Delta'_j D_t^r u\|_{L^p(\mathbf{R}_+^N)}^p + \sum_{\substack{0 \leq k+r \leq m \\ -m+\delta k+r \in \mathbf{N}}} 2^{jp(s+k)} \|\Delta'_j t^{-m+\delta k+r} D_t^r u\|_{L^p(\mathbf{R}_+^N)}^p \\ & \leq C \left\{ 2^{jps} \|\Delta'_j L^0 u\|_{L^p(\mathbf{R}_+^N)}^p + \sum_{\ell=0}^{m_+-1} 2^{jp(s+(m-\ell)/\delta-1/\delta p)} \|\Delta'_j \gamma_\ell u\|_{L^p(\mathbf{R}^{N-1})}^p \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} & \|u\|_{W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} \\ & \leq C \left\{ \|L^0 u\|_{L^p(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} + \sum_{\ell=0}^{m_+-1} \|\gamma_\ell u\|_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})} + R_0 \right\}. \end{aligned} \quad (2.7)$$

where

$$R_0 = \sum_{r=0}^m \|\Delta'_0 D_t^r u\|_{L^p(\mathbf{R}_+^N)} + \sum_{\substack{0 \leq k+r \leq m \\ -m+\delta k+r \in \mathbf{N}}} \|\Delta'_0 t^{-m+\delta k+r} D_t^r u\|_{L^p(\mathbf{R}_+^N)}.$$

In the same way we obtain (2.7) for $p = +\infty$. Since $\text{supp } u \subset K$, we estimate from above the remainder term R_0 ,

$$\begin{aligned} R_0 & \leq \|\Delta'_0 D_t^m u\|_{L^p(\mathbf{R}_+^N)} + C_K \sum_{r=0}^{m-1} \|\Delta'_0 D_t^r u\|_{L^p(\mathbf{R}_+^N)} \\ & \leq \|\Delta'_0 D_t^m u\|_{L^p(\mathbf{R}_+^N)} + C_K \sum_{r=0}^{m-1} \|D_t^r u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s+(m-r)/\delta-1}(\mathbf{R}^{N-1}))} \end{aligned}$$

Lemma 1.16 yields a constant $C > 0$ such that for any $\varepsilon > 0$,

$$R_0 \leq \|\Delta'_0 D_t^m u\|_{L^p(\mathbf{R}_+^N)} + C_K C \left\{ \varepsilon \|D_t^m u\|_{L^p(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} + \varepsilon^{-r/(m-r)} \|u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s+m/\delta-1}(\mathbf{R}^{N-1}))} \right\}. \quad (2.8)$$

Now, equation $L^0 u = f$ gives

$$\Delta'_0 D_t^m u = \Delta'_0 L^0 u - \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} a_{\alpha'j} \Delta'_0 t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j u$$

hence

$$\begin{aligned} \|\Delta'_0 D_t^m u\|_{L^p(\mathbf{R}_+^N)} &\leq \|L^0 u\|_{L^p(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} \\ &\quad + C \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j \in \mathbf{N}}} \|t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s-|\alpha'|+(m-j)/\delta-1}(\mathbf{R}^{N-1}))}. \end{aligned}$$

Now $\text{supp } u \subset K$ yields

$$\begin{aligned} \|\Delta'_0 D_t^m u\|_{L^p(\mathbf{R}_+^N)} &\leq \|L^0 u\|_{L^p(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} \\ &\quad + C_K \sum_{j=0}^{m-1} \|D_t^j u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s+(m-j)/\delta-1}(\mathbf{R}^{N-1}))}. \end{aligned}$$

Lemma 1.16 together with (2.8) and (2.7) give Proposition 2.9. $\square$

Theorem 2.2 will be a consequence of Proposition 2.9 and the following estimate:
For every compact set K of $\overline{\mathbf{R}_+^N}$, there exists a constant $C_K > 0$ such that

$$\|u\|_{W_\delta^{m+s,p}(\mathbf{R}_+^N)} \leq C_K \left\{ \|Lu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \|u\|_{W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} \right\} \quad (2.9)$$

holds for any $u \in W_\delta^{m+s,p}(\mathbf{R}_+^N)$, $\text{supp } u \subset K$.

Proof of (2.9). Let us take $0 < s < 1$. We decompose

$$Lu = a(x) D_t^m u + \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j \geq 0}} a_{\alpha'j}(x) t^{[-m+\delta|\alpha'|+j]^+} D_{x'}^{\alpha'} D_t^j u + \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j < 0}} a_{\alpha'j}(x) D_{x'}^{\alpha'} D_t^j u$$

where the function a is $C^\infty(\overline{\mathbf{R}_+^N})$ and does not vanishe on $\overline{\mathbf{R}_+^N}$.

(i) With Lemma 1.17 we have for any $|\alpha'| + j \leq m$, $j < m$ and $-m + \delta|\alpha'| + j = 0$ (thus $|\alpha'| = (m - j)/\delta$)

$$\begin{aligned} &\|D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \\ &\leq C \left\{ \|D_{x'}^{\alpha'} D_t^j u\|_{L^p(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} + \|D_{x'}^{\alpha'} D_t^{j+1} u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s-1}(\mathbf{R}^{N-1}))} \right\} \\ &\leq C \left\{ \|D_t^j u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}^{N-1}))} + \|D_t^{j+1} u\|_{L^p(\mathbf{R}_+; B_{p,p}^{s+(m-j)/\delta-1}(\mathbf{R}^{N-1}))} \right\} \\ &\leq C \|u\|_{W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))} \quad (\text{since } \delta \geq 1). \end{aligned} \quad (2.10)$$

In the case $-m + \delta|\alpha'| + j \in \mathbf{N} \setminus \{0\}$, Lemma 1.17 yields

$$\begin{aligned} & \|t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \\ & \leq C \left\{ \|t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j u\|_{L^p(\mathbf{R}_+, B_{p,p}^s(\mathbf{R}^{N-1}))} \right. \\ & \quad + \|t^{-m+\delta|\alpha'|+j-1} D_{x'}^{\alpha'} D_t^j u\|_{L^p(\mathbf{R}_+, B_{p,p}^{s-1}(\mathbf{R}^{N-1}))} \\ & \quad \left. + \|t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^{j+1} u\|_{L^p(\mathbf{R}_+, B_{p,p}^{s-1}(\mathbf{R}^{N-1}))} \right\}. \end{aligned}$$

Lemma 1.16 gives a constant $C > 0$ such that for any $j = 0, \dots, m-2$

$$\|t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \leq C \|u\|_{W_\delta^{m,p}(\mathbf{R}_+, B_{p,p}^s(\mathbf{R}^{N-1}))}. \quad (2.11)$$

For $j = m-1$ (thus $|\alpha'| = 1$ and $\delta \geq 2$) Lemma 1.17 and $\text{supp } u \subset K$ give

$$\|t^{\delta-1} D_{x'}^{\alpha'} D_t^{m-1} u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \leq C_K \|u\|_{W_\delta^{m,p}(\mathbf{R}_+, B_{p,p}^s(\mathbf{R}^{N-1}))}. \quad (2.12)$$

Inequalities (2.10), (2.11) and (2.12) yield a constant $C_K > 0$ such that

$$\|t^{-m+\delta|\alpha'|+j} D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \leq C_K \|u\|_{W_\delta^{m,p}(\mathbf{R}_+, B_{p,p}^s(\mathbf{R}^{N-1}))} \quad (2.13)$$

holds for any $|\alpha'| + j \leq m$, $j < m$ and $-m + \delta|\alpha'| + j \in \mathbf{N}$.

(ii) In the case $|\alpha'| + j \leq m$, $j < m$, $-m + \delta|\alpha'| + j > 0$ and $-m + \delta|\alpha'| + j \notin \mathbf{N}$, we set $-m + \delta|\alpha'| + j = k + \mu$ with $k \in \mathbf{N}$ and $0 < \mu < 1$, (thus $[-m + \delta|\alpha'| + j]^+ = k + 1$). Using Lemma 1.17 we prove in the same way as in **(i)**,

$$\|t^{[-m+\delta|\alpha'|+j]^+} D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \leq C_K \|u\|_{W_\delta^{m,p}(\mathbf{R}_+, B_{p,p}^s(\mathbf{R}^{N-1}))}. \quad (2.14)$$

(iii) On the other hand, equation $Lu = f$ gives

$$D_t^m u = b(x)Lu - \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j \geq 0}} b_{\alpha'j}(x) t^{[-m+\delta|\alpha'|+j]^+} D_{x'}^{\alpha'} D_t^j u - \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j < 0}} b_{\alpha'j}(x) D_{x'}^{\alpha'} D_t^j u.$$

We deduce from (2.13) and (2.14)

$$\begin{aligned} \|D_t^m u\|_{B_{p,p}^s(\mathbf{R}_+^N)} & \leq C_K \left\{ \|Lu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \|u\|_{W_\delta^{m,p}(\mathbf{R}_+, B_{p,p}^s(\mathbf{R}^{N-1}))} \right. \\ & \quad \left. + \sum_{\substack{|\alpha'|+j \leq m \\ j < m \\ -m+\delta|\alpha'|+j < 0}} \|b_{\alpha'j}(x) D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} \right\}. \end{aligned}$$

To finish the proof of (2.9) we have to estimate from above the term

$$\|b_{\alpha'j}(x) D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)}.$$

In the case $|\alpha'| \geq (m-j)/\delta - 1$, $s + (m-j)/\delta - 1 \leq s + |\alpha'| < s + (m-j)/\delta$, the following compact inequality

$$\begin{aligned} \|b_{\alpha'j}(x)D_{x'}^{\alpha'}D_t^ju\|_{B_{p,p}^s(\mathbf{R}_+^N)} &\leq C\|D_t^ju\|_{B_{p,p}^{s+|\alpha'|}(\mathbf{R}_+^N)} \\ &\leq C\left\{\varepsilon\|D_t^ju\|_{B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}_+^N)} + C_\varepsilon\|D_t^ju\|_{B_{p,p}^{s+(m-j)/\delta-1}(\mathbf{R}_+^N)}\right\} \end{aligned}$$

and the second part of Lemma 1.16 yield

$$\|b_{\alpha'j}(x)D_{x'}^{\alpha'}D_t^ju\|_{B_{p,p}^s(\mathbf{R}_+^N)} \leq C\left\{\varepsilon\|D_t^mu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + C_\varepsilon\|u\|_{B_{p,p}^{s+m/\delta-1}(\mathbf{R}_+^N)}\right\}.$$

In the case $|\alpha'| < (m-j)/\delta - 1$, we apply Lemma 1.16 for $s' = s + |\alpha'| - (m-j)/\delta$ instead of s

$$\begin{aligned} \|b_{\alpha'j}(x)D_{x'}^{\alpha'}D_t^ju\|_{B_{p,p}^s(\mathbf{R}_+^N)} &\leq C\|D_t^ju\|_{B_{p,p}^{s+|\alpha'|}(\mathbf{R}_+^N)} \\ &\leq C\left\{\varepsilon\|D_t^mu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + C_\varepsilon\|u\|_{B_{p,p}^{s+m/\delta-1}(\mathbf{R}_+^N)}\right\}. \end{aligned}$$

Therefore

$$\|D_t^mu\|_{B_{p,p}^s(\mathbf{R}_+^N)} \leq C_K\left\{\|Lu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \|u\|_{W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))}\right\}. \quad (2.15)$$

The above inequality together with Corollary 1.19 yield

$$\|D_t^ju\|_{B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}_+^N)} \leq C_K\left\{\|Lu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \|u\|_{W_\delta^{m,p}(\mathbf{R}_+; B_{p,p}^s(\mathbf{R}^{N-1}))}\right\}. \quad (2.16)$$

Inequalities (2.13) (2.14) (2.15) and (2.16) make the proof of (2.9) complete for $0 < s < 1$. To prove (2.9) in the general case $s = q + r$ with $q \in \mathbf{N}$ and $0 < r < 1$, we do an induction on q .

2.5 Proof of Theorem 2.5. For $h \in \mathbf{R} \setminus \{0\}$, $|h|$ small and $i = 1, \dots, N-1$ consider the difference quotients of a function v ,

$$\rho_{ih}v(t, x') = \frac{1}{h}(\tau_{ih}v(t, x') - v(t, x'))$$

where $\tau_{ih}v(t, x') = v(t, x_1, \dots, x_i + h, \dots, x_{N-1})$. Set

$$\begin{aligned} \Phi_{ih}(\sigma) &= F(t, x_1, \dots, x_i + \sigma h, \dots, x_{N-1}; \\ &\quad t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u + \sigma h \rho_{ih}(t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u))_{|\alpha'|+j \leq m}. \end{aligned}$$

From problem (2.3) we deduce $\int_0^1 \frac{d\Phi_{ih}}{d\sigma}(\sigma) d\sigma = 0$ and then the following family of equations $L_{ih}(\rho_{ih}u) = f_{ih}$, where

$$L_{ih} = \sum_{|\alpha'|+j \leq m} a_{\alpha'j}^{ih}(t, x') t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j$$

$$a_{\alpha'j}^{ih}(t, x') = \int_0^1 \frac{\partial F}{\partial p_{\alpha'j}} \left(t, x_1, \dots, x_i + \sigma h, \dots, x_{N-1}; \right. \\ \left. t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u + \sigma h \rho_{ih}(t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u) \right) d\sigma$$

$$f_{ih}(t, x') = - \int_0^1 \frac{\partial F}{\partial x_i} \left(t, x_1, \dots, x_i + \sigma h, \dots, x_{N-1}; \right. \\ \left. t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u + \sigma h \rho_{ih}(t^{[-m+\delta|\alpha'|+j]^+} \partial_{x'}^{\alpha'} \partial_t^j u) \right) d\sigma.$$

Note that the coefficients $a_{\alpha'j}^{ih}$ and the function f_{ih} belong to $C_{\text{loc}}^{\ell+\mu}(\mathbf{R}_+^N)$, uniformly as $|h| \rightarrow 0$, if $u \in W_{\delta, \text{loc}}^{m+(\ell+\mu), +\infty}(\mathbf{R}_+^N)$, $\ell \in \mathbf{N}$, while this hypothesis and Theorem 2.2 (for L_{ih} and $\rho_{ih}u$) give a uniform bound on the $W_{\delta}^{m+(\ell+\mu), +\infty}(K)$ -norm of $\rho_{ih}u$ as $|h| \rightarrow 0$, K is a compact set of $\mathbf{R}_+^N$. Therefore if $u \in W_{\delta, \text{loc}}^{m+(\ell+\mu), +\infty}(\mathbf{R}_+^N)$, $\ell \in \mathbf{N}$, then $\partial_{x_i} u \in W_{\delta, \text{loc}}^{m+(\ell+\mu), +\infty}(\mathbf{R}_+^N)$ and thus $u \in W_{\delta, \text{loc}}^{m+(\ell+1+\mu), +\infty}(\mathbf{R}_+^N)$. This argument can be iterated repeatedly to yield Theorem 2.5.

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Résumé substantiel en français. Le but de ce travail est d'établir les estimations à priori dans les espaces de Besov construits sur L^p , $1 \leq p \leq +\infty$, pour les solutions d'une classe de problèmes aux limites linéaires associés aux opérateurs (2.1) elliptiques singuliers du type Hermite et Tricomi. Pour atteindre ce but, on commence par donner des inégalités d'interpolation avec poids du type (1.1) pour des fonctions L^p vectorielles à valeurs dans les espaces de Besov-Peetre $B_{p,q}^s(\mathbf{R}^N)$ ou bien dans les espaces de Campanato $\mathcal{L}^{p,\lambda}(\mathbf{R}^N)$. Une application de ces inégalités est donnée à l'étude de la régularité des distributions tempérées en fonction de celle de leurs dérivées à poids et dérivées normales. De telles inégalités interviennent dans l'étude de certains espaces de Sobolev à poids et dans l'étude de la régularité des solutions de certains problèmes aux limites elliptiques singuliers (voir [4, 10, 11, 12, 15]).

Voici le résultat principal de cet article.

Théorème 2.2. Soit s un réel > 0 et $1 \leq p \leq +\infty$. Sous les hypothèses (H_1) , (H_2) et (H_3) , pour tout compact K de $\mathbf{R}_+^N$, il existe une constante $C_K > 0$, telle que pour tout $u \in W_{\delta}^{m+s,p}(\mathbf{R}_+^N)$ avec $\text{supp } u \subset K$ on ait

$$\begin{aligned} & \|u\|_{W_{\delta}^{m+s,p}(\mathbf{R}_+^N)} \\ & \equiv \sum_{\substack{|\alpha'|+j \leq m \\ -m+\delta|\alpha'|+j \geq 0}} \|t^{[-m+\delta|\alpha'|+j]^+} D_{x'}^{\alpha'} D_t^j u\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \sum_{j=0}^m \|D_t^j u\|_{B_{p,p}^{s+(m-j)/\delta}(\mathbf{R}_+^N)} \\ & \leq C \left\{ \|Lu\|_{B_{p,p}^s(\mathbf{R}_+^N)} + \sum_{\ell=0}^{m-1} \|\gamma_{\ell} u\|_{B_{p,p}^{s+(m-\ell)/\delta-1/\delta p}(\mathbf{R}^{N-1})} + \|u\|_{B_{p,p}^{s+m/\delta-1}(\mathbf{R}_+^N)} \right\} \end{aligned}$$

l'opérateur L étant défini par (2.1), $x = (t, x') \in \mathbf{R}_+ \times \mathbf{R}^{N-1}$, $m \in \mathbf{N}$, δ étant un réel vérifiant $-m + \delta m \in \mathbf{N}$, et $[-m + \delta |\alpha'| + j]^+$ désigne le plus petit entier $\geq -m + \delta |\alpha'| + j$.

Ce théorème prolonge les résultats de [8], [9] et [16]. Pour éclairer cela, on va donner quelques exemples.

Exemples 2.3.

1. Lorsque $\delta = 1$, on retrouve le cas des opérateurs elliptiques, cf. [2].
2. Lorsque $m = \delta = 2$, on obtient l'opérateur d'Hermite dont le modèle est donné sur $\mathbf{R}_+^2$ par

$$L = D_t^2 + t^2 D_{x'}^2 + \lambda D_{x'}; \quad (t, x') \in \mathbf{R}_+ \times \mathbf{R} \text{ et } \lambda \in \mathbf{C}.$$

Si $\lambda \neq \pm(4k - 1)$, $k \in \mathbf{N} \setminus \{0\}$, et si u est une solution du problème

$$\begin{cases} u \in B_{p,p}^s(\mathbf{R}_+^2) \\ Lu = f \in B_{p,p}^s(\mathbf{R}_+^2) \\ \gamma_0 u = g \in B_{p,p}^{s+1-1/2p}(\mathbf{R}) \end{cases}$$

alors u appartient à

$$W_{2,\text{loc}}^{2+s,p}(\mathbf{R}_+^2) = \{ u \in B_{p,p,\text{loc}}^{s+1}(\mathbf{R}_+^2) : D_t^2 u, t^2 D_{x'}^2 u, t D_t D_{x'} u \in B_{p,p,\text{loc}}^s(\mathbf{R}_+^2) \}.$$

Si $p = +\infty$, on retrouve la régularité Höldérienne donnée dans [8] et si $p < +\infty$ on retrouve la régularité L^p donnée dans [9].

3. Lorsque $m = 2$ et $\delta = 3/2$, on obtient l'opérateur de Tricomi dont le modèle est donné sur $\mathbf{R}_+^2$ par

$$L = D_t^2 + t D_{x'}^2; \quad (t, x') \in \mathbf{R}_+ \times \mathbf{R}.$$

Si u est une solution du problème

$$\begin{cases} u \in B_{p,p}^s(\mathbf{R}_+^2) \\ Lu = f \in B_{p,p}^s(\mathbf{R}_+^2) \\ \gamma_0 u = g \in B_{p,p}^{s+4/3-2/3p}(\mathbf{R}) \end{cases}$$

alors u appartient à

$$W_{3/2,\text{loc}}^{2+s,p}(\mathbf{R}_+^2) = \{ u \in B_{p,p,\text{loc}}^{s+4/3}(\mathbf{R}_+^2) : D_t^2 u, t D_{x'}^2 u, t D_t D_{x'} u \in B_{p,p,\text{loc}}^s(\mathbf{R}_+^2) \}.$$

Si $p = +\infty$ on retrouve la régularité Höldérienne donnée dans [16] et si $p < +\infty$ on retrouve la régularité L^p donnée dans [9] et [16].

Pour montrer le Théorème 2.2, on aura besoin des inégalités d'interpolation du type suivant :

Théorème 1.7. Soit $s \in \mathbf{R}$ et $\lambda \geq 0$. Sous les hypothèses (1.2) à (1.4), il existe une constante $C > 0$ telle que pour tout $u \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$ on ait

$$\left\| |t|^\gamma D_t^j u \right\|_{L^r(B_{r,r}^{2sa(1-a)+\gamma-j})} \leq C \left\| |t|^\alpha D_t^m u \right\|_{L^p(B_{p,p}^{s(1-a)+\alpha-m})}^a \left\| |t|^\beta u \right\|_{L^q(B_{q,q}^{sa+\beta})}^{1-a}$$

et

$$\left\| |t|^\gamma D_t^j u \right\|_{L^r(\mathcal{L}^{r,\lambda,2sa(1-a)+\gamma-j})} \leq C \left\| |t|^\alpha D_t^m u \right\|_{L^p(\mathcal{L}^{p,\lambda,s(1-a)+\alpha-m})}^a \left\| |t|^\beta u \right\|_{L^q(\mathcal{L}^{q,\lambda,sa+\beta})}^{1-a}.$$

Une autre application de ces inégalités est donnée à la régularité des distributions tempérées en fonction de celle de leurs dérivées normales et dérivées à poids.

Corollaire 1.9. Soit $s > -m$ et supposons que les hypothèses (1.5) à (1.7) soient satisfaites. Dans ces conditions, si $u \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^N)$ vérifie

$$(1) \quad |t|^\beta u \in L^q(B_{q,q}^{(s+m)/2(1-a)+\beta}),$$

$$(2) \quad |t|^\alpha D_t^m u \in L^p(B_{p,p}^{(s+m)/2a+\alpha-m}), \text{ et}$$

$$(3) \quad D_t^m u \in B_{r,r}^s(\mathbf{R}^n \times \mathbf{R}^N)$$

alors $u \in B_{r,r}^{s+m}(\mathbf{R}^n \times \mathbf{R}^N)$.

Exemple 1.10. On retrouve en particulier certains résultats de [10] et [19]. Lorsque $n = 1$, $s > 0$, $p = q = r$, $\alpha = 0$, $\beta = m = 2$ et $a = 1/2$ on obtient : si $u(t, x') \in \mathcal{S}'(\mathbf{R} \times \mathbf{R}^N)$ vérifie $t^2 u \in L^p(\mathbf{R}; B_{p,p}^{s+4}(\mathbf{R}^N))$ et $D_t^2 u \in B_{p,p}^s(\mathbf{R} \times \mathbf{R}^N)$ alors $u \in B_{p,p}^{s+2}(\mathbf{R} \times \mathbf{R}^N)$.

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