

EXPANSIONS IN GENERALIZED SPHERICAL HARMONICS IN $\mathbb{R}^{k+1}$

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RÉSUMÉ. Nous investiguons les problèmes intérieurs et extérieurs de Dirichlet pour l'équation de Dunkl sur la sphère de dimension k . Il est démontré que la solution est unique et peut être exprimée en séries sphériques harmoniques généralisées. Un outil essentiel dans la preuve est l'estimation de noyaux des espaces des harmoniques sphériques généralisées de Xu [15].

ABSTRACT. The interior and exterior Dirichlet problems for the Dunkl equation on the k -dimensional unit sphere is investigated. It is shown that the solution is unique and can be expanded in series of generalized spherical harmonics. An estimate for the reproducing kernels of spaces of generalized spherical harmonics due to Xu [15] is an important tool in the proofs.

1. Introduction. In this paper, we consider Dunkl equation

$$\Delta_\rho u = \sum_{j=0}^k \mathcal{D}_j^2 u = 0, \quad (1)$$

where $\mathcal{D}_j$ is a modification of the j -th partial derivative

$$\mathcal{D}_j u(x) = \partial_j u(x) + \left(\rho_j - \frac{1}{2} \right) \frac{u(x) - u(\sigma_j x)}{x_j}, \quad x = (x_0, \dots, x_k) \in \mathbb{R}^{k+1}, \quad (2)$$

and σ_j denotes the reflection $\sigma_j(x_0, \dots, x_k) = (x_0, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_k)$. The parameters $\rho_0, \dots, \rho_k$ in the functional-difference equation (1) are assumed to be positive. In the rest of this paper, we assume $\rho_j \geq 1/2$ for all j . We consider these parameters to be fixed and usually do not indicate the dependence on $\rho_0, \dots, \rho_k$ in our notations. Equation (1) reduces to the Laplace equation

$$\Delta u = 0 \quad (3)$$

when $\rho_0 = \rho_1 = \dots = \rho_k = 1/2$. Let the open ball

$$B_R = \{x \in \mathbb{R}^{k+1} : |x| < R\}, \quad |x| = (x_0^2 + \dots + x_k^2)^{1/2}. \quad (4)$$

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For a differentiable function $f : B_R \rightarrow \mathbb{C}$ and $j = 0, \dots, k$, let

$$\mathcal{D}_j f(x) = \partial_j f(x) + \left(\rho_j - \frac{1}{2} \right) \frac{f(x) - f(\sigma_j x)}{x_j}. \quad (5)$$

Then $\mathcal{D}_j^2 f(x) = \partial_j^2 f(x) + \frac{2\rho_j - 1}{x_j} \partial_j f(x) - \frac{\rho_j - \frac{1}{2}}{x_j^2} (f(x) - f(\sigma_j x))$.

We note that, for every $j = 0, \dots, k$, and every function f that is defined and m times continuously differentiable on the ball (4), the function $\mathcal{D}_j f$ can be extended to a function that is $m - 1$ times continuously differentiable on the same ball. So the operator $\mathcal{D}_j$ maps $\mathcal{C}^m(B_R)$ to $\mathcal{C}^{m-1}(B_R)$.

Recall that a spherical harmonic of degree n is a homogeneous polynomial u in the variables $x_0, \dots, x_k$ of degree n that satisfies equation (3). Note that in some of the literature the domain of definition of spherical harmonics is restricted to the k -dimensional unit sphere $\mathbf{S}^k$. A generalized spherical harmonic of degree n is a homogeneous polynomial of degree n that satisfies Equation (1). Several results in this direction are already given in Dunkl [5], Xu [15] and Volkmer [13]. Generalized spherical harmonics of distinct degrees are orthogonal over the sphere $\mathbf{S}^k$ with respect to the weight function

$$|x_0|^{2\rho_0-1} |x_1|^{2\rho_1-1} \dots |x_k|^{2\rho_k-1}. \quad (6)$$

Generalized spherical harmonics are complete in the sense that

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}$ denotes the Hilbert space of functions on $\mathbf{S}^k$ which are square integrable with respect to the weight (6) and $\mathcal{H}_n = \mathcal{H}_n^{k+1}$ is the finite dimensional vector space of generalized spherical harmonics of degree n . If P_n denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_n$, then every function $f \in \mathcal{H}$ has the Fourier expansion

$$f \sim \sum_{n=0}^{\infty} P_n f. \quad (7)$$

The paper is organized as follows. The study of generalized spherical harmonics is discussed in Section 2. We introduce the generalized Laplace-Beltrami operator in Section 3. In Section 4, we prove the uniform convergence of the expansion (7) when f is sufficiently smooth. Our analysis is based on an estimate for the kernel of the integral operator P_n that is given in Xu [15]. In sections 5, 6 and 7, we investigate interior Dirichlet problems for equation (1) on the k -dimensional sphere $\mathbf{S}^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : \sum_{i=0}^k x_i^2 = 1\}$ by infinite series expansions in generalized spherical harmonics. We show that such problems have a uniquely determined solution which can be found by series expansion starting with the expansion (7) of the boundary function. We provide an error estimate for the remainder term in the series expansion of a solution of the Dirichlet problem in Section 7. Finally, we investigate the exterior Dirichlet problems: its existence, uniqueness and error estimate in Section 8.

2. Generalized Spherical Harmonics. Let $\mathcal{P}_n^{k+1}$ denote the space of homogeneous polynomials of degree n in $k+1$ variables. Its dimension (see [8]) is

$$\dim \mathcal{P}_n^{k+1} = \binom{n+k}{n}. \quad (8)$$

Let $\mathcal{H}_n^{k+1} = \mathcal{H}_n^{k+1}(\rho)$, $\rho = (\rho_0, \dots, \rho_k)$, denote the space of generalized spherical harmonics of degree n , that is, homogeneous polynomials of degree n in $k+1$ variables $x_0, \dots, x_k$ that solve (1). Its dimension (see [15]) is

$$\dim \mathcal{H}_n^{k+1} = \binom{n+k-1}{k-1} + \binom{n+k-2}{k-1}. \quad (9)$$

The proof of this formula is similar to the proof of the theorem given in [8]. Let d be the weight function

$$d(x) = D|x_0|^{2\rho_0-1} \cdots |x_k|^{2\rho_k-1}, \quad x = (x_0, \dots, x_k) \in \mathbf{S}^k, \quad (10)$$

where the constant D is determined in such a way that the integral of $d(x)$ over $\mathbf{S}^k$ is 1. Let $L_d^2(\mathbf{S}^k)$ be the corresponding Hilbert space of square integrable functions with respect to this weight. Let $\langle f, g \rangle_d$ and $\|f\|_d$ be the notations for the inner product and norm in $L_d^2(\mathbf{S}^k)$, respectively. It is shown in [5] that if $m \neq n$, the spaces $\mathcal{H}_n^{k+1}$ and $\mathcal{H}_m^{k+1}$ are orthogonal over $\mathbf{S}^k$ with respect to the weight function (10).

The spaces $\mathcal{H}_n^{k+1}(\rho)$ depend on the parameters ρ . However, the following lemma shows that the direct sum of the spaces $\mathcal{H}_n^{k+1}(\rho)$ restricted to $\mathbf{S}^k$ for $n = 0, \dots, N$ does not depend on ρ .

Lemma 2.1. *Let $\mathcal{Q}_N^{k+1}$ be the space of all functions defined on the sphere $\mathbf{S}^k$ that are restrictions of polynomials in $k+1$ variables of degree at most N . Then $\mathcal{Q}_N^{k+1}$ is the direct sum of the spaces $\mathcal{H}_n^{k+1}(\rho)$ restricted to $\mathbf{S}^k$ for $n = 0, \dots, N$.*

Proof. Let $\overline{\mathcal{P}}_N^{k+1}$ be the space of all polynomials of degree at most N . The dimension of this space is

$$\dim \overline{\mathcal{P}}_N^{k+1} = \sum_{n=0}^N \binom{n+k}{k} = \binom{N+k+1}{k+1}.$$

Consider the map R that restricts polynomials in $\overline{\mathcal{P}}_N^{k+1}$ to functions defined on $\mathbf{S}^k$. The map R is injective for $f(x) = \|x\|^n Rf\left(\frac{x}{\|x\|}\right)$ and $Rf = 0$ implies $f = 0$. Every polynomial p can be written in the form

$$p(x) = (x_0^2 + \cdots + x_k^2 - 1)q(x) + x_0q_1(x_1, \dots, x_k) + q_0(x_1, \dots, x_k),$$

where q_0, q_1, q are polynomials. If p is in the kernel of R , then $x_0q_1 + q_0 = 0$ on $\mathbf{S}^k$. Since $x \in \mathbf{S}^k$ implies $\sigma_0 x \in \mathbf{S}^k$, it follows that $q_0 = q_1 = 0$ on the unit ball in $\mathbb{R}^k$. This implies that $q_0 = q_1 = 0$ identically. Hence the kernel of R has dimension

$$\dim(\ker R) = \dim \overline{\mathcal{P}}_{N-2}^{k+1} = \binom{N+k-1}{k+1}.$$

The range of R is $\mathcal{Q}_N^{k+1}$ and has dimension

$$\dim R(\mathcal{P}_N^{k+1}) = \binom{N+k+1}{k+1} - \binom{N+k-1}{k+1}.$$

This agrees with the sum of the dimensions of the spaces $\mathcal{H}_n^{k+1}$, $n = 0, \dots, N$, because

$$\sum_{n=0}^N \binom{n+k-1}{k-1} + \sum_{n=0}^N \binom{n+k-2}{k-1} = \binom{N+k+1}{k+1} - \binom{N+k-1}{k+1}. \quad \square$$

By Lemma 2.1, the linear hull $\mathcal{M}$ of the spaces $\mathcal{H}_n^{k+1}$, $n \in \mathbf{N}_0$, contains all functions on $\mathbf{S}^k$ which are restrictions of polynomials in $x_0, \dots, x_k$. Therefore, every continuous function on $\mathbf{S}^k$ can be uniformly approximated by functions in $\mathcal{M}$. This implies that $\mathcal{M}$ is dense in $L_d^2(\mathbf{S}^k)$. Hence we can write

$$L_d^2(\mathbf{S}^k) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^{k+1},$$

where we consider $\mathcal{H}_n^{k+1}$ as a linear subspace of $L_d^2(\mathbf{S}^k)$. Let P_n be the orthogonal projection of $L_d^2(\mathbf{S}^k)$ onto $\mathcal{H}_n^{k+1}$. Every function $f \in L_d^2(\mathbf{S}^k)$ can be expanded in an orthogonal series

$$f = \sum_{n=0}^{\infty} P_n f. \quad (11)$$

The projection P_n can be written as an integral operator

$$P_n f(x) = \int_{\mathbf{S}^k} d(y) L_n(x, y) f(y) dy, \quad x \in \mathbf{S}^k, \quad (12)$$

where the kernel L_n is given by $L_n(x, y) = \sum_{j=1}^N h_j(x) \overline{h_j(y)}$ for $x, y \in \mathbf{S}^k$, and $h_1, \dots, h_N$ is an orthonormal basis of $\mathcal{H}_n^{k+1}$. In [15], Theorem 3.3 gives a formula for the kernel L_n (denoted by P_n in [15]) which leads to the estimate

$$|L_n(x, y)| \leq \frac{n+\lambda}{\lambda} \binom{n+2\lambda-1}{n}, \quad x, y \in \mathbf{S}^k, \quad (13)$$

where $\lambda = \rho_0 + \dots + \rho_k - 1$.

Lemma 2.2. *Let $h \in \mathcal{H}_n^{k+1}$. Then*

$$|h(x)|^2 \leq \frac{n+\lambda}{\lambda} \binom{n+2\lambda-1}{n} \|h\|_d^2, \quad x \in \mathbf{S}^k \quad (14)$$

where $\|h\|_d$ is the norm of h in $L_d^2(\mathbf{S}^k)$.

Proof. Let $h_1, \dots, h_N$ be an orthonormal basis of $\mathcal{H}_n^{k+1}$ for a given n . Then

$$L_n(x, y) = \sum_{m=1}^N h_m(x) \overline{h_m(y)}$$

and so

$$L_n(x, x) = \sum_{m=1}^N |h_m(x)|^2. \quad (15)$$

Now write $h(x) = \sum_{m=1}^N c_m h_m(x)$. Then, by the Cauchy-Schwarz inequality,

$$|h(x)|^2 \leq \sum_{m=1}^N |h_m(x)|^2 \sum_{m=1}^N |c_m|^2.$$

Using (13) and (15), we obtain the claimed inequality. $\square$

In the following, we prove that the expansion (11) converges uniformly on $\mathbf{S}^k$ provided that f is smooth enough. First, we will show that generalized spherical harmonics are eigenfunctions of the generalized Laplace-Beltrami operator.

3. The Generalized Laplace-Beltrami Operator. We define an operator $\Delta_{\rho,S}: C^2(\mathbf{S}^k) \rightarrow C^0(\mathbf{S}^k)$ which generalizes the classical Laplace-Beltrami operator Δ_S . Let $f \in C^2(\mathbf{S}^k)$. Then, for $0 \neq x \in \mathbb{R}^{k+1}$,

$$F(x) = f\left(\frac{x}{|x|}\right), \quad (16)$$

is twice continuously differentiable, and $\Delta_{\rho,S}f$ is defined as the restriction of $\Delta_{\rho}F$ to $\mathbf{S}^k$. Note that $\Delta_{\rho,S}$ maps $C^m(\mathbf{S}^k)$ to $C^{m-2}(\mathbf{S}^k)$. If g is differentiable at a point $0 \neq x \in \mathbb{R}^{k+1}$, the radial derivative $\partial_r g$ at x is defined as

$$\partial_r g(x) = \frac{1}{|x|} \left(\frac{d}{dx} g(tx) \right) \Big|_{t=1}.$$

Let $f(x)$ be twice continuously differentiable in $R_1 < |x| < R_2$ with $R_1 < 1 < R_2$. Let F be defined by (16). Let $\Delta_S f$ be defined as the restriction of ΔF to $\mathbf{S}^k$. Then it can be shown by direct calculation that

$$\Delta_S f = \Delta f - k \partial_r f - \partial_r^2 f \text{ on } \mathbf{S}^k. \quad (17)$$

From equation (1), we write the Dunkl operator as follows.

$$\Delta_{\rho} f(x) = \Delta f(x) + \sum_{j=0}^k \frac{2\rho_j - 1}{x_j} \partial_j f(x) - \sum_{j=0}^k \frac{\rho_j - \frac{1}{2}}{x_j^2} (f(x) - f(\sigma_j x)). \quad (18)$$

Using (18) and (17), a short calculation gives

$$\Delta_{\rho,S} f = \Delta_{\rho} f - (2|\rho|_1 - 1) \partial_r f - \partial_r^2 f \text{ on } \mathbf{S}^k, \quad (19)$$

where

$$|\rho|_1 = \rho_0 + \rho_1 + \cdots + \rho_k. \quad (20)$$

If, in addition, f is homogeneous of degree $\gamma \in \mathbb{R}$ in $\mathbb{R}^{k+1} - \{0\}$, then (19) becomes

$$\Delta_{\rho,S} f = \Delta_{\rho} f - \gamma(\gamma + 2|\rho|_1 - 2)f \text{ on } \mathbf{S}^k. \quad (21)$$

We obtain that generalized spherical harmonics are eigenfunctions of the generalized Laplace-Beltrami operator $\Delta_{\rho,S}$:

Theorem 3.1. *If f_n is a generalized spherical harmonic of degree n , then*

$$-\Delta_{\rho,S} f_n = n(n + 2|\rho|_1 - 2) f_n \text{ on } \mathbf{S}^k.$$

In particular, if f_n is a classical spherical harmonic of degree n , we obtain the following well known result: f_n is an eigenfunction of the Laplace-Beltrami operator Δ_S ,

$$-\Delta_S f_n = n(n + k - 1) f_n. \quad (22)$$

The operator $\Delta_{\rho,S}$ is symmetric in $L_d^2(\mathbf{S}^k)$:

$$\langle \Delta_{\rho,S} f, g \rangle_d = \langle f, \Delta_{\rho,S} g \rangle_d \text{ for } f, g \in C^2(\mathbf{S}^k). \quad (23)$$

This can be proven by using a special form of Green's identity for ddx [5].

4. Uniform Convergence of the Expansion in Generalized Spherical Harmonics. We use Theorem 3.1 to prove the following proposition.

Proposition 4.1. *Let $f \in C^\ell(\mathbf{S}^k)$, where ℓ is an even positive integer. Let $f \sim \sum_{n=0}^\infty f_n$ be the expansion of f in generalized spherical harmonics. Then, for all $N \in \mathbf{N}_0$,*

$$\left\| f - \sum_{n=0}^N f_n \right\|_d \leq \frac{1}{((N+1)(N+2|\rho|_1-1))^{\ell/2}} \left\| \Delta_{\rho,S}^{\ell/2} f \right\|_d.$$

Proof. Using symmetry of $\Delta_{\rho,S}$ and Theorem 3.1, we obtain

$$\langle \Delta_{\rho,S} f, f_n \rangle_d = \langle f, \Delta_{\rho,S} f_n \rangle_d = -n(n + 2|\rho|_1 - 2) \langle f, f_n \rangle_d.$$

Let $g = \Delta_{\rho,S}^{\ell/2} f$. It follows that

$$\langle g, f_n \rangle_d = (-n(n + 2|\rho|_1 - 2))^{\ell/2} \langle f, f_n \rangle_d.$$

Note that $\langle f, f_n \rangle_d = \langle f_n, f_n \rangle_d$. Hence, with $e_n = f_n / \|f_n\|_d$ (if $f_n \neq 0$),

$$\|f_n\|_d = \frac{1}{(n(n + 2|\rho|_1 - 2))^{\ell/2}} |\langle g, e_n \rangle_d|.$$

Since $\{e_n\}$ is an orthonormal system, Bessel's inequality gives

$$\left\| f - \sum_{n=0}^N f_n \right\|_d^2 = \sum_{n=N+1}^\infty \|f_n\|_d^2 \leq \frac{1}{((N+1)(N+2|\rho|_1-1))^\ell} \|g\|_d^2.$$

This is the claimed inequality. $\square$

We now prove uniform convergence for the expansion (11) if f is smooth enough.

Theorem 4.1. Let $f \in C^\ell(\mathbf{S}^k)$, where ℓ is an even positive integer with

$$\ell > \rho_0 + \cdots + \rho_k. \quad (24)$$

Then the expansion (11) of f in generalized spherical harmonics converges uniformly on $\mathbf{S}^k$.

Proof. Consider the partial sums $q_N = \sum_{n=0}^N P_n f$. By Proposition 4.1, the L_d^2 -norm of the remainder term is estimated by

$$\|f - q_N\|_d \leq K_1(N+1)^{-\ell}, \quad (25)$$

where K_1 is a constant independent of N . Every generalized spherical harmonic of degree $N+1$ is orthogonal to q_N with respect to the weight (10). Hence, using (12),

$$\begin{aligned} P_{N+1}f(x) &= \int_{\mathbf{S}^k} d(y)f(y)L_{N+1}(x,y) \, dy \\ &= \int_{\mathbf{S}^k} d(y)(f(y) - q_N(y))L_{N+1}(x,y) \, dy. \end{aligned}$$

Now (25) and the Cauchy-Schwarz inequality give

$$|P_{N+1}f(x)|^2 \leq K_2(N+1)^{-2\ell} \int_{\mathbf{S}^k} d(y)|L_{N+1}(x,y)|^2 \, dy.$$

It is easy to see that

$$\int_{\mathbf{S}^k} d(y)|L_n(x,y)|^2 \, dy = L_n(x,x).$$

By (13),

$$|L_n(x,x)| \leq K_3 n^{2\rho_0 + \cdots + 2\rho_k - 2},$$

where the constant K_3 is independent of n and $x \in \mathbf{S}^k$. Combining our estimates, we obtain

$$|P_{N+1}f(x)| \leq K_4(N+1)^{-\ell-1+\rho_0+\cdots+\rho_k}.$$

By assumption (24), this implies uniform convergence of the expansion (11). $\square$

5. The Dirichlet Problem for the Unit Ball: Existence. We consider the following Dirichlet problem (boundary value problem) in the unit ball $\mathcal{B}$. For a given function $f \in L_d^2(\mathbf{S}^k)$, find a function $u \in C^2(\mathcal{B})$ that solves

$$\Delta_\rho u = 0 \text{ in } \mathcal{B}, \quad (26)$$

and that assumes the boundary value $f(x)$ on $\mathbf{S}^k$ in the sense that

$$u(tx) \rightarrow f(x) \text{ in } L_d^2(\mathbf{S}^k) \text{ as } 1 > t \rightarrow 1. \quad (27)$$

Such a function u will be called a solution of the Dirichlet problem (26) and (27). In the following theorem, we show that a solution of the Dirichlet problem exists and can be found by expansion of f in a series of generalized spherical harmonics. The following lemma of general nature will be useful in the proof. Let ∇ denote the gradient.

Lemma 5.1. *Let $g(x_0, \dots, x_k)$ be a real homogeneous polynomial of degree n . Then, for every $x \in \mathbf{S}^k$,*

$$|\nabla g(x)| \leq n \|g\|_\infty, \text{ where } \|g\|_\infty = \max\{|g(y)| : y \in \mathbf{S}^k\}.$$

Proof. The proof follows from a direct application of an orthogonal transformation and from the Bernstein-Szegö inequality (see Theorem 5.1.3 in [1]). $\square$

Theorem 5.1. *Let $f \in L_d^2(\mathbf{S}^k)$, and let f be expanded in a series of generalized spherical harmonics f_n of degree n :*

$$f \sim \sum_{n=0}^{\infty} f_n \text{ in } L_d^2(\mathbf{S}^k).$$

Let $\lambda = |\rho|_1 - 1$. Then $|f_n(x)| \leq c_n R^n$ for $|x| \leq R < 1$, where

$$c_n = \sqrt{\frac{n+\lambda}{\lambda} \binom{n+2\lambda-1}{n}}, \quad (28)$$

and

$$u(x) = \sum_{n=0}^{\infty} f_n(x), \quad x \in \mathcal{B}, \quad (29)$$

is a solution of the Dirichlet problem (26) and (27).

Proof. By Lemma 2.2,

$$|f_n(x)| \leq c_n \|f_n\|_d \text{ for } x \in \mathbf{S}^k, \quad (30)$$

where the sequence $c_n = \sqrt{\frac{\lambda+n}{\lambda} \binom{n+2\lambda-1}{n}}$ has polynomial growth in n .

If $|x| \leq R < 1$, then it follows that

$$|f_n(x)| \leq c_n R^n \|f_n\|_d. \quad (31)$$

By orthogonality of the sequence $\{f_n\}$, $\|f_n\|_d \leq \|f\|_d$. Hence, the series $\sum_n f_n(x)$ converges for $x \in \mathcal{B}$. Using Lemma 5.1, we have

$$|\partial_j f_n(x)| \leq n \|f_n\|_\infty \text{ for } x \in \mathbf{S}^k, \quad j = 0, \dots, k. \quad (32)$$

We assumed that f and so f_n is real-valued without loss of generality. It follows that $|\partial_j f_n(x)| \leq n c_n R^{n-1} \|f_n\|_\infty$ for $|x| \leq R$. Hence the series $\sum_n \partial_j f_n(x)$ converges uniformly in every compact subset of $\mathcal{B}$. Therefore, u is continuously differentiable and $\partial_j u = \sum_n \partial_j f_n$ for every j . By (32),

$$\left| \frac{f_n(x) - f_n(\sigma_j x)}{x_j} \right| \leq 2n \|f_n\|_\infty \text{ for } x \in \mathbf{S}^k. \quad (33)$$

This inequality shows that the series $\sum_n \mathcal{D}_j f_n(x)$ is also uniformly convergent on every compact subset of $\mathcal{B}$ and $\mathcal{D}_j u = \sum_{n=0}^{\infty} \mathcal{D}_j f_n$. Repeating this argument, we obtain that u is twice continuously differentiable in $\mathcal{B}$, and

$$\Delta_\rho u = \sum_{n=0}^{\infty} \Delta_\rho f_n = 0.$$

It remains to show that u satisfies the boundary conditions (27). For $x \in \mathbf{S}^k$ and $0 < t < 1$, let

$$u_t(x) = u(tx) = \sum_{n=0}^{\infty} t^n f_n(x).$$

Then $u_t - f \sim \sum_{n=0}^{\infty} (t^n - 1) f_n$, and so, by Parseval's equality,

$$\|u_t - f\|_d^2 = \sum_{n=0}^{\infty} (t^n - 1)^2 \|f_n\|_d^2. \quad (34)$$

Let $\varepsilon > 0$ be given. Choose $N \in \mathbf{N}_0$ such that $\sum_{n=N+1}^{\infty} \|f_n\|_d^2 < \varepsilon/2$. Then, by (34), $\|u_t - f\|_d^2 < \sum_{n=0}^N (t^n - 1)^2 \|f_n\|_d^2 + \varepsilon/2$. If $t < 1$ is close enough to 1, we obtain that $\|u_t - f\|_d^2 < \varepsilon$. Hence $u_t \rightarrow f$ as $t \rightarrow 1$ in $L_d^2(\mathbf{S}^k)$. $\square$

6. The Dirichlet Problem for the Unit Ball: Uniqueness. We first prove a series expansion of solutions of $\Delta_\rho u = 0$.

Theorem 6.1. *Let $u \in C^2(B_R)$ satisfy $\Delta_\rho u = 0$. Then there are generalized spherical harmonics f_n of degree n such that*

$$u(x) = \sum_{n=0}^{\infty} f_n(x), \quad x \in B_R,$$

where the infinite series $\sum_n f_n$ converges uniformly in every compact subset of B_R .

Proof. For $0 \leq t < R$ and $x \in \mathbf{S}^k$, define $u_t(x) = u(tx)$. Since $\Delta_\rho u = 0$, we have that $\Delta_\rho u_t = 0$. Hence (19) implies that

$$\Delta_{\rho,S} u_t = -(2|\rho|_1 - 1) \partial_r u_t - \partial_r^2 u_t \quad (35)$$

on the unit sphere. For every n , choose an orthonormal basis $\{f_{nm}\}_m$ for $\mathcal{H}_n^{k+1}$ with respect to the inner product $\langle \cdot, \cdot \rangle_d$. Define $c_{nm}(t) = \langle u_t, f_{nm} \rangle_d$. Then, by Theorem 3.1, (23) and (35),

$$\begin{aligned} n(n-2+2|\rho|_1) c_{nm}(t) &= -\langle u_t, \Delta_{\rho,S} f_{nm} \rangle_d \\ &= -\langle \Delta_{\rho,S} u_t, f_{nm} \rangle_d \\ &= (2|\rho|_1 - 1) \langle \partial_r u_t, f_{nm} \rangle_d + \langle \partial_r^2 u_t, f_{nm} \rangle_d. \end{aligned}$$

Since

$$\partial_r u_t = \left(\frac{d}{ds} u_{st} \right) \Big|_{s=1} \quad \text{on } \mathbf{S}^k,$$

interchanging integration and differentiation with respect to the parameter s gives

$$\langle \partial_r u_t, f_{nm} \rangle_d = \left(\frac{d}{ds} \langle u_{st}, f_{nm} \rangle_d \right) \Big|_{s=1} = t c'_{nm}(t).$$

Similarly, $\langle \partial_r^2 u_t, f_{nm} \rangle_d = t^2 c''_{nm}(t)$. Hence

$$n(n-2+2|\rho|_1)c_{nm}(t) = (2|\rho|_1-1)tc'_{nm}(t) + t^2c''_{nm}(t).$$

If $n \neq 2 - n - 2|\rho|_1$, then this differential equation for c_{nm} has the solution

$$c_{nm}(t) = a_{nm}t^n + b_{nm}t^{2-n-2|\rho|_1}$$

with constants a_{nm}, b_{nm} . The function $c_{nm}(t)$ is twice continuously differentiable on $[0, R)$. Since $|\rho|_1 > 0$, this implies that $b_{nm} = 0$ (in the case $n = 0$ we also used that $c'_{00}(0) = 0$). Hence

$$u(tx) \sim \sum_{n=0}^{\infty} \sum_m c_{nm}(t) f_{nm}(x) = \sum_{n=0}^{\infty} \sum_m a_{nm} t^n f_{nm}(x) \text{ in } L_d^2(\mathbf{S}^k)$$

for $0 \leq t < R$. Now we obtain the desired statement with

$$f_n = \sum_m a_{nm} f_{nm}.$$

The uniform convergence of the series $\sum_{n=0}^{\infty} f_n$ is shown as in the proof of Theorem 5.1. $\square$

We are now in a position to prove uniqueness of the solution of the Dirichlet problem (26) and (27).

Theorem 6.2. *For every $f \in L_d^2(\mathbf{S}^k)$, the solution u of the Dirichlet problem (26) and (27) is uniquely determined.*

Proof. Let u, v be solutions of (26) and (27). Set $z = u - v$. Then $z_t(x) = z(tx) \rightarrow 0$ in $L_d^2(\mathbf{S}^k)$ as $1 > t \rightarrow 1$. By Theorem 6.1, there are $f_n \in \mathcal{H}_n^{k+1}$ such that $z_t = \sum_{n=0}^{\infty} t^n f_n$ in $L_d^2(\mathbf{S}^k)$. Now as $1 > t \rightarrow 1$,

$$t^n \langle f_n, f_n \rangle_d = \langle z_t, f_n \rangle_d \rightarrow 0 \text{ as } 1 > t \rightarrow 1.$$

Hence $f_n = 0$ for all n . It follows that $z = 0$ and so $u = v$. $\square$

7. The Dirichlet Problem for the Unit Ball: Error Estimates. We have the following error estimate for the computation of the solution of the Dirichlet problem (26) and (27).

Theorem 7.1. *Let $f \in L_d^2(\mathbf{S}^k)$ be expanded in generalized spherical harmonics according to (11). Let $N \in \mathbf{N}_0$. Then*

$$\sum_{n=N}^{\infty} |P_n f(x)| \leq (1+R)^{\frac{1}{2}} (1-R)^{-\lambda-1} R^{\frac{N}{2}} \|f\|_d,$$

where $|x| = R < 1$ and $\lambda = |\rho|_1 - 1$.

Proof. For abbreviation write $b_n = c_n^2$ where c_n is given in (28). Then

$$b_n = \frac{n + \lambda}{\lambda} \binom{n + 2\lambda - 1}{n} = \binom{n + 2\lambda}{n} + \binom{n + 2\lambda - 1}{n - 1}.$$

Note that

$$\sum_{n=0}^{\infty} b_n t^n = (1 + t)(1 - t)^{-2\lambda - 1}, \quad -1 < t < 1.$$

Then, by Lemma 2.2, $|P_n f(x)| \leq b_n^{1/2} \|P_n f\|_d \leq b_n^{1/2} \|f\|_d$, $x \in \mathbf{S}^k$.

Since $P_n f$ is homogeneous of degree n , we obtain, for $x \in \mathbb{R}^{k+1}$ with euclidian norm $R < 1$,

$$\begin{aligned} \left(\sum_{n=N}^{\infty} |P_n f(x)| \right)^2 &\leq \left(\sum_{n=N}^{\infty} b_n^{1/2} R^n \|f\|_d \right)^2 \\ &\leq \|f\|_d^2 \sum_{n=N}^{\infty} b_n R^n \sum_{n=N}^{\infty} R^n \\ &\leq \|f\|_d^2 (1 + R)(1 - R)^{-2\lambda - 1} R^N (1 - R)^{-1}. \end{aligned}$$

This yields the desired inequality. $\square$

8. The Exterior Dirichlet Problem. We consider the following exterior Dirichlet problem in $\mathcal{B}_e = \mathbb{R}^{k+1} - \overline{\mathcal{B}}$ where $\mathcal{B}$ is the open unit ball. For a given function $f \in L_d^2(\mathbf{S}^k)$, find a function $u \in C^2(\mathcal{B}_e)$ that solves

$$\Delta_\rho u = 0 \text{ in } \mathcal{B}_e \tag{36}$$

and that assumes the boundary value $f(x)$ on $\mathbf{S}^k$ in the sense that

$$u(tx) \rightarrow f(x) \text{ in } L_d^2(\mathbf{S}^k) \text{ as } 1 < t \rightarrow 1. \tag{37}$$

We require in addition that u be bounded: $|u| \leq M$ for some constant M . Such a function u will be called a solution of the exterior Dirichlet problem (36) and (37).

Lemma 8.1. *If $h \in \mathcal{H}_n^{k+1}$, then $|x|^{-(n+2|\rho|_1-2)} h\left(\frac{x}{|x|}\right)$, $|x| \neq 0$, solves (I).*

Proof. Let $t = n + 2|\rho|_1 - 2$. Define $g(x) = r^{-t} h(x/r)$ for $x \neq 0$. The function g is homogeneous of degree $-t$. In the following, we apply equation (21), and Theorem 3.1 and obtain, on $\mathbf{S}^k$,

$$\begin{aligned} \Delta_\rho(g) &= \Delta_{\rho,S}g + (2|\rho|_1 - 1)\partial_r g + \partial_r^2 g, \\ &= \Delta_{\rho,S}h + (2|\rho|_1 - 1)(-t)h + (-t)(-t - 1)h, \\ &= \{(t - 2|\rho|_1 + 2)(-t) + (2|\rho|_1 - 1)(-t) + (-t)(-t - 1)\}h, \\ &= 0 \text{ on } \mathbf{S}^k. \end{aligned}$$

Since $\Delta_\rho g$ is homogeneous function, this implies that $\Delta_\rho g(x) = 0$ for all $x \neq 0$. $\square$

Now we prove the following theorem.

Theorem 8.1. *Let $f \in L_d^2(\mathbf{S}^k)$, and let f be expanded in a series of generalized spherical harmonics f_n of degree n :*

$$f \sim \sum_{n=0}^{\infty} f_n \text{ in } L_d^2(\mathbf{S}^k).$$

Set $g_n(x) = r^{-(n+2|\rho|_1-2)} f_n\left(\frac{x}{r}\right)$, where $|x| = r > 0$. Then

$$|g_n(x)| \leq c_n R^{-(n+2|\rho|_1-2)} \|f\|_d, \quad 1 < R \leq |x| < \infty,$$

where $c_n = \sqrt{\frac{\lambda+n}{\lambda} \binom{n+2\lambda-1}{n}}$ with $c_n^2 = b_n$, and

$$u(x) = \sum_{n=0}^{\infty} g_n(x), \quad x \in \mathcal{B}_e \tag{38}$$

is a solution of the exterior Dirichlet problem where the series (38) is uniformly convergent on compact subsets of $\mathcal{B}_e$.

Proof. If $1 < R \leq |x| < \infty$, then it follows from Lemma 2.2 and (30) that

$$|g_n(x)| \leq c_n R^{-(n+2|\rho|_1-2)} \|f_n\|_d.$$

By orthogonality of the sequence $\{f_n\}$, $\|f_n\|_d \leq \|f\|_d$. Hence, the series $\sum_n g_n(x)$ converges for $x \in \mathcal{B}_e$. By (32), it follows that

$$|\partial_j g_n(x)| \leq K n c_n R^{-(n+2|\rho|_1-2)} \|f_n\|_{\infty}$$

for $1 < R \leq |x| < \infty$ and for a constant K . Hence the series $\sum_n \partial_j g_n(x)$ converges uniformly in every compact subset of $\mathcal{B}_e$. Therefore, u is continuously differentiable and $\partial_j u = \sum_n \partial_j g_n$ for every j . The inequality (33) shows that the series $\sum_n \mathcal{D}_j g_n(x)$ is also uniformly convergent on every compact subset of $\mathcal{B}_e$ and $\mathcal{D}_j u = \sum_{n=0}^{\infty} \mathcal{D}_j g_n$. Repeating this argument, we obtain that u is twice continuously differentiable in $\mathcal{B}_e$, and

$$\Delta_{\rho} u = \sum_{n=0}^{\infty} \Delta_{\rho} g_n = 0.$$

It remains to show that u satisfies the boundary condition (37). For $x \in \mathbf{S}^k$ and $1 < t < \infty$, let

$$u_t(x) = u(tx) = \sum_{n=0}^{\infty} t^{-(n+2|\rho|_1-2)} g_n(x).$$

Then $u_t - f \sim \sum_{n=0}^{\infty} (t^{-(n+2|\rho|_1-2)} - 1) f_n$, and so, by Parseval's equality,

$$\|u_t - f\|_d^2 = \sum_{n=0}^{\infty} (t^{-(n+2|\rho|_1-2)} - 1)^2 \|f_n\|_d^2. \tag{39}$$

Let ε be given. Choose $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \|f_n\|_d^2 < \varepsilon/2$. Then, by (39),

$$\|u_t - f\|_d^2 < \sum_{n=0}^N (t^{-(n+2|\rho|_1-2)} - 1)^2 \|f_n\|_d^2 + \frac{\varepsilon}{2}.$$

If $t > 1$ is close enough to 1, we obtain that $\|u_t - f\|_d^2 < \varepsilon$. Hence $u_t \rightarrow f$ as $t \rightarrow 1$ in $L_d^2(\mathbf{S}^k)$. $\square$

We are now in a position to prove uniqueness of the solution of the exterior Dirichlet problem (36) and (37).

Theorem 8.2. *For every $f \in L_d^2(\mathbf{S}^k)$, the solution u of the Dirichlet problem (36) and (37) is uniquely determined.*

Proof. Let u, v be solutions of (36) and (37). Set $z = u - v$. Then $z_t = z(tx) \rightarrow 0$ in $L_d^2(S^k)$ as $1 < t \rightarrow 1$. By Theorem 8.1, there are $g_n \in \mathcal{H}_n^{k+1}$ such that $z_t = \sum_{n=0}^{\infty} t^{-(n+2|\rho|_1-2)} g_n$ in $L_d^2(S^k)$. Now as $1 < t \rightarrow 1$, $t^{-(n+2|\rho|_1-2)} \langle g_n, g_n \rangle_d \langle z_t, g_n \rangle_d \rightarrow 0$ as $1 < t \rightarrow 1$. Hence $g_n = 0$ for all n . It follows that $z = 0$ and so $u = v$. $\square$

We now show that the following theorem gives the error estimate for the remainder term $\sum_{n=N}^{\infty} g_n$ of this expansion inside the region $\mathcal{B}_e$.

Theorem 8.3. *Under the same assumptions and notations as in Theorem 8.1, it follows that*

$$\sum_{n=N}^{\infty} |g_n(x)| \leq (1 + R^{-1})^{\frac{1}{2}} (1 - R^{-1})^{-\lambda-1} R^{-\frac{N}{2}} R^{-2|\rho|_1+2} \|f\|_d,$$

where $|x| = R > 1$.

Proof. Note that $g_n(x) = R^{-2n-2|\rho|_1+2} f_n(x)$ if $|x| = R$. Then

$$\begin{aligned} \sum_{n=N}^{\infty} |g_n(x)| &= \sum_{n=N}^{\infty} R^{-2n-2|\rho|_1+2} |f_n(x)|, \\ &\leq \sum_{n=N}^{\infty} R^{-2n-2|\rho|_1+2} c_n \|f\|_d R^n, \\ &= R^{-2|\rho|_1+2} \|f\|_d \sum_{n=N}^{\infty} R^{-n} c_n, \\ &\leq R^{-2|\rho|_1+2} \|f\|_d (1 + R^{-1})^{\frac{1}{2}} (1 - R^{-1})^{-\lambda-1} R^{-\frac{N}{2}}. \quad \square \end{aligned}$$

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Résumé substantiel en français. Nous investiguons les problèmes intérieurs et extérieurs de Dirichlet pour l'équation de Dunkl sur la sphère de dimension k . Il est démontré que la solution est unique et peut être exprimée en séries sphériques harmoniques

généralisées. Un outil essentiel dans la preuve est l'estimation de kernels des espaces des harmoniques sphériques généralisées de Xu [15].

En résumé, cet article est organisé de la manière suivante. L'étude de fonctions harmoniques sphériques généralisées est présentée dans la deuxième partie de l'article. Nous avons introduit l'opérateur généralisé de Laplace-Beltrami dans la troisième partie. Dans la quatrième partie, nous avons démontré la convergence uniforme de l'expansion (7) quand f est suffisamment régulière. Notre analyse est basée sur l'estimation de kernel de l'opérateur intégral P_n de Xu [15]. Dans les cinquième, sixième et septième parties, nous avons investigué les problèmes intérieurs de Dirichlet pour l'équation (1) sur la sphère de dimension k du développement en séries infinies de polynômes harmoniques sphériques généralisées.

Nous avons démontré que les solutions de ces problèmes sont uniquement déterminées et peuvent être trouvées par des expansions de séries commençant par le développement (7) de conditions aux limites.

Nous avons donné l'estimation de l'erreur de reste exprimée en expansions de séries de la solution de problème de Dirichlet dans la septième partie.

Finalement, dans la huitième partie, nous avons investigué le problème extérieur de Dirichlet, son existence, son unicité et l'estimation de l'erreur.

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