

ON A QUESTION OF MARTINET CONCERNING THE 2-CLASS FIELD TOWER OF IMAGINARY QUADRATIC NUMBER FIELD

ELLIOT BENJAMIN

RÉSUMÉ. Soit k un corps de nombres quadratiques imaginaires dont le 2-sous-groupe de Sylow $C_{k,2}$, du groupe de classes C_k est de rang 4. Nous prouvons que la tour de classes de Hilbert de k est infinie si le 4-rang de C_k est égal à 2 et si l'une des deux conditions suivantes est satisfaite : ou bien 5 discriminants premiers négatifs divisent le discriminant de k ou bien le discriminant de k n'est pas congru à 4 modulo 8. Lorsque le 4-rang de C_k est égal à 1, nous donnons des conditions qui permettent de conclure que la 2-tour de classes de Hilbert de k est infinie. Il est connu que si le 4-rang de C_k est supérieur ou égal à 3 ou bien est égal à 2 avec exactement un discriminant premier négatif divisant le discriminant de k , alors la 2-tour de classes de Hilbert de k est infinie. Nos résultats tendent donc à supporter la conjecture que la 2-tour de classes de Hilbert d'un corps quadratique imaginaire k est infinie, lorsque le rang de $C_{k,2}$ est 4.

ABSTRACT. Let k be an imaginary quadratic number field with $C_{k,2}$, the 2-Sylow subgroup of its ideal class group C_k , of rank 4. We show that if the 4-rank of C_k is equal to 2, and either five negative prime discriminants divide the discriminant of k or the discriminant of k is not congruent to 4 mod 8, then k has infinite 2-class field tower. When the 4-rank of C_k is equal to 1, we state particular conditions that allow us to conclude that k has infinite 2-class field tower. Since it is known that if the 4-rank of C_k is greater than or equal to 3, or equal to 2 with exactly one negative prime discriminant dividing the discriminant of k , then k has infinite 2-class field tower, our results give support to the conjecture that all imaginary quadratic number fields k with $C_{k,2}$ of rank 4 have infinite 2-class field tower.

Introduction. Let k denote an algebraic number field and $C_{k,2}$ denote its 2-class group, *i.e.* the 2-Sylow subgroup of the ideal class group C_k (in the wide sense) of k ; denote by k_1 the Hilbert 2-class field of k . Let k_n (for n a nonnegative integer) be defined inductively as $k_0 = k$ and $k_{n+1} = (k_n)_1$. Then $k_0 \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k_n \subseteq \cdots$ is called the 2-class field tower of k . If n is the minimal integer such that $k_n = k_{n+1}$, then n is called the length of the tower. If no such n exists, then the tower is said to be of infinite length.

In 1964, Golod and Shafarevich (*cf.* [3]) established for the first time the existence of infinite p -class field towers, for p prime. In the case $p = 2$, their criterion (as refined

Reçu le 30 janvier 2001 et, sous forme définitive, le 10 février 2002.

by Gaschütz and Vinberg [7]) can be stated in the following way, where $E_{k,2}$ denotes the unit group of k mod its squares, E_k/E_k^2 : if $\text{rank } C_{k,2} \geq 2 + 2(\text{rank } E_{k,2} + 1)^{1/2}$ then k has infinite 2-class field tower. We shall refer to the above inequality as the Golod-Shafarevich inequality. We immediately see that for k imaginary quadratic with $\text{rank } C_{k,2} \geq 5$, or k real quadratic with $\text{rank } C_{k,2} \geq 6$, the Golod-Shafarevich inequality is satisfied and k thereby has infinite 2-class field tower. It is well known that for k imaginary quadratic with $\text{rank } C_{k,2} = 2$ or 3 , the 2-class field tower of k may be finite or infinite, and that if $\text{rank } C_{k,2} = 1$ then the 2-class field tower of k is finite and has length 1 (cf. [2, 5, 9, 10, 14]). It has been conjectured that for k imaginary quadratic with $\text{rank } C_{k,2} = 4$, k has infinite 2-class field tower (see Martinet: [10, 11]).

A partial result in this direction, as proved by Hajir, is that if k is an imaginary quadratic number field such that $C_{k,2}$ has 4-rank greater than or equal to 3, then k has infinite 2-class field tower (cf. [4, 5, 6]). We define d_j to be a negative prime discriminant to mean that $d_j < 0$ and $-d_j$ is prime, $d_j = -4$, or $d_j = -8$. We define a Δ_k -splitting is of the second kind to be a splitting of Δ_k , the discriminant of k , into two fundamental discriminants d_1 and d_2 such that the Kronecker symbols $(d_1/p) = 1$ and $(d_2/q) = 1$ for all primes p dividing d_2 and for all primes q dividing d_1 . In the case where $\Delta_k \not\equiv 4 \pmod{8}$ and exactly one negative prime discriminant divides $\Delta_k = -p_1 p_2 p_3 p_4 p_5$ where $k = Q(\sqrt{-p_1 p_2 p_3 p_4 p_5})$ (i.e. $C_{k,2}$ has rank 4), we have previously defined (cf. [1]) a Kronecker symbol configuration to be a complete list of Kronecker symbols (p_i/p_j) , $i < j$, $1 \leq i \leq 5$. In [1], we have denoted a Kronecker symbol configuration by listing all the Kronecker symbols (p_i/p_j) as above with $(p_i/p_j) = 1$ (respectively -1), where the remaining Kronecker symbols (p_i/p_j) , $i < j$, are assumed to be -1 (respectively). In our earlier work (cf. [1]), we have shown that if k is an imaginary quadratic number field such that $C_{k,2}$ has rank 4, C_k has 4-rank greater than or equal to 2, and exactly one negative prime discriminant divides Δ_k , then k has infinite 2-class field tower. In our present work we extend both Hajir's result and our earlier result, demonstrating our main theorem (see Theorem 1): *If k is an imaginary quadratic number field such that $C_{k,2}$ has rank 4 and C_k has 4-rank greater than or equal to 2, such that either five negative prime discriminants divide Δ_k or Δ_k is not congruent to 4 mod 8, then k has infinite 2-class field tower.* Our second theorem states that *if Δ_k is congruent to 4 mod 8 and exactly three negative prime discriminants divide Δ_k , then k has infinite 2-class field tower if the Kronecker symbols of the primes dividing Δ_k do not have a particular format* (see sub-cases 9 thru 12 of Theorem 2). We also give a partial criterion for obtaining imaginary quadratic number fields k with $C_{k,2}$ of rank 4 and C_k having 4-rank equal to 1 such that k has infinite 2-class field tower. This criterion is related to the nontrivial Δ_k -splitting of the second kind for k , and examples of such fields k are given in [1]. Our methods of proving our main result are essentially combinatorial, applied to our earlier results related to the Golod-Shafarevich inequality.

Preliminaries. We begin by stating the results from our earlier work that we will need (cf. [1]).

Lemma 1. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$. If for some prime $p_j \equiv 1 \pmod{4}$, or $p_j = 2$ in which case we further assume that 8 is a fundamental discriminant dividing Δ_k , we have the Kronecker symbols $(p_j/p_k) =$*

$(p_j/p_\ell) = (p_j/p_m)$, p_j, p_k, p_ℓ, p_m distinct primes, $p_j p_k p_\ell p_m \mid \Delta_k$, then k has infinite 2-class field tower.

We utilize the following notational convenience: if d_j is a negative prime discriminant, we let p_j denote the prime dividing d_j if $d_j \neq -4$ and $p_j = 1$ if $d_j = -4$.

Lemma 2. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$. Assume there exists a negative prime discriminant d_j , $d_j \mid \Delta_k$, such that $(-p_j/p_i) = 1$, $i \neq j$, for all primes $p_i \mid \Delta_k$, $1 \leq i \leq 5$. Then k has infinite 2-class field tower.*

Lemma 3. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$, at least two of the prime discriminants dividing Δ_k are positive, and $(p_1/p_3) = (p_2/p_3) = 1$ where p_1 and p_2 are distinct primes dividing positive prime discriminants dividing Δ_k , and p_3 is a prime dividing a positive or negative prime discriminant dividing Δ_k , $p_1 \neq p_3 \neq p_2$. Then k has infinite 2-class field tower.*

Lemma 4. *Let k be an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$, exactly one negative prime discriminant divides Δ_k , and $\Delta_k \equiv 4 \pmod{8}$. Then k has infinite 2-class field tower.*

Remark 1. Lemma 4 is an immediate corollary of Lemma 2.

Lemma 5. *Let k be an imaginary quadratic number field with $\text{rank } C_{k,2} = 4$, exactly one negative prime discriminant divides Δ_k , and $\Delta_k \not\equiv 4 \pmod{8}$. Then the following fields k have infinite 2-class field tower, where $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$:*

- A) 4-rank of C_k equal to 2, 3 or 4
- B) 4-rank of C_k equal to 1 and Kronecker symbol configuration of k not $(p_i/p_j) = (p_i/p_k) = 1$, where either p_j or p_k is the prime dividing the negative prime discriminant dividing Δ_k .

4-rank of C_k equal to 2. Before stating our main result, we establish this following lemma and corollary as a useful technique to determine that certain Δ_k -splittings are not Δ_k -splittings of the second kind (we thank one of the referees for the suggestion and proofs of Lemma 6 and Corollary 1).

Lemma 6. *Let d_1, d_2, d_3 be relatively prime negative discriminants and q_i the prime dividing d_i , $i = 1, 2, 3$. Then it is not possible that $(d_1 d_2 / q_3) = (d_2 d_3 / q_1) = (d_1 d_3 / q_2) = 1$.*

Proof. We assume that $(d_1 d_2 / q_3) = (d_2 d_3 / q_1) = (d_1 d_3 / q_2) = 1$. If the d_i are all odd then $d_i = -q_i$, $i = 1, 2, 3$ and $(d_1 / q_3) = (d_2 / q_3) = (q_3 / q_2) = -(d_3 / q_2) = 1$, as well as $(d_1 / q_3) = -(d_2 / q_1) = (d_1 / q_2)$. This implies that $-(d_1 d_3 / q_2) = 1$, which is a contradiction of our assumption. If $d_3 = -4$ and $q_3 = 2$, then $(d_1 d_3 / q_2) = -(q_1 / q_2) = 1$ and $(d_2 d_3 / q_1) = -(q_2 / q_1) = 1$, contradicting the law of quadratic reciprocity. Finally, if $d_3 = -8$ and $q_3 = 2$, then $d_1 \equiv d_2 \pmod{8}$. This implies that $(-2 / q_1) = (-2 / q_2)$, $(-2 / q_1) = (d_2 / q_1) = (q_1 / q_2)$, and $(-2 / q_2) = (d_1 / q_2) = (q_2 / q_1)$, once again contradicting the law of quadratic reciprocity; our lemma is therefore proved. $\square$

Corollary 1. *Let d_1, d_2, d_3 be relatively prime negative discriminants, $d_i \mid D_i$ for $i = 1, 2, 3$, and $(d_1d_2, D_3), (d_1d_3, D_3)$ be Δ_k -splittings into fundamental discriminants. Then (d_1d_2, D_3) and (d_1d_3, D_3) cannot both be Δ_k -splittings of the second kind.*

Proof. Assume that (d_1d_2, D_3) and (d_1d_3, D_3) are both Δ_k -splittings of the second kind. We apply the natural group structure on Δ_k -splittings of the second kind (cf. [6]) to obtain a third Δ_k -splitting of the second kind: (d_2d_3, D_1) . We therefore have $(d_1d_2/q_3) = (d_1d_3/q_2) = (d_2d_3/q_1)$, which contradicts Lemma 6; our corollary is therefore established. $\square$

We are now ready to state our main result. From our earlier work (see Lemma 4 and Lemma 5) we know that if k is an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$, 4-rank of C_k equal to 2, and exactly one negative prime discriminant divides Δ_k , then k has infinite 2-class field tower. For our first theorem we extend this result to the case where either five negative prime discriminants divide Δ_k or Δ_k is not congruent to 4 mod 8.

Theorem 1. *Let k be an imaginary quadratic number field with $\text{rank } C_{k,2} = 4$ and 4-rank of C_k equal to 2 such that either five negative prime discriminants divide Δ_k or Δ_k is not congruent to 4 mod 8. Then k has infinite 2-class field tower.*

Proof. If the 4-rank of C_k is equal to 3 or 4 then we know by Hajir ([5, 6]) that k has infinite 2-class field tower. We assume that the 4-rank of C_k is equal to 2, and examine the possible cases that can occur in our theorem in terms of how many negative prime discriminants divide Δ_k . If exactly one negative prime discriminant divides Δ_k then we know from our earlier work (cf. [1] and Lemma 4, Lemma 5) that k has infinite 2-class field tower. We proceed to examine the cases when exactly three or five negative prime discriminants divide Δ_k , and divide the five negative prime discriminant case further into sub-cases according to whether $\Delta_k \not\equiv 4 \pmod{8}$ or $\Delta_k \equiv 4 \pmod{8}$.

Case A: Five negative prime discriminants divide Δ_k , $\Delta_k \not\equiv 4 \pmod{8}$.

If C_k has 4-rank equal to 2, then Δ_k has exactly three nontrivial Δ_k -splittings of the second kind. We let $k = Q(\sqrt{-q_1q_2q_3q_4q_5})$ where if 8 does not divide Δ_k then the q_i 's are distinct primes congruent to 3 mod 4, $1 \leq i \leq 5$, and if 8 divides Δ_k then we assume (without loss of generality) that $q_i = 8$ for one particular q_i and the other q_i 's are distinct primes congruent to 3 mod 4. If $(-q_1, q_2q_3q_4q_5)$ (without loss of generality) is one of these nontrivial Δ_k -splittings of the second kind then we know from Lemma 2 that k has infinite 2-class field tower. Assume that $(-q_1, q_2q_3q_4q_5)$ is not a Δ_k -splitting of the second kind (we shall continue to utilize our above notation without loss of generality throughout this proof). Then we must have two Δ_k -splittings of the second kind which are of the following types: $(q_1q_2, -q_3q_4q_5)$ and $(q_1q_3, -q_2q_4q_5)$. However by Corollary 1, this cannot occur and therefore $(-q_1, q_2q_3q_4q_5)$ must be a Δ_k -splitting of the second kind. From Lemma 2, we conclude that k has infinite 2-class field tower.

Case B: Five negative prime discriminants divide Δ_k , $\Delta_k \equiv 4 \pmod{8}$.

We proceed in a similar manner as in Case A. We let $k = Q(\sqrt{-4q_2q_3q_4q_5})$ and assume that neither $(-4, q_2q_3q_4q_5)$ nor $(-q_2, 4q_3q_4q_5)$ (once again without loss of generality) are Δ_k -splittings of the second kind. We now have three possibilities that

may occur, each one consisting of two $\triangle_k$ -splittings of the second kind, as follows:

$$\begin{array}{lll} 1) (q_2q_3, -4q_4q_5) & 2) (4q_2, -q_3q_4q_5) & 3) (4q_2, -q_3q_4q_5) \\ (q_2q_4, -4q_3q_5) & (q_2q_3, -4q_4q_5) & (4q_3, -q_2q_4q_5) \end{array}$$

However, we see by Corollary 1 that none of these three possibilities can occur, and therefore either $(-4, q_2q_3q_4q_5)$ or $(-q_2, 4q_3q_4q_5)$ must be a $\triangle_k$ -splittings of the second kind. From Lemma 2 we conclude that k has infinite 2-class field tower.

Case C: Exactly three negative prime discriminants divide $\triangle_k$, $\triangle_k \not\equiv 4 \pmod{8}$.

In order to establish our results for the three negative prime discriminant case, we will make use of Lemmas 1 and 3, since we now have two positive prime discriminants to work with. We let $k = Q(\sqrt{-p_1p_2q_2q_3q_4q_5})$ where p_1 and p_2 are distinct positive prime discriminants dividing $\triangle_k$, and if -8 is not a fundamental discriminant dividing $\triangle_k$ then q_3, q_4, q_5 are distinct primes congruent to 3 mod 4, whereas if -8 is a fundamental discriminant dividing $\triangle_k$ then we assume (without loss of generality) that $q_i = 8$ for a particular q_i and the other q_i 's are distinct primes congruent to 3 mod 4, $1 \leq i \leq 3$. If C_k has 4-rank equal to 2 and we assume that $(-q_3, p_1p_2q_4q_5)$ (again without loss of generality) is not a $\triangle_k$ -splitting of the second kind, then we initially have the following thirteen sub-cases that may occur for $\triangle_k$ -splittings of the second kind:

$$\begin{array}{llll} 1) (p_1, -p_2q_3q_4q_5) & 2) (p_1, -p_2q_3q_4q_5) & 3) (p_1, -p_2q_3q_4q_5) & 4) (p_1, -p_2q_3q_4q_5) \\ (p_2, -p_1q_3q_4q_5) & (-p_1q_3, p_2q_4q_5) & (-p_2q_3, p_1q_4q_5) & (q_3q_4, -p_1p_2q_5) \\ 5) (p_1, -p_2q_3q_4q_5) & 6) (p_1p_2, -q_3q_4q_5) & 7) (p_1p_2, -q_3q_4q_5) & 8) (-p_1q_3, p_2q_4q_5) \\ (p_1p_2, -q_3q_4q_5) & (-p_1q_3, p_2q_4q_5) & (q_3q_4, -p_1p_2q_5) & (-p_1q_4, p_2q_5q_3) \\ 9) (-p_1q_3, p_2q_4q_5) & 10) (-p_1q_3, p_2q_4q_5) & 11) (-p_1q_3, p_2q_4q_5) & 12) (-p_1q_3, p_2q_4q_5) \\ (-p_2q_3, p_1q_4q_5) & (-p_2q_4, p_1q_5q_3) & (q_3q_4, -p_1p_2q_5) & (q_4q_5, -p_1p_2q_3) \\ 13) (q_3q_4, -p_1p_2q_5) & & & \\ (q_3q_5, -p_1p_2q_4) & & & \end{array}$$

We proceed to show that for each one of the above thirteen sub-cases, either we have a contradiction, which means that the sub-case cannot occur, or we are able to use Lemma 1 or Lemma 3 to show that k has infinite 2-class field tower.

Sub-case 1) If $(p_1, -p_2q_3q_4q_5)$ and $(p_2, -p_1q_3q_4q_5)$ are $\triangle_k$ -splittings of the second kind then we have $(p_1/p_2) = (p_1/p_4) = (p_1/q_3) = (p_2/q_3) = 1$. By Lemma 1 or Lemma 3, we are able to conclude that k has infinite 2-class field tower.

Sub-case 2) If $(p_1, -p_2q_3q_4q_5)$ and $(-p_1q_3, p_2q_4q_5)$ are $\triangle_k$ -splittings of the second kind then by the group structure of $\triangle_k$ -splittings of the second kind we see that $(-q_3, p_1p_2q_4q_5)$ is also a $\triangle_k$ -splitting of the second kind, which is a contradiction; sub-case 2 therefore does not occur.

Sub-cases 3), 4) and 5) If $(p_1, -p_2q_3q_4q_5)$ is a $\triangle_k$ -splitting of the second kind then we have $(p_1/p_2) = (p_1/q_3) = (p_1/q_4) = 1$ and Lemma 1 suffices to show that k has infinite 2-class field tower.

Sub-cases 6) and 7) If $(p_1p_2, -q_3q_4q_5)$ is a $\triangle_k$ -splitting of the second kind then, we have four possibilities for sets of Kronecker symbol equalities as follows:

$$\begin{array}{ll}
 A) \quad (p_1/q_3) = (p_2/q_3) = 1 & B) \quad (p_1/q_3) = (p_2/q_3) = -1 \\
 \quad (p_1/q_4) = (p_2/q_4) = 1 & \quad (p_1/q_4) = (p_2/q_4) = -1 \\
 \quad (p_1/q_5) = (p_2/q_5) = 1 & \quad (p_1/q_5) = (p_2/q_5) = 1 \\
 C) \quad (p_1/q_3) = (p_2/q_3) = 1 & D) \quad (p_1/q_3) = (p_2/q_3) = -1 \\
 \quad (p_1/q_4) = (p_2/q_4) = -1 & \quad (p_1/q_4) = (p_2/q_4) = 1 \\
 \quad (p_1/q_5) = (p_2/q_5) = -1 & \quad (p_1/q_5) = (p_2/q_5) = -1
 \end{array}$$

However, in each of the above four sets of Kronecker symbol equalities we are able to apply Lemma 3 to conclude that k has infinite 2-class field tower.

Sub-case 8) If $(-p_1q_3, p_2q_4q_5)$ is a $\triangle_k$ -splitting of the second kind then we have the following four possibilities for sets of Kronecker symbol equalities:

$$\begin{array}{ll}
 A) \quad (p_1/p_2) = (-q_3/p_2) = 1 & B) \quad (p_1/p_2) = (-q_3/p_2) = -1 \\
 \quad (p_1/q_4) = (-q_3/q_4) = 1 & \quad (p_1/q_4) = (-q_3/q_4) = -1 \\
 \quad (p_1/q_5) = (-q_3/q_5) = 1 & \quad (p_1/q_5) = (-q_3/q_5) = 1 \\
 C) \quad (p_1/p_2) = (-q_3/p_2) = 1 & D) \quad (p_1/p_2) = (-q_3/p_2) = -1 \\
 \quad (p_1/q_4) = (-q_3/q_4) = -1 & \quad (p_1/q_4) = (-q_3/q_4) = 1 \\
 \quad (p_1/q_5) = (-q_3/q_5) = -1 & \quad (p_1/q_5) = (-q_3/q_5) = -1
 \end{array}$$

However, each one of these four possible sets of Kronecker symbol equalities contradicts $(-p_1q_4, p_2q_5q_3)$ being a $\triangle_k$ -splitting of the second kind; sub-case 8 therefore does not occur.

In sub-cases 9), 10) and 12) we are not able to directly conclude that these sub-cases do not occur, or utilize Lemma 1 or Lemma 3 directly to show that k has infinite 2-class field tower. However, by using the group property of $\triangle_k$ -splittings of the second kind we are able to determine the third nontrivial $\triangle_k$ -splitting of the second kind, and consequently are still able to either conclude that these sub-cases do not occur or make use of Lemma 1 or Lemma 3 to conclude that k has infinite 2-class field tower. We note that by using this group property we see that sub-case 7 results in $(-q_5, p_1p_2q_3q_4)$ being a $\triangle_k$ -splitting of the second kind, which is another means (via Lemma 2) of demonstrating that k has infinite 2-class field tower in this sub-case. In actuality, sub-case 7 does not occur — given our initial assumption that $(-q_5, p_1p_2q_3q_4)$ is not a $\triangle_k$ -splitting of the second kind.

Sub-case 9) If $(-p_1q_3, p_2q_4q_5)$ and $(-p_2q_3, p_1q_4q_5)$ are $\triangle_k$ -splittings of the second kind then we add the third nontrivial $\triangle_k$ -splitting of the second kind, which is $(p_1p_2, q_3q_4q_5)$. We are thus back in sub-case 6, and from above we are able to apply Lemma 3 to conclude that k has infinite 2-class field tower.

Sub-case 10) If $(-p_1q_3, p_2q_4q_5)$ and $(-p_2q_4, p_1q_3q_5)$ are $\triangle_k$ -splittings of the second kind then we again add the third nontrivial $\triangle_k$ -splitting of the second kind, which

is $(-q_5, p_1 p_2 q_3 q_4)$. We thus see that sub-case 10 does not occur, given our initial assumption that $(-q_5, p_1 p_2 q_3 q_4)$ is not a $\triangle_k$ -splitting of the second kind.

Sub-case 11) If $(-p_1 q_3, p_2 q_4 q_5)$ is a $\triangle_k$ -splitting of the second kind then we have the same four sets of possibilities for Kronecker symbol inequalities as we had for sub-case 8:

$$\begin{array}{ll}
 A) \quad (p_1/p_2) = (-q_3/p_2) = 1 & B) \quad (p_1/p_2) = (-q_3/p_2) = -1 \\
 \quad (p_1/q_4) = (-q_3/q_4) = 1 & \quad (p_1/q_4) = (-q_3/q_4) = -1 \\
 \quad (p_1/q_5) = (-q_3/q_5) = 1 & \quad (p_1/q_5) = (-q_3/q_5) = 1 \\
 C) \quad (p_1/p_2) = (-q_3/p_2) = 1 & D) \quad (p_1/p_2) = (-q_3/p_2) = -1 \\
 \quad (p_1/q_4) = (-q_3/q_4) = -1 & \quad (p_1/q_4) = (-q_3/q_4) = 1 \\
 \quad (p_1/q_5) = (-q_3/q_5) = -1 & \quad (p_1/q_5) = (-q_3/q_5) = -1
 \end{array}$$

However, each one of these four possible sets of Kronecker symbol inequalities contradicts $(q_3 q_4, -p_1 p_2 q_5)$ being a $\triangle_k$ -splitting of the second kind; sub-case 11 therefore does not occur.

Sub-case 12) If $(-p_1 q_3, p_2 q_4 q_5)$ and $(q_4 q_5, -p_1 p_2 q_3)$ are $\triangle_k$ -splitting of the second kind then we again add the third nontrivial $\triangle_k$ -splitting of the second kind, which is $(p_2, p_1 q_3 q_4 q_5)$. We are thus back in sub-cases 3 and 4, and are able to make use of Lemma 1 to conclude that k has infinite 2-class field tower.

Sub-case 13) By Corollary 1, $(q_3 q_4, -p_1 p_2 q_5)$ and $(q_3 q_5, -p_1 p_2 q_4)$ cannot both be $\triangle_k$ -splittings of the second kind; sub-case 13 therefore does not occur.

Since we have exhausted all possible sub-cases of nontrivial $\triangle_k$ -splittings of the second kind when C_k has 4-rank equal to 2 for Case C of Theorem 1, our result is established for the three negative prime discriminant case when $\triangle_k \not\equiv 4 \pmod{8}$ and our theorem is proved. $\square$

In order to state our result for the three negative prime discriminant case where $\triangle_k \equiv 4 \pmod{8}$, we let $k = Q(\sqrt{-4p_1 p_2 q_1 q_2})$ where p_1 and p_2 are distinct primes congruent to 1 mod 4, and q_1 and q_2 are distinct primes congruent to 3 mod 4. We note that if $\triangle_k \not\equiv 4 \pmod{8}$ in the three negative prime discriminants case then by Theorem 1 we know that k has infinite 2-class field tower.

Theorem 2. *Let k be an imaginary quadratic number field with rank $C_{k,2} = 4$ and 4-rank of C_k equal to 2 such that exactly three negative prime discriminants divide $\triangle_k$, and $\triangle_k \equiv 4 \pmod{8}$. If the Kronecker symbols of the primes dividing $\triangle_k$ do not have the format $(p_1/2) = (p_2/2) = (p_1/q_3) = (p_1/q_4) = (p_2/q_3) = (p_2/q_4) = -1$ then k has infinite 2-class field tower.*

Proof. We assume that neither $(-4, p_1 p_2 q_3 q_4)$ nor $(-q_3, 4p_1 p_2 q_4)$ (again without loss of generality) are $\triangle_k$ -splittings of the second kind. We now initially have twenty five

sub-cases to consider for $\triangle_k$ -splittings of the second kind:

- | | | | |
|--------------------------|--------------------------|--------------------------|--------------------------|
| 1) $(p_1, -4p_2q_3q_4)$ | 2) $(p_1, -4p_2q_3q_4)$ | 3) $(p_1, -4p_2q_3q_4)$ | 4) $(p_1, -4p_2q_3q_4)$ |
| $(p_2, -4p_1q_3q_4)$ | $(-p_1q_3, 4p_2q_4)$ | $(-4p_1, p_2q_3q_4)$ | $(-p_2q_3, 4p_1q_4)$ |
| 5) $(p_1, -4p_2q_3q_4)$ | 6) $(p_1, -4p_2q_3q_4)$ | 7) $(p_1, -4p_2q_3q_4)$ | 8) $(p_1, -4p_2q_3q_4)$ |
| $(-4p_2, p_1q_3q_4)$ | $(q_3q_4, -4p_2p_1)$ | $(4q_3, -p_1p_2q_4)$ | $(p_1p_2, -4q_3q_4)$ |
| 9) $(p_1p_2, -4q_3q_4)$ | 10) $(p_1p_2, -4q_3q_4)$ | 11) $(p_1p_2, -4q_3q_4)$ | 12) $(p_1p_2, -4q_3q_4)$ |
| $(-p_1q_3, 4p_2q_4)$ | $(-4p_1, p_2q_3q_4)$ | $(q_3q_4, -4p_1p_2)$ | $(4q_3, -p_1p_2q_4)$ |
| 13) $(-p_1q_3, 4p_2q_4)$ | 14) $(-p_1q_3, 4p_2q_4)$ | 15) $(-p_1q_3, 4p_2q_4)$ | 16) $(-p_1q_3, 4p_2q_4)$ |
| $(-p_1q_4, 4p_2q_3)$ | $(-4p_1, p_2q_3q_4)$ | $(-p_2q_3, 4p_1q_4)$ | $(-4p_2, p_1q_3q_4)$ |
| 17) $(-p_1q_3, 4p_2q_4)$ | 18) $(-p_1q_3, 4p_2q_4)$ | 19) $(-p_1q_3, 4p_2q_4)$ | 20) $(-p_1q_3, 4p_2q_4)$ |
| $(-p_2q_4, 4p_1q_3)$ | $(q_3q_4, -4p_1p_2)$ | $(4q_3, -p_1p_2q_4)$ | $(4q_4, -p_1p_2q_3)$ |
| 21) $(-4p_1, p_2q_3q_4)$ | 22) $(-4p_1, p_2q_3q_4)$ | 23) $(-4p_1, p_2q_3q_4)$ | 24) $(q_3q_4, -4p_2p_1)$ |
| $(-4p_2, p_1q_3q_4)$ | $(q_3q_4, -4p_1p_2)$ | $(4q_3, -p_1p_2q_4)$ | $(4q_3, -p_1p_2q_4)$ |
| 25) $(4q_3, -p_1p_2q_4)$ | | | |
| $(4q_4, -p_1p_2q_3)$ | | | |

As in Case C of Theorem 1, we proceed to demonstrate that for each one of the above twenty five possibilities, either we have a contradiction, which means that the sub-case cannot occur, or with two exceptions we are able to use Lemma 1 or Lemma 3 to show that k has infinite 2-class field tower, the exceptions being the particular Kronecker symbol format $(p_1/2) = (p_2/2) = (p_1/q_3) = (p_1/q_4) = (p_2/q_3) = (p_2/q_4) = -1$ for the sub-cases which satisfy either both sub-case 9 and sub-case 15, or both sub-case 10 and sub-case 21.

Sub-cases 1) thru 8) Since in sub-cases 1 thru 8 we have $(p_1, -4p_2q_3q_4)$ occurring as a $\triangle_k$ -splitting of the second kind, we have $(p_1/p_2) = (p_1/q_3) = (p_1/q_4) = 1$ and we are able to use Lemma 1 to conclude that k has infinite 2-class field tower. However, we see that in actuality sub-cases 2 and 3 do not occur, as applying the group structure of $\triangle_k$ -splitting of the second kind contradicts our initial assumption that neither $(-4, p_1p_2q_3q_4)$ nor $(q_3, 4p_1p_2q_4)$ are $\triangle_k$ -splitting of the second kind.

Sub-cases 9) thru 12) In sub-cases 9 thru 12 we have $(p_1p_2, -4q_3q_4)$ occurring as a $\triangle_k$ -splitting of the second kind, and we therefore have the following four possibilities for sets of Kronecker symbol equalities:

- | | |
|-----------------------------|------------------------------|
| A) $(p_1/2) = (p_2/2) = 1$ | B) $(p_1/2) = (p_2/2) = 1$ |
| $(p_1/q_3) = (p_2/q_3) = 1$ | $(p_1/q_3) = (p_2/q_3) = -1$ |
| $(p_1/q_4) = (p_2/q_4) = 1$ | $(p_1/q_4) = (p_2/q_4) = -1$ |
| C) $(p_1/2) = (p_2/2) = -1$ | D) $(p_1/2) = (p_2/2) = -1$ |
| $(p_1/q_3) = (p_2/q_3) = 1$ | $(p_1/q_3) = (p_2/q_3) = -1$ |
| $(p_1/q_4) = (p_2/q_4) = 1$ | $(p_1/q_4) = (p_2/q_4) = -1$ |

We observe that sub-cases 11 and 12 do not occur, since by applying the group structure of $\triangle_k$ -splitting of the second kind we see that sub-case 11 implies that $(-4, p_1 p_2 q_3 q_4)$ is a $\triangle_k$ -splitting of the second kind and sub-case 12 implies that $(-q_4, 4p_1 p_2 q_3)$ is a $\triangle_k$ -splitting of the second kind; both of these do not occur by our initial assumptions. For the equalities in parts A, B, and C of sub-cases 9 and 10 we are able to apply Lemma 3 to conclude that k has infinite 2-class field tower. For the equalities in part D of sub-cases 9 and 10 we are not able to apply Lemma 3, and we therefore add the third nontrivial $\triangle_k$ -splitting of the second kind, from applying the group structure of $\triangle_k$ -splitting of the second kind. We observe that sub-case 9 implies sub-case 15 and sub-case 10 implies sub-case 21.

As far as our methods can be used, we are not able to demonstrate that either sub-cases 9 and 15 cannot occur or sub-cases 10 and 21 cannot both occur. Neither are we able to utilize any of our previous lemmas to conclude that k has infinite 2-class field tower when the equalities in part D of sub-cases 9 and 10 are satisfied.

We now continue with the remainder of our sub-cases.

Sub-cases 13) thru 20) In sub-cases 13 thru 20 we have $(-p_1 q_3, 4p_2 q_4)$ occurring as a $\triangle_k$ -splitting of the second kind, and we therefore have the following four possibilities for sets of Kronecker symbol equalities:

$$\begin{array}{ll}
 \text{A)} & (p_1/2) = (-q_3/2) = 1 \\
 & (p_1/p_2) = (q_3/p_2) = 1 \\
 & (p_1/q_4) = (-q_3/q_4) = 1 \\
 \text{B)} & (p_1/2) = (-q_3/2) = -1 \\
 & (p_1/p_2) = (q_3/p_2) = 1 \\
 & (p_1/q_4) = (-q_3/q_4) = 1 \\
 \text{C)} & (p_1/2) = (-q_3/2) = 1 \\
 & (p_1/p_2) = (q_3/p_2) = -1 \\
 & (p_1/q_4) = (-q_3/q_4) = -1 \\
 \text{D)} & (p_1/2) = (-q_3/2) = -1 \\
 & (p_1/p_2) = (q_3/p_2) = -1 \\
 & (p_1/q_4) = (-q_3/q_4) = -1
 \end{array}$$

We observe that sub-cases 13, 14, 18 and 19 cannot occur, since every one of the above four possibilities contradicts each of $(-p_1 q_4, 4p_2 q_3)$, $(-4p_1, p_2 q_3 q_4)$, $(q_3 q_4, -4p_1 p_2)$, and $(4q_3, -p_1 p_2 q_4)$ being a $\triangle_k$ -splitting of the second kind. We observe that sub-cases 16 and 17 cannot occur, since by applying the group structure of $\triangle_k$ -splitting of the second kind we see that sub-case 16 implies that $(-q_4, 4p_1 p_2 q_3)$ is a $\triangle_k$ -splitting of the second kind and sub-case 17 implies that $(-4, p_1 p_2 q_3 q_4)$, is a $\triangle_k$ -splitting of the second kind; both of these do not occur by our initial assumptions.

For sub-case 20, since $(4q_4, -p_1 p_2 q_3)$ is a $\triangle_k$ -splitting of the second kind, we must have $(p_1/q_4) = (p_2/q_4) = 1$ and we therefore conclude by Lemma 3 that k has infinite 2-class field tower.

For sub-case 15 by applying the group structure of $\triangle_k$ -splitting of the second kind we find that sub-case 15 implies sub-case 9 (as previously observed).

Sub-cases 21) thru 23) In sub-cases 21 thru 23 we have $(-4p_1, p_2 q_3 q_4)$ occurring as a $\triangle_k$ -splitting of the second kind, and we therefore must have $(p_1/p_2) = 1$, $(p_1/q_3) = (p_1/q_4) = -1$. In sub-case 22, by applying the group structure of $\triangle_k$ -splitting of the second kind we see that $(p_2, -4p_1 q_3 q_4)$ is a $\triangle_k$ -splitting of the second kind, and we therefore can apply Lemma 1, to conclude that k has infinite 2-class field tower.

In sub-case 23, since $(-4q_3, p_1p_2q_4)$ is a Δ_k -splitting of the second kind, we must have $(p_1/q_3) = 1$, which is a contradiction and therefore sub-case 23 cannot occur.

As we have previously observed, sub-case 21 implies sub-case 10.

Sub-cases 24) and 25) By Corollary 1 we see that sub-cases 24 and 25 cannot occur.

Since we have exhausted all possible sub-cases of nontrivial Δ_k -splittings of the second kind when C_k has 4-rank equal to 2 and k satisfies the assumptions of Theorem 2, our result is established for the three negative prime discriminant case when $\Delta_k \equiv 4 \pmod{8}$. $\square$

4-rank of C_k equal to 1. For the case when the 4-rank of C_k is equal to 1, we are able to state the following lemma, where p_1 and p_2 represents positive prime discriminants dividing Δ_k , $-q_1$ represents a negative prime discriminant dividing Δ_k , and either r_i , $2 \leq i \leq 5$, represents a positive prime discriminant dividing Δ_k or $-r_i$ represents a negative prime discriminant dividing Δ_k .

Lemma 6. *Let k be an imaginary quadratic number field with rank $C_{k,2} = 4$ and 4-rank of C_k equal to 1. Assume that the nontrivial Δ_k -splitting of the second kind for k is either (without loss of generality) $(-q_1, r_2r_3r_4r_5)$, $(p_1, -r_2r_3r_4r_5)$, or $(p_1p_2, -r_3r_4r_5)$. Then k has infinite 2-class field tower.*

Proof. If the nontrivial Δ_k -splitting of the second kind for k is $(-q_1, r_2r_3r_4r_5)$ then k has infinite 2-class field tower by Lemma 2. If the nontrivial Δ_k -splitting of the second kind for k is $(p_1, -r_2r_3r_4r_5)$ then k has infinite 2-class field tower by Lemma 1. If the nontrivial Δ_k -splitting of the second kind for k is $(p_1p_2, -r_3r_4r_5)$ then we have the following four possibilities for sets of Legendre symbol equalities:

$$\begin{array}{ll}
 \text{A)} & (p_1/r_3) = (p_2/r_3) = 1 \\
 & (p_1/r_4) = (p_2/r_4) = 1 \\
 & (p_1/r_5) = (p_2/r_5) = 1 \\
 \text{B)} & (p_1/r_3) = (p_2/r_3) = 1 \\
 & (p_1/r_4) = (p_2/r_4) = -1 \\
 & (p_1/r_5) = (p_2/r_5) = -1 \\
 \text{C)} & (p_1/r_3) = (p_2/r_3) = -1 \\
 & (p_1/r_4) = (p_2/r_4) = 1 \\
 & (p_1/r_5) = (p_2/r_5) = -1 \\
 \text{D)} & (p_1/r_3) = (p_2/r_3) = -1 \\
 & (p_1/r_4) = (p_2/r_4) = -1 \\
 & (p_1/r_5) = (p_2/r_5) = 1
 \end{array}$$

We therefore are able to apply Lemma 3 for each of the above four possibilities to conclude that k has infinite 2-class field tower; our result is thus established. $\square$

Remark 2. We note at this point that the fields k which have infinite 2-class field tower in Theorem 1, Theorem 2 and Lemma 6 also have the property that the ranks of the 2-class groups of these fields tend to infinity since they possess an unramified quadratic extension which satisfies the Golod-Shafarevich inequality (cf. [1, 4, 10]).

Remark 3. For examples of imaginary quadratic number fields k with rank $C_{k,2} = 4$ and 4-rank of C_k equal to 1 or 2 such that k has infinite 2-class field tower, please see [1]. We see that the example in [1] of 4-rank of $C_k = 2$ which has infinite 2-class field tower, $k = Q(\sqrt{-3, 989, 045}) = Q(\sqrt{-5 \cdot 11 \cdot 29 \cdot 41 \cdot 61})$, satisfies Theorem 1.

We note that the nontrivial Δ_k -splitting of the second kind for k in all of the examples in [1] for which the 4-rank of C_k equals 1 satisfies one of the conditions

of Lemma 6 (for example $k = Q(\sqrt{-184,008}) = Q(\sqrt{-2 \cdot 3 \cdot 11 \cdot 17 \cdot 41})$, $k = Q(\sqrt{-531,867}) = Q(\sqrt{-3 \cdot 7 \cdot 19 \cdot 31 \cdot 43})$). An example of a sub-case in Theorem 2 for which we are not able to determine if k has infinite 2-class field tower is the following: $k = Q(\sqrt{-4 \cdot 5 \cdot 7 \cdot 13 \cdot 67}) = Q(\sqrt{-121,940})$. We see that k satisfies sub-cases 9 and 15 of Theorem 2 with $(5/2) = (13/2) = (5/7) = (5/67) = (13/7) = (13/67) = -1$, and we are therefore not able to conclude that k has infinite 2-class field tower.

For ease of reference we summarize our findings in the following table:

4-rank of C_k	Exact number of negative prime discriminants dividing Δ_k	$\Delta_k \equiv 4 \pmod{8}$	$\Delta_k \not\equiv 4 \pmod{8}$	Infinite 2-class field tower
4	1, 3, or 5	✓	✓	yes
3	1, 3, or 5	✓	✓	yes
2	1 or 5	✓	✓	yes
2	3		✓	yes
2	3	✓		?
1	1	✓		yes
1	1		✓	?
1	3 or 5	✓	✓	?
0	1	cannot occur	✓	?
0	3 or 5	✓	✓	?

TABLE 1. Infinite 2-class Field Tower for Imaginary Quadratic Number Fields with 2-class Group of Rank 4

In conclusion, we see that the conjecture concerning the 2-class field tower of k being infinite holds whenever the 4-rank of C_k is greater than or equal to 3, or equal to 2 when either one or five negative prime discriminants divide Δ_k or Δ_k is not congruent to 4 mod 8, assuming that $\text{rank } C_{k,2} = 4$. With particular conditions on the nontrivial Δ_k -splitting of the second kind for k , the conjecture also holds for certain families of fields k when the 4-rank of C_k is equal to 1. However, the complete resolution of the conjecture concerning all imaginary quadratic number fields k with $\text{rank } C_{k,2} = 4$ is still an open question.

Acknowledgement. I would like to express my gratitude to both Chip Snyder and Franz Lemmermeyer for a number of helpful suggestions throughout this paper.

Résumé substantiel en français. Soit k un corps de nombres algébriques et soit $C_{k,2}$ son 2-groupe de classes, c'est-à-dire le 2-sous-groupe de Sylow du groupe C_k de classes (au sens large) de k . Dénons par k_1 le 2-corps de classes de k . Pour $n \geq 0$, définissons k_n par $k_0 = k$ et $k_{n+1} = (k_n)_1$. Alors $k_0 \subseteq k_1 \subseteq k_2 \subseteq \cdots \subseteq k_n \subseteq \cdots$ est appelée la tour des 2-corps de classes de k . Si n est l'entier minimal pour lequel $k_n = k_{n+1}$, alors

on dit que n est la longueur de la tour des 2-corps de classes. Si un tel entier n n'existe pas, alors on dit que la tour des 2-corps de classes est infinie.

Il est bien connu que si k est un corps quadratique imaginaire dont le rang de $C_{k,2}$ est ≥ 5 , alors la tour des 2-corps de classes de k est infinie. Si $\text{rang}(C_{k,2}) = 2$ ou 3, alors la tour des 2-corps de classes peut être finie ou infinie. Si $\text{rang}(C_{k,2}) = 1$, alors la longueur de cette tour est 1. Par contre, si $\text{rang}(C_{k,2}) = 4$, on ne sait pas si la tour des 2-corps de classes de k est toujours infinie. Dans un de ses articles, Martinet a posé la question de savoir si c'est toujours le cas.

Hajir a obtenu un résultat partiel dans cette direction : si k est un corps quadratique imaginaire dont le 4-rang de $C_{k,2}$ est ≥ 3 , alors la tour des 2-corps de classes de k est infinie. Dans un travail antérieur, nous avons montré que si k est un corps quadratique imaginaire dont le rang de $C_{k,2}$ est 4, dont le 4-rang de C_k est ≥ 2 , et dont le discriminant Δ_k est divisible par un seul discriminant premier négatif, alors la tour des 2-corps de classes de k est infinie. On dit que d est un discriminant premier négatif si $d < 0$ et $-d$ est un premier ou vaut 4 ou 8. Dans cet article, nous généralisons à la fois le résultat de Hajir et notre résultat antérieur ; plus précisément, nous montrons que la tour des 2-corps de classes de k est infinie, si k est un corps quadratique imaginaire ayant les propriétés suivantes : le rang de $C_{k,2}$ est 4, le 4-rang de C_k est ≥ 2 et ou bien Δ_k est divisible par cinq discriminants premiers ou bien $\Delta_k \not\equiv 4 \pmod{8}$. Nous démontrons aussi que si $\Delta_k \equiv 4 \pmod{8}$ et si exactement 3 discriminants premiers négatifs divisent Δ_k , alors la tour des 2-corps de classes de k est infinie lorsque les symboles de Kronecker des premiers divisant Δ_k n'ont pas une certaine configuration. Finalement, nous exhibons des conditions suffisantes pour obtenir des corps quadratiques imaginaires dont le rang de $C_{k,2}$ est 4, dont le 4-rang de C_k est 1 et dont la tour des 2-corps de classes est infinie. Les méthodes utilisées dans nos preuves sont essentiellement combinatoires et sont basées sur des résultats antérieurs reliés à l'inégalité de Golod-Shafarevich.

REFERENCES

1. E. Benjamin, *On imaginary quadratic number fields with 2-class group of rank 4 and infinite 2-class field tower*, Pacific J. Math. **201** (2001), 257–266.
2. E. Benjamin, F. Lemmermeyer and C. Snyder, *Imaginary quadratic fields K with cyclic $Cl_2(K^1)$* , J. Number Theory **67** (1997), 229–245.
3. E. Golod and I. Shafarevich, *On class field towers*, Izv. Akad. Nauk SSSR **28** (1964), 261–272 (in Russian); English translation in, Amer. Math. Soc. Transl. **48** (1965), 91–102.
4. F. Hajir, *On the growth of p -class groups in p -class field towers*, J. Algebra **188** (1997), 256–271.
5. F. Hajir, *On a theorem of Koch*, Pacific J. Math. **176** (1996); no. 1, 15–18.
6. F. Hajir, *Correction to "On a theorem of Koch"*, Pacific J. Math. **196** (2000); no. 2, 507–508.
7. H. Koch, *Zum Satz von Golod-Schafarewitsch*, Math. Nachr. **42** (1969), 321–333.
8. J. C. Lagarias, *On determining the 4-rank of the ideal class group of quadratic field*, J. Number Theory **12** (1980), 191–196.
9. F. Lemmermeyer, *On 2-class field towers of some imaginary quadratic number fields*, Abh. Math. Sem. Univ. Hamburg **67** (1997), 205–214.
10. J. Martinet, *Tours de corps de classes et estimations de discriminants*, Invent. Math. **44** (1978), 65–73.
11. J. Martinet, *Discriminant de corps de nombres*, Journées Arithmétiques 1980 (JV Armitage, ed.), vol. 1404, Cambridge Univ. Press, 1987, pp. 151–193.

12. L. Rédei, *Arithmetischer beweis des satzes über die anzahl der durch uier teilbaren invarianten der absoluten klassengruppe in quadratischen zahlkörper*, Crelle **171** (1934), 55–60.
13. L. Rédei and H. Reichardt, *Die anzahl der durch 4 teilbaren invarianten der klassengruppe eines beliebigen quadratischen zahlkörpers*, J. Reine Angew. Math. **170** (1966), 69–74.
14. B. Schmithals, *Konstruktion imaginärquadratischer körper mit unendlichem klassenkörperturm*, Arch. Math. (Basel) **34** (1980), 307–312.

DR. ELLIOT BENJAMIN
DEPARTMENT OF MATHEMATICS
UNITY COLLEGE
UNITY, MAINE 04988-9502
U.S.A.
E-MAIL: benjamin@mint.net