

**LINEAR QUADRATIC OPTIMAL CONTROL OF  
LINEAR TIME INVARIANT SYSTEMS WITH DELAYS  
IN STATE, CONTROL AND OBSERVATION VARIABLES**

JAMILA KARRAKCHOU

RÉSUMÉ. Cet article traite du problème de contrôle linéaire quadratique associé à un système mathématique qui se trouve un bon modèle pour une famille assez large de systèmes héréditaires. L'approche HUM, basée sur la construction d'une structure hilbertienne et l'inversion d'un isomorphisme, est utilisée et le contrôle optimal est caractérisé par la résolution d'une équation algébrique dans un espace approprié.

ABSTRACT. The paper deals with the problem of linear quadratic optimal control associated to a coupled mathematical system which provides a good model for an important families of linear time-invariant hereditary systems. The HUM approach based on the construction of an appropriate Hilbert space and the inversion of an isomorphism is used and the optimal control is characterized by the solution of an algebraic equation.

**1. Introduction.** In this paper we study the quadratic optimal control problem associated to a simple mathematical system which provides a good model for important families of linear time-invariant hereditary systems. It accounts for families of delay differential equations, integro-differential equations, integral equations, functional differential equations of retarded and neutral types, difference equations, *etc.* This model has been studied in details by Delfour *et al.* [1], and Delfour and Karakchou [6], and a complete theory was given. It is shown that of the more important particular cases of the considered model are the system of neutral type and the system with delay on state, control and observation.

The system considered is

$$\begin{cases} Mx_t - Ny_t = B_0u_t, & y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = B_1u_t, & y(0) = \phi^0, u_0 = \mu, \end{cases}$$

and the problem is to find a control  $u(\cdot)$  in  $L^2(0, T; \mathbb{R}^m)$  minimizing the functional

$$J(u) = Gy(T) \cdot y(T) + \int_0^T \{Qy(t) \cdot y(t) + Cx(t) \cdot x(t) + Ru(t) \cdot u(t)\} dt,$$

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with the classical hypothesis on  $G, Q, C$ , and  $R$ . In our knowledge, this problem has never been completely solved. Using a state space formulation, it is possible to construct a state in a product space and put the system considered in the form of an evolution equation, without delays, in an infinite dimensional Banach space to bring such systems in line with the standard control theory of an infinite dimensional evolution systems. The presence of pure delays in the control operator makes it an unbounded operator (in the state space formulation). This creates new difficulties in the modelling and the solution of this linear quadratic problem. Another difficulty is the presence of  $x(t)$  in the cost functional which makes impossible to characterize the cost functional with the state considered. The problem has been studied by Karrakchou [8] in the particular case  $H = 0, G = 0$ , and  $C = 0$ . The control is given by the resolution of some Riccati equation. In [5], Delfour has studied the problem for delay system with delay on the control. It is shown that the decoupling operator is a linear continuous transformation on the product space considered, but this is not sufficient to obtain a Riccati differential equation, and it is necessary to embed the optimal control problem to a larger family with initial data and right-hand side in a “bigger” space.

The object of this paper is to study the problem in the more general form given below. Classical techniques allowed us to show that the solution of the problem exists, is unique, and can be characterized as a feedback law determined by an optimality system. For decoupling this system we use techniques similar to HUM (Hilbert Uniqueness Method) based on the construction of an appropriate Hilbert space and the inversion of an isomorphism (see Lions [13]). The optimal control is then derived from the solution of an algebraic equation in an appropriate Hilbert space. The idea was used in Karrakchou and Namir [9] and Karrakchou *et al.* [3, 10, 11] for distributed systems with delay on the control in both the continuous and the discrete cases. This approach presents the advantage that classical numerical methods, such as Galerkin method, could easily be applied to approximate the solution.

**2. Preliminary results.** Given the integers  $n \geq 1, k \geq 1, m \geq 1$ , a real number  $h$  called the memory of the system,  $0 < h \leq \infty$ , and continuous linear maps

$$\begin{aligned} L : K(-h, 0; \mathbb{R}^k) &\longrightarrow \mathbb{R}^n, & M : K(-h, 0; \mathbb{R}^k) &\longrightarrow \mathbb{R}^k, \\ N : K(-h, 0; \mathbb{R}^n) &\longrightarrow \mathbb{R}^k, & H : K(-h, 0; \mathbb{R}^n) &\longrightarrow \mathbb{R}^n, \\ B_0 : K(-h, 0; \mathbb{R}^m) &\longrightarrow \mathbb{R}^n, & B_1 : K(-h, 0; \mathbb{R}^m) &\longrightarrow \mathbb{R}^k, \end{aligned}$$

where

$$K(-h, 0; \mathbb{R}^n) = \begin{cases} C(-h, 0; \mathbb{R}^n), & \text{if } h < +\infty, \\ C_0(-h, 0; \mathbb{R}^n), & \text{if } h = +\infty. \end{cases}$$

We shall consider the following coupled system of equations

$$\begin{cases} Mx_t - Ny_t = g(t), & y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = f(t), & y(0) = \phi^0, \end{cases} \quad (1)$$

where

$$\begin{aligned} g : [0, \infty[ &\longrightarrow \mathbb{R}^k, & f : [0, \infty[ &\longrightarrow \mathbb{R}^n, \\ \phi^1 : I(-h, 0) &\longrightarrow \mathbb{R}^k, & \phi^2 : I(-h, 0) &\longrightarrow \mathbb{R}^n, & \phi^0 &\in \mathbb{R}^n, \\ x : I(-h, \infty) &\longrightarrow \mathbb{R}^k, & y : I(-h, \infty) &\longrightarrow \mathbb{R}^n, \end{aligned}$$

$I(a, b) = [a, b] \cap \mathbb{R}$  and  $\forall t > 0$ , the function  $x_t$  is defined on  $I(-h, 0) = [-h, 0] \cap \mathbb{R}$  by  $x_t(\theta) = x(t + \theta)$ .

In this section we give two preliminary lemmas, the fundamental theorem of existence, uniqueness and continuity of solutions with respect to data, and we introduce the transposed system and give a very important technical lemma who allows to bind the solution of the original system to that of the adjoint one. For the proofs, see [1].

### 2.1. Preliminary lemmas.

**Lemma 1.** *Fix the length of the memory  $h$ ,  $0 < h \leq \infty$ , and the real number  $p$ ,  $1 \leq p < \infty$ .*

(i) *There exists an  $n \times k$  matrix  $\lambda$  of regular Borel measures such that*

$$L\phi = \int_{-h}^0 d_\theta \lambda \phi(\theta).$$

(ii) *Fix a real number  $T > 0$ . For each  $x$  in  $C_c(-\infty, T; \mathbb{R}^k)$ , the function  $\mathcal{L}x$  is defined as follows:*

$$\mathcal{L}x(t) = Lx_t, \quad x_t(\theta) = x(t + \theta), \quad \theta \in I(-h, 0), \quad t \in [0, T].$$

*It is continuous on  $[0, T]$  and generates the continuous linear map*

$$\mathcal{L} : C_c(-\infty, T; \mathbb{R}^k) \longrightarrow L^p(0, T; \mathbb{R}^n).$$

(iii) *The above map has a continuous linear extension from  $C_c(-\infty, T; \mathbb{R}^k)$  to  $L^p(-\infty, T; \mathbb{R}^k)$ :*

$$\mathcal{L} : L^p(-\infty, T; \mathbb{R}^k) \longrightarrow L^p(0, T; \mathbb{R}^n)$$

*such that*

$$\|\mathcal{L}x\|_{L^p(0, t)} \leq |\lambda| \cdot \|x\|_{L^p(-h, t)}, \quad \forall t \in [0, T]$$

*where  $|\lambda|$  is the total variation of the matrix  $\lambda$ .  $\square$*

Similar constructions can be undertaken for the maps  $M, N, H, B_0$  and  $B_1$ .

**Lemma 2.** *Let  $p$ ,  $1 \leq p < \infty$ , be a real number and*

$$M : K(-h, 0; \mathbb{R}^k) \longrightarrow \mathbb{R}^k$$

be the continuous linear map to which we associate the  $k \times k$  matrix  $\mu$  of regular Borel measures and the continuous map  $\mathcal{M} : L^p(-\infty, T; \mathbb{R}^k) \longrightarrow L^p(0, T; \mathbb{R}^k)$  (cf. Lemma 1).

For  $T > 0$ , define the continuous linear operator

$$\mathcal{M}e_+^0 : L^p(0, T; \mathbb{R}^k) \longrightarrow L^p(0, T; \mathbb{R}^k), \quad (\mathcal{M}e_+^0)x = \mathcal{M}(e_+^0x),$$

where  $e_+^0x$  is the extension by 0 to  $I(-\infty, T)$  of the function  $x$ .

In an analogous fashion to  $\mathcal{M}$ ,  $\forall t, 0 < t \leq T$ , define the family of continuous linear operators

$$\mathcal{M}^t : L^p(-\infty, t; \mathbb{R}^k) \longrightarrow L^p(0, t; \mathbb{R}^k),$$

(in particular  $\mathcal{M}^T = \mathcal{M}$ ).

(i) For all  $t \in ]0, T]$  and  $x$  in  $L^p(0, T; \mathbb{R}^k)$ ,

$$\mathcal{M}^t e_+^0(x|_{[0, t]}) = (\mathcal{M}e_+^0x)|_{[0, t]}.$$

(ii) If for all  $t \in ]0, T]$ ,  $\mathcal{M}^t e_+^0$  is an isomorphism, then there exists a constant  $C_T$  such that

$$\|x\|_{L^p(0, t)} \leq C_T \|\mathcal{M}^t e_+^0x\|_{L^p(0, t)}.$$

(iii) If  $M$  has an isolated atom at the point 0, that is, if there exists a  $k \times k$  invertible matrix  $M_0$  of real numbers and a  $k \times k$  matrix  $\mu_0$  of regular real Borel measures such that

$$M\phi = M_0\phi(0) + \int_{-h}^0 d_\theta \mu \phi(\theta), \quad \forall \phi \in K(-h, 0; \mathbb{R}^k),$$

and

$$\lim_{\epsilon \rightarrow 0} |\mu_0|([-\epsilon, 0]) = 0,$$

then  $\mathcal{M}^t e_+^0$  is an isomorphism of  $L^p(0, T; \mathbb{R}^k)$  onto itself for all  $t > 0$ .  $\square$

**Remark 3.** For convenience, we shall often use the notation  $Lx_t$  instead of  $\mathcal{L}x(t)$  for  $x \in L^p(-h, T; \mathbb{R}^k)$ . It will be the same thing for the maps  $M, N, H, B_0$  and  $B_1$ .

## 2.2. Existence and uniqueness result.

**Theorem 4.** Fix a real number  $p, 1 \leq p < \infty$ , and the maps  $H, L, M$  and  $N$ . Assume that for all  $t > 0$ , the operator  $\mathcal{M}^t e_+^0$  is an isomorphism. Then

(1) for all

$$\begin{aligned} \phi &= (\phi^0, \phi^1, \phi^2) \in Z^p = \mathbb{R}^n \times L^p(-h, 0; \mathbb{R}^n) \times L^p(-h, 0; \mathbb{R}^k), \\ f &\in L_{\text{loc}}^p(0, \infty; \mathbb{R}^n) \text{ and } g \in L_{\text{loc}}^p(0, \infty; \mathbb{R}^k), \end{aligned}$$

the system of equations

$$\begin{aligned} Mx_t - Ny_t &= g(t), \quad y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t &= f(t), \quad y(0) = \phi^0, \end{aligned}$$

has a unique solution  $(x, y)$  in  $L^p_{\text{loc}}(0, \infty; \mathbb{R}^k) \times W^{1,p}_{\text{loc}}(0, \infty; \mathbb{R}^n)$ . Moreover for each  $T > 0$ , there exists a constant  $C(T) > 0$  such that

$$\|x\|_{L^p(0,T)} + \|y\|_{W^{1,p}(0,T)} \leq C(T) \left[ \|\phi\|_{Z^p} + \|g\|_{L^p(0,T)} + \|f\|_{L^p(0,T)} \right].$$

(2) For  $f = g = 0$ , for  $t \geq 0$ , define on  $Z^p$  the map

$$S(t) : Z^p \longrightarrow Z^p, \quad S(t)\phi = (y(t), y_t, x_t).$$

The family  $\{S(t) : t \geq 0\}$  is a strongly continuous semigroup of class  $C_0$  on  $Z^p$ . Its infinitesimal generator  $A$  is defined on

$$D(A) = \left\{ (\phi^0, \phi^1, \phi^2) \in Z^p \left| \begin{array}{l} \phi^1 \in W^{1,p}(-h, 0; \mathbb{R}^n) \\ \phi^2 \in W^{1,p}(-h, 0; \mathbb{R}^k) \\ M\phi^2 = N\phi^1, \phi^0 = \phi^1(0) \end{array} \right. \right\},$$

by

$$A(\phi^0, \phi^1, \phi^2) = (H\phi^1 + L\phi^2, D\phi^1, D\phi^2). \quad \square$$

### 2.3. Transposition and adjoint system.

**Definition 5.** For each continuous linear map

$$L : K(-h, 0; \mathbb{R}^k) \longrightarrow \mathbb{R}^n,$$

we define the transposed operator  $L^T$  as follows

$$L^T \phi = \int_{-h}^0 d_\theta \lambda^T \phi(\theta),$$

where  $\lambda^T$  is the transposed (in the usual sense) of the matrix  $\lambda$ .

**Definition 6.** Given  $q, 1 \leq q < \infty$ . The “transposed” system associated with system (1) is defined as follows

$$\begin{cases} M^T z_t - L^T w_t = g(t), & w_0 = \psi^1, z_0 = \psi^2, \\ \dot{w}(t) - H^T w_t - N^T z_t = f(t), & w(0) = \psi^0. \end{cases} \quad (2)$$

System (2) enjoys exactly the same properties as system (1) and Theorem 4 directly applies. When  $f = g = 0$  the solution  $(z, w)$  of system (2) also generates a strongly continuous semigroup  $\{S^T(t)\}$ .

*Remark 7.* The “transposed” semigroup  $\{S^T(t)\}$  and its generator  $A^T$  are not to be confused with the usual (topological) adjoint semigroup  $\{S^*(t)\}$  and its generator  $A^*$ . They are different, but there is an interesting intertwining relation between them (see [6]).

**Definition 8.** Fix the real number  $s, 1 < s < \infty$ . Given  $\psi = (\psi^0, \psi^1, \psi^2) \in Z^s = \mathbb{R}^n \times L^s(-h, 0; \mathbb{R}^n) \times L^s(-h, 0; \mathbb{R}^k)$ ,  $f \in L^s(0, T; \mathbb{R}^n)$  and  $g \in L^s(0, T; \mathbb{R}^k)$ . The adjoint system is defined by the system of equations

$$\begin{cases} M^T p^t - L^T q^t = g(t), & q^T = \psi^1, p^T = \psi^2, \\ -[\dot{q}(t) + H^T q^t + N^T p^t] = f(t), & q(T) = \psi^0, \end{cases} \quad (3)$$

where for any function  $y$  in  $L^s(0, T+h; \mathbb{R}^l)$  ( $l \geq 1$  an integer)

$$y^t : I(-h, 0) \longrightarrow \mathbb{R}^l$$

is defined by

$$y^t(\theta) = g(t - \theta).$$

It is readily seen that finding a solution  $(p, q)$  to (3) is equivalent to finding a solution  $(z, w)$  to the “transposed system”

$$\begin{cases} M^T z_t - L^T w_t = \tilde{g}(t), & w_0 = \psi^1, z_0 = \psi^2, \\ \dot{w}(t) - H^T w_t - N^T z_t = \tilde{f}(t), & w(0) = \psi^0, \end{cases}$$

where  $(z, w, \tilde{f}, \tilde{g})$  are related to  $(p, q, f, g)$  by the relations

$$\begin{aligned} p(t) &= z(T - t), & q(t) &= w(T - t), \\ \tilde{g}(t) &= g(T - t), & \tilde{f}(t) &= f(T - t). \end{aligned}$$

**Remark 9.** The operator  $\mathcal{L}^* : C(0, T+h; \mathbb{R}^k) \longrightarrow L^p(0, T; \mathbb{R}^n)$  defined by  $\mathcal{L}^* x(t) = L^T x^t$  has a continuous extension to  $L^p(0, T+h; \mathbb{R}^k)$ . For convenience, we will note always  $L^T y^t$  instead of  $\mathcal{L}^* x(t)$ . It is the same for maps  $M, N, H, B_0$  and  $B_1$ .

The following Lemma will play an important role in the characterization of the optimality system.

**Lemma 10.** Let  $L$  be a continuous linear map from  $K(-h, 0; \mathbb{R}^k)$  to  $\mathbb{R}^n$ ,  $p, 1 < p < \infty$ , and  $q, q^{-1} + p^{-1} = 1$ , be real numbers. Then

(1) for all  $z$  in  $L^q(0, T; \mathbb{R}^n)$  and  $x$  in  $L^p(0, T; \mathbb{R}^k)$ ,

$$\int_0^T z(T-t) \cdot (\mathcal{L} e_0^+ x)(t) dt = \int_0^T (\mathcal{L}^T e_0^+ z)(T-s) \cdot x(s) ds,$$

(2) for all  $p$  in  $L^q(0, T+h; \mathbb{R}^n)$ ,  $p^T = 0$  and  $x$  in  $L^p(-h, T; \mathbb{R}^k)$ ,  $x_0 = 0$

$$\int_0^T p(t) \cdot L x_t dt = \int_0^T L^T p^t \cdot x(t) dt. \quad \square$$

**3. Optimality system.** In all the next development, we suppose  $p = 2$  to have an Hilbert space structure. Let us consider the system

$$(S) \quad \begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = \phi^0, u_0 = \mu, \end{cases} \quad (4)$$

and the functional cost

$$J(u) = Gy(T) \cdot y(T) + \int_0^T \{Qy(t) \cdot y(t) + Cx(t) \cdot x(t) + Ru(t) \cdot u(t)\} dt, \quad (5)$$

where  $G, Q$ , and  $C$ , are self-adjoint non negative matrices and  $R$  is a coercive matrix.

The problem considered is to find a control  $u^* \in L^2(0, T; \mathbb{R}^m)$  minimizing  $J(u)$ , where  $(x, y)$  is the solution of system (S) corresponding to the initial conditions  $(\phi^0, \phi^1, \phi^2) \in Z^2, \mu \in L^2(-h, 0; \mathbb{R}^m)$  and the control function  $u \in L^2(0, T; \mathbb{R}^m)$ .

**Proposition 11.** *Let  $(\phi^0, \phi^1, \phi^2) \in Z^2 = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n) \times L^2(-h, 0; \mathbb{R}^k)$  and  $\mu \in L^2(-h, 0; \mathbb{R}^m)$ . Suppose that the operator  $M$  satisfies the hypothesis given in Theorem 4. Then the optimal control is given by the optimality system*

$$\begin{cases} (S) \begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = \phi^0, u_0 = \mu, \end{cases} \\ (S^*) \begin{cases} -M^T p^t + L^T q^t + Cx(t) = 0, & q^T = 0, p^T = 0, \\ \dot{q}(t) + H^T q^t + N^T p^t + Qy(t) = 0, & q(T) = Gy(T), \\ u^*(t) = -R^{-1}[B_0^T p^t + B_1^T q^t]. \end{cases} \end{cases}$$

*Proof.* Under the hypothesis given in Theorem 4, for all  $u \in L^2(0, T; \mathbb{R}^m)$  the solution of system (S) exists and it is unique. Classical techniques (see [4, 12, 15]) allowed us to see that the optimal control  $u^*$  is given by

$$J'(u^*)v = 0, \quad \forall v \in L^2(0, T; \mathbb{R}^m).$$

But,

$$\frac{1}{2} J'(u)v = Gy(T) \cdot \tilde{y}(T) + \int_0^T \{Qy(t) \cdot \tilde{y}(t) + Cx(t) \cdot \tilde{x}(t) + Ru(t) \cdot v(t)\} dt$$

where  $(\tilde{x}, \tilde{y})$  is the solution of the system

$$\begin{cases} M\tilde{x}_t - N\tilde{y}_t = B_0 v_t, & (\tilde{y}(0), \tilde{y}_0, \tilde{x}_0) = 0, \\ \dot{\tilde{y}}(t) - H\tilde{y}_t - L\tilde{x}_t = B_1 v_t, & v_0 = 0. \end{cases} \quad (6)$$

Introduce the adjoint system

$$\begin{cases} -M^T p^t + L^T q^t + Cx(t) = 0, & q^T = 0, p^T = 0, \\ \dot{q}(t) + H^T q^t + N^T p^t + Qy(t) = 0, & q(T) = Gy(T), \end{cases} \quad (7)$$

then  $J'(u^*)v = 0$  is equivalent to

$$\begin{aligned} Gy(T) \cdot \tilde{y}(T) - \int_0^T (\dot{q}(t) + H^T q^t + N^T p^t) \cdot \tilde{y}(t) \, dt \\ + \int_0^T (M^T p^t - L^T q^t) \cdot \tilde{x}(t) \, dt + \int_0^T Ru^*(t) \cdot v(t) \, dt = 0. \end{aligned}$$

Using integration by parts, system (6) and the part (2) of Lemma 10, we show that

$$- \int_0^T N \tilde{y}_t \cdot p(t) \, dt + \int_0^T M \tilde{x}_t \cdot p(t) \, dt + \int_0^T q(t) \cdot B_1 v_t \, dt + \int_0^T Ru^*(t) \cdot v(t) \, dt = 0.$$

Again using system (6), we have

$$\int_0^T B_0 v_t \cdot p(t) \, dt + \int_0^T q(t) \cdot B_1 v_t \, dt + \int_0^T Ru^*(t) \cdot v(t) \, dt = 0.$$

Then

$$\int_0^T [B_0^T p^t + B_1^T q^t + Ru^*(t)] \cdot v(t) \, dt = 0, \quad \forall v \in L^2(0, T; \mathbb{R}^m).$$

Finally the optimal control is given by

$$u^*(t) = -R^{-1}[B_0^T p^t + B_1^T q^t]. \quad \square$$

**4. Resolution of the problem.** The next step consists in decoupling the Hamiltonian system. For this, we will use techniques similar to the Hilbert Uniqueness Method, introduced first by Lions [13] to study the exact controllability for hyperbolic systems and generalized or “adapted” by other authors to study different concepts for different dynamical systems [2, 3, 9], *etc.* We show that introducing a new norm on the state space, the optimal control is derived from the resolution of some algebraic equation in an appropriate Hilbert space.

Let us define the space

$$F = \mathbb{R}^n \times L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^k).$$

For all  $\varphi = (\varphi^0, \varphi^1, \varphi^2) \in F$ , the system

$$\begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}} \varphi^2(t) = 0, & q^T = 0, \, p^T = 0, \\ \dot{q}(t) + H^T q^t + N^T p^t + Q^{\frac{1}{2}} \varphi^1(t) = 0, & q(T) = G^{\frac{1}{2}} \varphi^0, \end{cases} \quad (8)$$

has a unique solution  $(q, p) \in W^{1,2}(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^k)$ . It is continuous with respect to data (*cf.* Theorem 4).

Define now on  $F$  the functional  $\|\cdot\|_1$  by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}}(B_0^T p^t + B_1^T q^t) \right|^2 \, dt.$$

It is a norm on  $F$  that is equivalent to the usual one. This is due to the fact that the operators  $B_0^T$  and  $B_1^T$  are bounded on  $L^p(0, T + h; \mathbb{R}^k)$  (resp.  $L^p(0, T + h; \mathbb{R}^n)$ ), and that the solution of the transposed system (8) is continuous with respect to data. In what follows, we will see that this norm can be associated with an inner product on  $F$ , and that the space  $(F, \|\cdot\|_1)$  is a Hilbert space.

Let  $\Lambda$  be the operator defined on  $F$  by:

- 1- for  $\varphi \in F$ , we resolve the system (8)
- 2- we let

$$u(t) = R^{-1}(B_0^T p^t + B_1^T q^t), \quad (9)$$

- 3- using this control  $u(t)$ , we resolve the system with initial conditions equal to zero

$$\begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = 0, x_0 = 0, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = 0, u_0 = 0, \end{cases} \quad (10)$$

- 4- we finally take

$$\Lambda\varphi = \varphi + (G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(\cdot), C^{\frac{1}{2}}x(\cdot)), \quad (11)$$

where  $(x, y)$  is the solution of system (10). It is easy to see that the operator  $\Lambda$  is symmetric and positive definite.

**Proposition 12.** For all  $\varphi, \psi \in F$ , we have

- 1-  $\langle \Lambda\varphi, \psi \rangle = \langle \varphi, \Lambda\psi \rangle$ ,
- 2-  $\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$ ,

where  $\langle \cdot, \cdot \rangle$  is the scalar product in the space  $F$ .

*Proof.* 1- Given  $\varphi, \psi \in F$ . Let  $(y, x, p, q, u)$  be the solution of system (8), (9), (10) corresponding to  $\varphi$ , and  $(\tilde{y}, \tilde{x}, \tilde{p}, \tilde{q}, \tilde{u})$  be the solution corresponding to  $\psi$ . We have

$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \langle \varphi, \psi \rangle + \left\langle (G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(\cdot), C^{\frac{1}{2}}x(\cdot)), \psi \right\rangle \\ &= \langle \varphi, \psi \rangle + G^{\frac{1}{2}}y(T) \cdot \psi^0 + \int_0^T Q^{\frac{1}{2}}y(t) \cdot \psi^1(t) dt + \int_0^T C^{\frac{1}{2}}x(t) \cdot \psi^2(t) dt, \end{aligned}$$

and by Equation (8) applied to  $\psi$  we have

$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \langle \varphi, \psi \rangle + G^{\frac{1}{2}}y(T) \cdot \psi^0 - \int_0^T y(t) \cdot (\dot{\tilde{q}}(t) + H^T \tilde{q}^t + N^T \tilde{p}^t) dt \\ &\quad + \int_0^T x(t) \cdot (M^T \tilde{p}^t - L^T \tilde{q}^t) dt. \end{aligned}$$

By integration by parts and using Equation (10), we obtain

$$\begin{aligned} \langle \Lambda\varphi, \psi \rangle &= \langle \varphi, \psi \rangle + \int_0^T \{(Hy_t + Lx_t + B_1 u_t) \cdot \tilde{q}(t) - y(t) \cdot (H^T \tilde{q}^t + N^T \tilde{p}^t)\} dt \\ &\quad + \int_0^T x(t) \cdot (M^T \tilde{p}^t - L^T \tilde{q}^t) dt. \end{aligned}$$

Then using part (2) of Lemma 10 and the first equation of system (10), we have

$$\begin{aligned}\langle \Lambda \varphi, \psi \rangle &= \langle \varphi, \psi \rangle + \int_0^T u(t) \cdot (B_1^T \tilde{q}^t + B_0^T \tilde{p}^t) dt \\ &= \langle \varphi, \psi \rangle + \int_0^T u(t) \cdot R \tilde{u}(t) dt.\end{aligned}$$

This proves (1).

2- For  $\psi = \varphi$ , we have

$$\begin{aligned}\langle \Lambda \varphi, \varphi \rangle &= \langle \varphi, \varphi \rangle + \int_0^T u(t) \cdot R u(t) dt \\ &= \langle \varphi, \varphi \rangle + \int_0^T R^{-1} (B_0^T p^t + B_1^T q^t) \cdot (B_0^T p^t + B_1^T q^t) dt \\ &= \langle \varphi, \varphi \rangle + \int_0^T \left| R^{-\frac{1}{2}} (B_0^T p^t + B_1^T q^t) \right|^2 dt \\ &= \|\varphi\|_1^2. \quad \square\end{aligned}$$

It follows from the last proposition that  $\Lambda$  is a continuous and coercive isomorphism. Hence the main result is then given by the following theorem

**Theorem 13.** *The optimal control of the problem (4)-(5) is given by*

$$u(t) = R^{-1} (B_0^T \bar{p}^t + B_1^T \bar{q}^t),$$

where  $(\bar{q}, \bar{p})$  is given by :

1- solve the “zero-controlled” system

$$\begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = \phi^1, x_0 = \phi^2, u(t) = 0 \text{ for } t > 0, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = \phi^0, u_0 = \mu, \end{cases} \quad (12)$$

2- solve the equation

$$\Lambda \varphi = -(G^{\frac{1}{2}} y(T), Q^{\frac{1}{2}} y(\cdot), C^{\frac{1}{2}} x(\cdot)), \quad (13)$$

3- solve the adjoint system

$$\begin{cases} -M^T \bar{p}^t + L^T \bar{q}^t + C^{\frac{1}{2}} \varphi^2(t) = 0, & \bar{q}^T = 0, \bar{p}^T = 0, \\ \dot{\bar{q}}(t) + H^T \bar{q}^t + N^T \bar{p}^t + Q^{\frac{1}{2}} \varphi^1(t) = 0, & \bar{q}(T) = G^{\frac{1}{2}} \varphi^0. \end{cases} \quad (14)$$

Moreover, the optimal cost is given by

$$J(u^*) = \|\varphi\|_1^2.$$

*Proof.* By Proposition 12,  $\Lambda$  is an isomorphism on  $F$ . Then  $\varphi$  is uniquely defined.

Let  $(y(t, (\phi, \mu), u), x(t, (\phi, \mu), u))$  be the solution of system  $(S)$  corresponding to the initial condition  $(\phi, \mu)$  and the control  $u$ . By definition of  $\Lambda$  we have

$$\begin{aligned} \varphi + (G^{\frac{1}{2}}y(T, 0, u), Q^{\frac{1}{2}}y(\cdot, 0, u), C^{\frac{1}{2}}x(\cdot, 0, u)) \\ = -(G^{\frac{1}{2}}y(T, (\phi, \mu), 0), Q^{\frac{1}{2}}y(\cdot, (\phi, \mu), 0), C^{\frac{1}{2}}x(\cdot, (\phi, \mu), 0)) \end{aligned}$$

where  $u$  is given by Equation (9).

Then, by linearity

$$\varphi = -(G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(\cdot), C^{\frac{1}{2}}x(\cdot)),$$

where  $(y, x)$  is the solution of system  $(S)$ .

Equation (14) is then written

$$\begin{cases} -M^T \bar{p}^t + L^T \bar{q}^t - Cx(t) = 0, & \bar{q}^T = 0, \bar{p}^T = 0, \\ \dot{\bar{q}}(t) + H^T \bar{q}^t + N^T \bar{p}^t - Qy(t) = 0, & \bar{q}(T) = Gy(T). \end{cases}$$

By unicity of the solution we have  $(\bar{q}, \bar{p}) = -(q, p)$  with  $(q, p)$  is the solution of the adjoint system  $(S^*)$ .

Then

$$u(t) = -R^{-1}(B_0^T p^t + B_1^T q^t),$$

and the optimality system is satisfied.

Moreover,

$$\begin{aligned} J(u^*) &= G^{\frac{1}{2}}y(T) \cdot G^{\frac{1}{2}}y(T) + \int_0^T \{Q^{\frac{1}{2}}y(t) \cdot Q^{\frac{1}{2}}y(t) + C^{\frac{1}{2}}x(t) \cdot C^{\frac{1}{2}}x(t)\} dt \\ &\quad + \int_0^T Ru(t) \cdot u(t) dt \\ &= |\varphi^0|^2 + \int_0^T |\varphi^1(t)|^2 dt + \int_0^T |\varphi^2(t)|^2 dt \\ &\quad + \int_0^T (B_0^T p^t + B_1^T q^t) \cdot R^{-1}(B_0^T p^t + B_1^T q^t) dt \\ &= |\varphi|^2 + \int_0^T \left| R^{-\frac{1}{2}}(B_0^T p^t + B_1^T q^t) \right|^2 dt \\ &= |\varphi|_1^2. \end{aligned}$$

This completes the proof.  $\square$

*Remark 14.* In order to obtain the optimal control  $u^*$  one has to solve Equation (13). However, in general we do not know an explicit form of the operator  $\Lambda^{-1}$ . But since the bilinear continuous form:  $(\varphi, \psi) \in F \times F \longrightarrow (\Lambda\varphi, \psi)$  is coercive, the Galerkin method can easily be applied to approximate the solution.

**5. Particular cases.** In this section, we present some important cases of delay systems with delay on the control, delay on the control and observation, and systems of neutral type. We will see that our approach can be applied to those systems and the optimal control can be characterized by the solution of algebraic equations.

**5.1. Systems with delay on state and control.** The problem considered is: minimize for  $u(\cdot) \in L^2(0, T; \mathbb{R}^n)$  the functional

$$J(u) = Gy(T) \cdot y(T) + \int_0^T \{Qy(t) \cdot y(t) + Ru(t) \cdot u(t)\} dt, \quad (15)$$

where  $y$  is the solution of the delayed system

$$\dot{y}(t) - Hy_t = Bu_t, \quad y(0) = \phi^0, \quad y_0 = \phi^1, \quad u_0 = \mu. \quad (16)$$

Such problem has been considered by Delfour [5] in the case  $G = 0$ . Using a state space approach, the optimal control was characterized by the resolution of some Riccati equation. The fact that  $G = 0$  was necessary to have the desired regularity about the adjoint state. The approach used here allows us to include the final state in the cost functional.

Let us put  $x(t) = y(t)$ . Equation (16) can be written as

$$\begin{cases} Mx_t - Ny_t = 0, & y_0 = \phi^1, \quad x_0 = 0, \\ \dot{y}(t) - Hy_t = Bu_t, & y(0) = \phi^0, \quad u_0 = \mu, \end{cases}$$

where  $M$  and  $N$  are defined by  $M\phi = N\phi = \phi(0)$ . Then, the problem considered here could be treated as a particular case of the original problem (4)-(5).

Let  $F = \mathbb{R}^n \times L^2(0, T; \mathbb{R}^n)$ . For  $\varphi = (\varphi^0, \varphi^1) \in F$ , we define  $\Lambda\varphi$  by

$$\Lambda\varphi = \varphi + (G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(\cdot)),$$

where  $y$  is given by the solution of the system

$$\begin{cases} \dot{q}(t) + H^T q^t + Q^{\frac{1}{2}} \varphi^1(t) = 0, & q(T) = G^{\frac{1}{2}} \varphi^0, \quad q^T = 0, \\ u(t) = R^{-1} B^T q^t, \\ \dot{y}(t) - Hy_t = Bu_t, & y(0) = 0, \quad y_0 = 0, \quad u_0 = 0. \end{cases}$$

The operator  $\Lambda$  is an isomorphism on  $F$ , and we have

$$\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2,$$

where  $\|\cdot\|_1$  is a norm defined on  $F$  by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}} B^T q^t \right|^2 dt.$$

Now, we can characterize the optimal control by the following theorem.

**Theorem 15.** *The optimal control solution of the problem (15), (16) is given by the resolution of the algebraic system*

$$\begin{cases} \dot{y}(t) - Hy_t = Bu_t, & y(0) = \phi^0, \quad y_0 = \phi^1, \quad u_0 = \mu, \quad u(t) = 0 \text{ for } t > 0, \\ \Lambda\varphi = -(G^{\frac{1}{2}}y(T), Q^{\frac{1}{2}}y(\cdot)), \\ \dot{q}(t) + H^T q^t + Q^{\frac{1}{2}} \varphi^1(t) = 0, & q(T) = G^{\frac{1}{2}} \varphi^0, \\ u^*(t) = R^{-1} B^T q^t. \end{cases}$$

**5.2. Systems with delay on state, control and observation.** Let us consider the problem: minimize on  $L^2(0, T; \mathbb{R}^n)$  the functional

$$J(u) = Gy(T) \cdot y(T) + \int_0^T \{|Ey_t + Du_t|^2 + Ru(t) \cdot u(t)\} dt \quad (17)$$

where  $y$  is the solution of the delayed system

$$\dot{y}(t) - Hy_t = Bu_t, \quad y(0) = \phi^0, \quad y_0 = \phi^1, \quad u_0 = \mu \quad (18)$$

and the operators  $E$  and  $D$  are linear and continuous.

Define the variable  $x(t)$  by

$$x(t) = Ey_t + Du_t.$$

The problem (17)-(18) can now be written as: minimize on  $L^2(0, T; \mathbb{R}^n)$  the functional

$$J(u) = Gy(T) \cdot y(T) + \int_0^T \{|x(t)|^2 + Ru(t) \cdot u(t)\} dt$$

where  $(x, y)$  is the solution of the system

$$\begin{aligned} x(t) - Ey_t &= Du_t, \quad x_0 = 0, \quad u_0 = \mu, \\ \dot{y}(t) - Hy_t &= Bu_t, \quad y(0) = \phi^0, \quad y_0 = \phi^1. \end{aligned}$$

It is again a particular case of problem (4)-(5), with  $M\phi = \phi(0)$ ,  $Q = 0$  and  $C = I$ . Conditions of applications of Theorem 4 and Theorem 13 are all satisfied and the optimal control is characterized by the inversion of an isomorphism in an appropriate Hilbert space.

Let  $F = \mathbb{R}^n \times L^2(0, T; \mathbb{R}^k)$ . For  $\varphi = (\varphi^0, \varphi^1) \in F$ , we define  $\Lambda\varphi$  by

$$\Lambda\varphi = \varphi + (G^{\frac{1}{2}}y(T), x(\cdot))$$

where  $(x, y)$  is the solution of the system

$$\begin{cases} \begin{cases} p(t) = \varphi^1(t), & q^T = p^T = 0, \\ \dot{q}(t) + H^T q^t + E^T p^t = 0, & q(T) = G^{\frac{1}{2}}\varphi^0, \\ u(t) = R^{-1}(D^T p^t + B^T q^t), \end{cases} \\ \begin{cases} x(t) - Ey_t = Du_t, & x_0 = 0, \quad u_0 = 0, \\ \dot{y}(t) - Hy_t = Bu_t, & y(0) = 0, \quad y_0 = 0. \end{cases} \end{cases}$$

The operator  $\Lambda$  is an isomorphism on  $F$ , and we have

$$\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$$

where  $\|\cdot\|_1$  is a norm defined on  $F$  by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}}(D^T p^t + B^T q^t) \right|^2 dt.$$

Now, we can characterize the optimal control by the following theorem

**Theorem 16.** *The optimal control solution of the problem (17), (18) is given by the resolution of the algebraic system*

$$\begin{cases} \begin{cases} x(t) - Ey_t = Du_t, & u_0 = \mu, \quad u(t) = 0 \text{ for all } t > 0, \\ \dot{y}(t) - Hy_t = Bu_t, & y(0) = \phi^0, \quad y_0 = \phi^1, \end{cases} \\ \Lambda\varphi = -(G^{\frac{1}{2}}y(T), x(\cdot)), \\ \begin{cases} p(t) = \varphi^1(t), & q^T = p^T = 0, \\ \dot{q}(t) + H^T q^t + E^T p^t = 0, & q(T) = G^{\frac{1}{2}}\varphi^0, \\ u^*(t) = R^{-1}(D^T p^t + B^T q^t). \end{cases} \end{cases}$$

**5.3. System of neutral type.** Let us consider the neutral system

$$\frac{d}{dt} Mx_t = Lx_t + Bu_t, \quad x_0 = \phi, \quad u_0 = \mu, \quad (19)$$

where  $M$  is a continuous linear operator with the classical hypothesis that the corresponding matrix of regular Borel measures has an isolated atom at 0.

To this equation, we associate the cost functional

$$J(u) = \int_0^T \{Cx(t) \cdot x(t) + Ru(t) \cdot u(t)\} dt. \quad (20)$$

Introducing the new variable  $y(t) = Mx_t$ , Equation (19) can be written as the system

$$\begin{aligned} Mx_t - y(t) &= 0, \quad x_0 = \phi, \\ \dot{y}(t) - Lx_t &= Bu_t, \quad y(0) = M\phi, u_0 = \mu, \end{aligned}$$

and the problem (19)-(20) is again a particular case of the original one (4)-(5). Our approach can then be applied.

Let  $F = L^2(0, T; \mathbb{R}^k)$ . For  $\varphi \in F$ , we define  $\Lambda\varphi$  by

$$\Lambda\varphi = \varphi + C^{\frac{1}{2}}x(\cdot)$$

where  $x$  is the solution of the system

$$\begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}}\varphi = 0, & q^T = p^T = 0, \\ \dot{q}(t) + p(t) = 0, & q(T) = 0, \\ u(t) = R^{-1}B^T q^t, \\ \begin{cases} Mx_t - y(t) = 0, & x_0 = 0, \quad u_0 = 0, \\ \dot{y}(t) - Lx_t = Bu_t, & y(0) = 0. \end{cases} \end{cases}$$

The operator  $\Lambda$  is an isomorphism on  $F$ , and we have

$$\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$$

where  $\|\cdot\|_1$  is a norm defined on  $F$  by

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}} B^T q^t \right|^2 dt.$$

Now, we can characterize the optimal control by the following theorem.

**Theorem 17.** *The optimal control solution of the problem (19), (20) is given by the solution of the algebraic system*

$$\begin{cases} \begin{cases} Mx_t - y(t) = 0, & x_0 = \phi, \\ \dot{y}(t) - Lx_t = Bu_t, & y(0) = M\phi, u_0 = \mu, u(t) = 0 \text{ for } t > 0, \end{cases} \\ \Lambda\varphi = -C^{\frac{1}{2}}x(\cdot), \\ \begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}}\varphi(t) = 0, & q^T = p^T = 0, \\ \dot{q}(t) + p(t) = 0, & q(T) = 0, \end{cases} \\ u^*(t) = R^{-1}B^T q^t. \end{cases}$$

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**Résumé substantiel en français.** Dans cet article, on étudie le problème de contrôle linéaire quadratique associé à un système mathématique très simple, qui se trouve un bon modèle pour une classe assez large de systèmes héréditaires, équations différentielles à retard, équations intégro-différentielles, équations intégrales, équations différentielles fonctionnelles à retard et de type neutre, *etc.*

Ce modèle a été étudié par Delfour *et al.* [1] et Delfour et Karakchou [6], et une analyse complète a été donnée. Notons que parmi les cas particuliers les plus importants de ce modèle, les systèmes de type neutre et les systèmes à retard, avec retard sur l'état, le contrôle et l'observation.

Le système considéré est de la forme

$$\begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = \phi^0, u_0 = \mu, \end{cases} \quad (21)$$

et le problème de contrôle associé, est la minimisation de la fonctionnelle de coût

$$J(u) = Gy(T) \cdot y(T) + \int_0^T \{Qy(t) \cdot y(t) + Cx(t) \cdot x(t) + Ru(t) \cdot u(t)\} dt, \quad (22)$$

où  $G$ ,  $Q$  et  $C$ , sont des opérateurs symétriques positifs et  $R$  est coercitif.

Des techniques classiques nous permettent de montrer que le contrôle optimal est caractérisé par un système d'optimalité, utilisant un système adjoint. Une adaptation de la méthode HUM (Hilbert Uniqueness Method) basée sur la construction d'une structure hilbertienne et l'inversion d'un isomorphisme (voir Lions [13]) est alors utilisée, et le contrôle optimal est caractérisé par la résolution d'une équation algébrique dans un espace de Hilbert approprié. L'idée a été utilisée par Karakchou et Namir [9] et Karakchou *et al.* [3, 10, 11], pour des systèmes distribués avec retard sur la variable de contrôle dans les deux cas, continu et discret.

L'étude faite dans cet article est basée sur les trois points suivants :

- système d'optimalité ;
- construction d'une structure hilbertienne ;
- inversion d'un isomorphisme sur le nouvel espace de Hilbert.

**Système d'optimalité.** Les techniques classiques d'introduction d'un état adjoint nous permettent de montrer que le contrôle optimal est donné par la résolution d'un système d'optimalité

**Proposition 11.** Soit  $(\phi^0, \phi^1, \phi^2) \in Z^2 = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n) \times L^2(-h, 0; \mathbb{R}^k)$  et  $\mu \in L^2(-h, 0; \mathbb{R}^m)$ . Supposons que l'opérateur  $M$  vérifie les hypothèses données au théorème 4. Alors le contrôle optimal est donné par la résolution du système d'optimalité

$$\begin{cases} (S) \begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = \phi^1, x_0 = \phi^2, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = \phi^0, u_0 = \mu, \end{cases} \\ (S^*) \begin{cases} -M^T p^t + L^T q^t + C^T z(t) = 0, & q^T = 0, p^T = 0, \\ \dot{q}(t) + H^T q^t + N^T p^t + Qy(t) = 0, & q(T) = Gy(T), \\ u^*(t) = -R^{-1}[B_0^T p^t + B_1^T q^t]. \end{cases} \end{cases}$$

**Structure hilbertienne.** Considérons l'espace

$$F = \mathbb{R}^n \times L^2(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^k).$$

Pour tout  $\varphi = (\varphi^0, \varphi^1, \varphi^2) \in F$ , le système

$$\begin{cases} -M^T p^t + L^T q^t + C^{\frac{1}{2}} \varphi^2(t) = 0, & q^T = 0, p^T = 0 \\ \dot{q}(t) + H^T q^t + N^T p^t + Q^{\frac{1}{2}} \varphi^1(t) = 0, & q(T) = G^{\frac{1}{2}} \varphi^0, \end{cases} \quad (23)$$

a une solution unique  $(q, p) \in W^{1,2}(0, T; \mathbb{R}^n) \times L^2(0, T; \mathbb{R}^k)$ . Elle est continue par rapport aux conditions initiales (cf. théorème 4).

Définissons maintenant sur  $F$  la fonctionnelle  $\|\cdot\|_1$  par

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \int_0^T \left| R^{-\frac{1}{2}}(B_0^T p^t + B_1^T q^t) \right|^2 dt.$$

C'est une norme sur  $F$  équivalente à la norme usuelle, puisque les opérateurs  $B_0^T$  et  $B_1^T$  sont continus sur  $L^2(0, T+h; \mathbb{R}^k)$  (resp.  $L^2(0, T+h; \mathbb{R}^n)$ ), et la solution du système transposé (23) est continue par rapport aux données du système.

Dans la suite nous verrons que cette norme peut être associée à un produit scalaire sur  $F$  et que l'espace  $(F, \|\cdot\|_1)$  est un espace de Hilbert.

Soit  $\Lambda$  l'opérateur défini sur  $F$  par:

1- pour  $\varphi \in F$ , on résout le système (23) ;

2- on pose

$$u(t) = R^{-1}(B_0^T p^t + B_1^T q^t); \quad (24)$$

3- utilisant cette fonction de contrôle  $u(t)$ , on résout le système avec conditions initiales nulles

$$\begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = 0, x_0 = 0, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = 0, u_0 = 0; \end{cases} \quad (25)$$

4- finalement, on pose

$$\Lambda \varphi = \varphi + (G^{\frac{1}{2}} y(T), q^{\frac{1}{2}} y(\cdot), C^{\frac{1}{2}} x(\cdot)), \quad (26)$$

où  $(x, y)$  est la solution du système (25).

Il est facile de voir que l'opérateur  $\Lambda$  est symétrique et défini positif.

**Proposition 12.** Pour tout  $\varphi, \psi \in F$ , on a

- 1-  $\langle \Lambda\varphi, \psi \rangle = \langle \varphi, \Lambda\psi \rangle$  ;
- 2-  $\langle \Lambda\varphi, \varphi \rangle = \|\varphi\|_1^2$  ;

où  $\langle \cdot, \cdot \rangle$  est le produit scalaire dans  $F$ .

Il découle de cette proposition que l'opérateur  $\Lambda$  est un isomorphisme continu et coercitif. Le résultat principal est alors donné par le théorème suivant.

**Théorème 13.** Le contrôle optimal solution du problème (21), (22) est donné par

$$u(t) = R^{-1}(B_0^T \bar{p}^t + B_1^T \bar{q}^t),$$

où le couple  $(\bar{q}, \bar{p})$  est construit de la manière suivante :

- 1- résoudre le système non contrôlé

$$\begin{cases} Mx_t - Ny_t = B_0 u_t, & y_0 = \phi^1, x_0 = \phi^2, u(t) = 0 \text{ pour } t > 0, \\ \dot{y}(t) - Hy_t - Lx_t = B_1 u_t, & y(0) = \phi^0, u_0 = \mu; \end{cases}$$

- 2- résoudre l'équation

$$\Lambda\varphi = -(G^{\frac{1}{2}}y(T), q^{\frac{1}{2}}y(\cdot), C^{\frac{1}{2}}x(\cdot)) ;$$

- 3- résoudre le système adjoint

$$\begin{cases} -M^T \bar{p}^t + L^T \bar{q}^t + C^{\frac{1}{2}} \varphi^2(t) = 0, & \bar{q}^T = 0, \bar{p}^T = 0, \\ \dot{q}(t) + H^T \bar{q}^t + N^T \bar{p}^t + Q^{\frac{1}{2}} \varphi^1(t) = 0, & \bar{q}(T) = G^{\frac{1}{2}} \varphi^0; \end{cases}$$

De plus, le coût optimal est donné par

$$J(u^*) = \|\varphi\|_1^2.$$

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JAMILA KARRAKCHOU

ECOLE MOHAMADIA D'INGÉNIEURS

LABORATOIRE D'ETUDES ET RECHERCHE EN MATHÉMATIQUES APPLIQUÉES

BP 765 RABAT

MAROC

E-MAIL: j.karakchou@emi.ac.ma