

CALCULUS OF THE FUNDAMENTAL MATRIX FOR GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

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RÉSUMÉ. On donne deux méthodes pour le calcul de la matrice fondamentale associée aux équations différentielles linéaires généralisées sur l'espace de Banach des fonctions à variation bornée. Le résultat principal de l'article généralise dans ce cadre la formule de Peano-Baker.

ABSTRACT. Two methods are given for the calculus of the fundamental matrix for the generalized linear differential equations on the Banach space of functions of bounded variation. The main result extends the Peano-Baker formula in this framework.

1. Introduction. The theory of generalized linear differential equations in the space of functions of bounded variation was developed by Kurzweil, [6, 7], Halanay and Moro [4], Schwabik, Tvrdý and Vejvoda [12], on the basis of the Perron-Stieltjes integral. Tvrdý [13, 14] extended this theory for equations with right hand member in the class of regulated functions. Similar problems in the space of regulated functions were studied by Hönig [5], Fichmann [3] and Barbanti [1], where the interior (Dushnik) integral was employed. This theory was used by Tvrdý [14] and Prepelitã *et al.* [8, 9, 10] in the study of generalized dynamical and acausal systems.

An important device in the study of such systems is the (generalized) fundamental matrix.

In this paper two methods are given for the effective calculus of the (generalized) fundamental matrix. A Peano-Baker type formula is proposed and the uniform convergence of the matrix series is proved.

2. Generalized linear differential equations and the Perron-Stieltjes integral.

Throughout this paper, $[a, b] \subset \mathbb{R}$ is a fixed interval; BV , BV^n and $BV^{n \times n}$ denote the Banach spaces of functions of bounded variation on $[a, b]$ and taking values in $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ respectively (the norm in BV is $\|f\|_{BV} = |f(a)| + \text{var}_a^b f$).

A function $f : [a, b] \rightarrow \mathbb{R}$ which possesses finite limits $f(t+)$ and $f(t-)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions on $[a, b]$ is denoted by G ; this set endowed with the supremal norm is a Banach space. Obviously, $BV \subset G$. The Banach space of

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n -vector functions with entries in G is denoted by G^n . G_L^n denotes the subspace of all functions in G^n which are left-continuous on $[a, b]$.

We denote by Δ^+ and Δ^- the operators defined by $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(t) = f(t) - f(t-)$, $t \in (a, b)$, $f \in G$; similar operators are defined for $f \in BV^{n \times n}$.

The Perron-Stieltjes integral is introduced by an equivalent definition, due to Kurzweil [6]. This integral includes the σ -Riemann-Stieltjes integral (see Schwabik [11]).

An arbitrary function $\delta : [a, b] \rightarrow \mathbb{R}^+$ is called a *gauge* on $[a, b]$. A *partition* P of $[a, b]$ is a sequence of points in $[a, b]$,

$$P = \{t_0, \tau_1, t_1, \tau_2, \dots, t_{j-1}, \tau_j, t_j, \dots, \tau_k, t_k\}$$

such that $t_0 = a, t_k = b$ and $t_{j-1} < t_j, t_{j-1} \leq \tau_j \leq t_j, j = 1, \dots, k$.

Given a gauge δ on $[a, b]$, the partition P of $[a, b]$ is said to be δ -*fine* if $[t_{j-1}, t_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$, $j = 1, \dots, k$. Given two functions $f, g : [a, b] \rightarrow \mathbb{R}$ and a partition P we associate the integral sum

$$S_{f,g}(P) = \sum_{j=1}^k f(\tau_j)(g(t_j) - g(t_{j-1})).$$

Definition 2.1. A number $I \in \mathbb{R}$ is called the *Perron-Stieltjes (Kurzweil) integral* of f with respect to g from a to b and is denoted by $I = \int_a^b f dg$ or $I = \int_a^b f(t) dg(t)$ if for any $\epsilon > 0$, there exists a gauge δ on $[a, b]$ such that $|I - S_{f,g}(P)| < \epsilon$ for all δ -fine partitions P of $[a, b]$.

If $f \in BV$ and $g \in G$ then the integrals $\int_a^b f dg$ and $\int_a^b g df$ exist (see [13], Theorem 2.8).

Let us summarize some properties of the Perron-Stieltjes integral that we need in the sequel.

Proposition 2.2. If f is nonnegative, g is nondecreasing on $[a, b]$ and $\int_a^b f dg$ exists, then $\int_a^b f dg \geq 0$.

Proof. Since $f(t) \geq 0, \forall t \in [a, b]$ and $g(t_j) - g(t_{j-1}) \geq 0, j = 1, \dots, k$ it results that $S_{f,g}(P) \geq 0$ for all δ -fine partition P of $[a, b]$. But by definition, for any $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that $|\int_a^b f dg - S_{f,g}(P)| < \epsilon$, hence $\int_a^b f dg \geq 0$.

For $f = f_1 - f_2$ or $g = g_1 - g_2$, $f_i : [a, b] \rightarrow \mathbb{R}, g_i : [a, b] \rightarrow \mathbb{R}, i = 1, 2$, we have

Corollary 2.3. If $f_1(t) \geq f_2(t), \forall t \in [a, b]$, g is nondecreasing on $[a, b]$ and the integrals $\int_a^b f_1 dg, \int_a^b f_2 dg$ exist, then $\int_a^b f_1 dg \geq \int_a^b f_2 dg$.

Corollary 2.4. If $f(t) \geq 0, \forall t \in [a, b]$, the function $g_1 - g_2$ is nondecreasing on $[a, b]$ and the integrals $\int_a^b f dg_1$ and $\int_a^b f dg_2$ exist, then $\int_a^b f dg_1 \geq \int_a^b f dg_2$.

The following four propositions are obtained from [13] Theorem 2.7 and [12] Corollary I.4.27, Theorem I.4.29 and Lemma I.4.21, respectively.

Theorem 2.5. If $f_n : [a, b] \rightarrow \mathbb{R}, n = 1, 2, \dots, \lim_{n \rightarrow \infty} f_n = f$ uniformly on $[a, b]$, $g \in G$ and $\int_a^b f_n dg$ exists for all $n = 1, 2, \dots$ then the limit $\lim_{n \rightarrow \infty} \int_a^b f_n dg$ as well as $\int_a^b f dg$ exist and they are equal.

Theorem 2.6. If $f, g \in BV$ then

$$\left| \int_a^b f dg \right| \leq \int_a^b |f(t)| d(\text{var}_a^t g) \quad (2.1)$$

Theorem 2.7. If $h : [a, b] \rightarrow \mathbb{R}$ is nonnegative, nondecreasing and continuous from the left in $[a, b]$, then, for any $k \in \mathbb{N}$,

$$\int_a^b h^k(t) dh(t) \leq \frac{1}{k+1} [h^{k+1}(b) - h^{k+1}(a)] \quad (2.2)$$

Proposition 2.8. If $f \in BV$ then $\int_a^b df = f(b) - f(a)$.

A generalized linear differential equation (GLDE) is denoted by the symbol

$$dx = d[A]x + dg \quad (2.3)$$

where $A \in BV^{n \times n}$ and $g \in G^n$.

Definition 2.9. A function $x : [a, b] \rightarrow \mathbb{R}^n$ is said to be a *solution* of the GLDE (2.3) on $[a, b]$ if for any $t, t_0 \in [a, b]$ the equality

$$x(t) = x(t_0) + \int_{t_0}^t d[A(s)]x(s) + g(t) - g(t_0) \quad (2.4)$$

holds.

Remark 2.10. Let us consider the particular case of absolutely continuous functions A and g having the expressions $A(t) = \int_a^t B(s)ds$, $g(t) = \int_a^t h(s)ds$, $t \in [a, b]$, where $B : [a, b] \rightarrow \mathbb{R}^{n \times n}$ and $h : [a, b] \rightarrow \mathbb{R}^n$ are continuous on $[a, b]$. Since $\int_a^t d[A(s)]x(s) = \int_a^t d[\int_a^s B(s)ds]x(s) = \int_a^t B(s)x(s)ds$ (see [12], Remark I.4.35) it results from (2.4) that the GLDE (2.3) is equivalent to the linear ordinary differential equation

$$\dot{x}(t) = B(t)x(t) + h(t) \quad (2.5)$$

hence GLDE (2.3) is a generalization of (2.5).

Given $t_0 \in [a, b]$ and $x_0 \in \mathbb{R}^n$, the solution which verifies the initial condition

$$x(t_0) = x_0 \quad (2.6)$$

is said to be the solution of the initial value problem (2.4), (2.6).

One associates to the nonhomogeneous GLDE (2.4) the homogeneous equation

$$dx = d[A]x. \quad (2.7)$$

In [12], Theorem III.2.10, the generalized fundamental matrix of the homogeneous equation (2.7) was introduced:

Theorem 2.11. *If the matrix valued function $A \in BV^{n \times n}$ has the properties*

$$\det(I + \Delta^+ A(t)) \neq 0 \quad (2.8)$$

$$\det(I - \Delta^- A(t)) \neq 0 \quad (2.9)$$

then there exists a unique matrix valued function $U : [a, b] \times [a, b] \rightarrow \mathbb{R}^{n \times n}$ such that

$$U(t, s) = I + \int_s^t d[A(r)]U(r, s) \quad (2.10)$$

for any $t, s \in [a, b]$. The fundamental matrix U has the following properties, for any $r, s, t \in [a, b]$: There exists $M > 0$ such that $\|U(t, s)\| + \text{var}_a^b U(t, \cdot) + \text{var}_a^b U(\cdot, s) + v(U) \leq M < \infty$; here $v(U)$ denotes the bidimensional (Vitali) variation of U on $[a, b] \times [a, b]$.

$$U(t, s) = U(t, r)U(r, s), \quad (2.11)$$

$$U(t, t) = I, \quad (2.12)$$

$$U(t+, s) = [I + \Delta^+ A(t)]U(t, s), U(t-, s) = [I - \Delta^- A(t)]U(t, s), \quad (2.13)$$

$$U(t, s+) = U(t, s)[I + \Delta^+ A(s)]^{-1}, U(t, s-) = U(t, s)[I - \Delta^- A(s)]^{-1}, \quad (2.14)$$

$$U(t, s)^{-1} = U(s, t). \quad (2.15)$$

On the basis of the generalized fundamental matrix one obtains the variation of parameters formula (see [12], Theorem III.2.13. and [14], Remark 2.3).

Theorem 2.12. *If $A \in BV^{n \times n}$ verifies the conditions (2.8) and (2.9), then for any $t_0 \in [a, b]$, $x_0 \in \mathbb{R}^n$, $g \in G^n$ the solution of the initial value problem (2.4), (2.6) is given by the formula*

$$x(t) = U(t, t_0)x_0 + g(t) - g(t_0) - \int_{t_0}^t d_s[U(t, s)](g(s) - g(t_0)), t \in [a, b] \quad (2.16)$$

and $x \in G^n$.

If $g \in BV^n$ then $x \in BV^n$; if A is continuous on the left on $[a, b]$ and $g \in G_L^n$, then $x \in G_L^n$.

3. Peano-Baker formula for generalized fundamental matrices.

In this section we establish a Peano-Baker type formula using a recurrence construction suggested by (2.10) and we emphasize sufficient conditions for the convergence of the corresponding matrix series.

Theorem 3.1. *If the matrix function $A = [a_{ij}]_{i,j=1}^n \in BV^{n \times n}$ is continuous on the left on $[a, b]$ and it verifies the assumption*

$$\det(I + \Delta^+ A(t)) \neq 0, \forall t \in [a, b] \quad (3.1)$$

then for any $t_0 \in [a, b]$ the sequence of matrices $\{M_k(t, t_0)\}_{k \geq 0}$ defined recursively by

$$M_0(t, t_0) = I \quad (3.2)$$

$$M_k(t, t_0) = I + \int_{t_0}^t d[A(s)]M_{k-1}(s, t_0) \quad (3.3)$$

converges uniformly with respect to t on $[t_0, b]$ and

$$\lim_{k \rightarrow \infty} M_k(t, t_0) = U(t, t_0). \quad (3.4)$$

Proof. If A is continuous on the left on $[a, b]$ then $\Delta^- A(t) = A(t) - A(t-) = 0_n$ (the null matrix of order n) hence $\det(I - \Delta^- A(t)) = \det I = 1 \neq 0$, that is the condition (2.9) holds; the assumption (3.1) implies the existence of the fundamental matrix $U(t, s)$, $t, s \in [a, b]$ (see Theorem 2.11).

Let us denote by $E_{ij}(A)$ the element a_{ij} of the matrix A . For a point $t_0 \in [a, b]$ arbitrary fixed, let us consider a function $h : [a, b] \rightarrow \mathbb{R}$ having the properties:

- i) h is positive, nondecreasing and continuous on the left on $[a, b]$,
 - ii) $h(t_0) = 0$,
 - iii) for any $s, t \in [t_0, b]$, $s < t$, $h(t) - h(s) \geq \max_{1 \leq i, j \leq n} \text{var}_s^t a_{ij}$
- (3.5)

For example the function

$$h(t) = \begin{cases} 0 & \text{for } t \in [a, t_0] \\ \sum_{i=1}^n \sum_{j=1}^n \text{var}_{t_0}^t a_{ij} & \text{for } t \in (t_0, b] \end{cases}$$

has the above three properties.

From the inequality (3.5), it results that the function $h(t) - \text{var}_{t_0}^t a_{ij}$ is nondecreasing for any $t \in [t_0, b]$ and any $i, j \in \mathbb{N}$, $1 \leq i, j \leq n$.

We shall establish by induction an inequality verified by the elements of the matrices $M_k(t, t_0)$.

By using Theorem 2.6 with $f \equiv 1$, Corollary 2.4 and the fact that $h(t) - \text{var}_{t_0}^t a_{ij}$ is nondecreasing as in the Proposition 2.8, we obtain:

$$\begin{aligned} \left| E_{ij} \left(\int_{t_0}^t dA(s) \right) \right| &= \left| \int_{t_0}^t dE_{ij}(A(s)) \right| = \left| \int_{t_0}^t da_{ij}(s) \right| \\ &\leq \int_{t_0}^t d(\text{var}_{t_0}^t a_{ij}) \leq \int_{t_0}^t dh = h(t) - h(t_0) = h(t). \end{aligned}$$

Using this inequality, Theorem 2.6, Corollaries 2.4 and 2.3 and Theorem 2.7, it results, for any $1 \leq i, j, l \leq n$:

$$\begin{aligned} \left| \int_{t_0}^t da_{il}(s_1) \int_{t_0}^{s_1} da_{lj}(s_2) \right| &\leq \int_{t_0}^t d(\text{var}_{t_0}^{s_1} a_{il}) \left| \int_{t_0}^{s_1} da_{lj}(s_2) \right| \\ &\leq \int_{t_0}^t d(\text{var}_{t_0}^{s_1} h(s_1)) \leq \int_{t_0}^t dh(s_1) h(s_1) \leq \frac{1}{2} (h^2(t) - h^2(t_0)) = \frac{1}{2!} h^2(t). \end{aligned}$$

Then

$$\left| E_{ij} \left(\int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \right) \right| \leq \sum_{l=1}^n \left| \int_{t_0}^t da_{il}(s_1) \int_{t_0}^{s_1} da_{lj}(s_2) \right| \leq \frac{n}{2!} h^2(t).$$

Let us assume that

$$\left| E_{ij} \left(\int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k) \right) \right| \leq \frac{n^{k-1}}{k!} h^k(t). \quad (3.6)$$

Then, using again the above mentioned Theorems, we obtain

$$\begin{aligned} & \left| E_{ij} \left(\int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k) \int_{t_0}^{s_k} dA(s_{k+1}) \right) \right| \\ & \leq \sum_{l=1}^n \left| E_{il} \left(\int_{t_0}^t dA(s_1) \right) E_{lj} \left(\int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k) \int_{t_0}^{s_k} dA(s_{k+1}) \right) \right| \\ & \leq \sum_{l=1}^n \int_{t_0}^t d(\text{var}_{t_0}^{s_1} a_{il}) \left| E_{lj} \left(\int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k) \int_{t_0}^{s_k} dA(s_{k+1}) \right) \right| \\ & \leq \sum_{l=1}^n \int_{t_0}^t d(\text{var}_{t_0}^{s_1} a_{il}) \frac{n^{k-1}}{k!} h^k(s_1) \leq n \frac{n^{k-1}}{k!} \int_{t_0}^t dh(s_1) h^k(s_1) \\ & \leq \frac{n^k}{k!} \frac{1}{k+1} (h^{k+1}(t) - h^{k+1}(t_0)) = \frac{n^k}{(k+1)!} h^{k+1}(t) \end{aligned}$$

hence (3.6) is true for any $k \in \mathbb{N}$.

From (3.2) and (3.3) it results:

$$\begin{aligned} M_k(t, t_0) = I + \int_{t_0}^t dA(s_1) + \int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) + \cdots \\ + \int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k). \end{aligned} \quad (3.7)$$

Therefore, for any $t \in [t_0, b]$, $1 \leq i, j \leq n$ and $k \in \mathbb{N}$ we have:

$$\begin{aligned} |E_{ij}(M_k(t, t_0))| & \leq 1 + \left| E_{ij} \left(\int_{t_0}^t dA(s_1) \right) \right| + \left| E_{ij} \left(\int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \right) \right| + \cdots \\ & \quad + \left| E_{ij} \left(\int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k) \right) \right| \\ & \leq 1 + h(t) + \frac{n}{2!} h^2(t) + \cdots + \frac{n^{k-1}}{k!} h^k(t) \end{aligned}$$

since h is nondecreasing on $[t_0, b]$.

The last sequence of positive real numbers is convergent since,

$$\lim_{k \rightarrow \infty} \left(1 - \frac{1}{n} + \frac{1}{n} \left(1 + \frac{nh(b)}{1!} + \frac{n^2 h^2(b)}{2!} + \cdots + \frac{n^k h^k(b)}{k!} \right) \right) = 1 - \frac{1}{n} + \frac{1}{n} \cdot e^{nh(b)}$$

and it results that the sequence $\{E_{ij}(M_k(t, t_0))\}_{k \geq 0}$ is uniformly convergent with respect to t on $[t_0, b]$ and the matrix sequence $\{M_k(t, t_0)\}_{k \geq 0}$ has the same property. Since $A \in BV^{n \times n}$, it results from (3.2), (3.3) and [12], Corollary I.4.27 that $M_k(t, t_0) \in BV^{n \times n}$ for any $k \in \mathbb{N}$.

One obtains from Theorem 2.5 that the integrals $\int_{t_0}^t d[A(s)]M_k(s, t_0)$ exist for any $k \geq 1$ and

$$\lim_{k \rightarrow \infty} \int_{t_0}^t d[A(s)]M_k(s, t_0) = \int_{t_0}^t d[A(s)] \left(\lim_{k \rightarrow \infty} M_k(s, t_0) \right).$$

By taking the limits as $k \rightarrow \infty$ in (3.3) we obtain

$$\lim_{k \rightarrow \infty} M_k(t, t_0) = I + \int_{t_0}^t d[A(s)] \left(\lim_{k \rightarrow \infty} M_k(s, t_0) \right)$$

that is $\lim_{k \rightarrow \infty} M_k(t_0, t)$ verifies the equation (2.10). By uniqueness it results that

$$U(t, t_0) = \lim_{k \rightarrow \infty} M_k(t, t_0).$$

Taking the limit as $k \rightarrow \infty$ in (3.7) we obtain

Corollary 3.2. *If $A \in BV^{n \times n}$ is continuous on the left and it satisfies the assumption (3.1), then the fundamental matrix $U(t, t_0)$ is given by Peano-Baker formula*

$$\begin{aligned} U(t, t_0) = I &+ \int_{t_0}^t dA(s_1) + \int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) + \cdots \\ &+ \int_{t_0}^t dA(s_1) \int_{t_0}^{s_1} dA(s_2) \cdots \int_{t_0}^{s_{k-1}} dA(s_k) + \cdots \end{aligned} \quad (3.8)$$

4. Calculus of the fundamental matrix for a class of system matrices of bounded variation.

The inequality (2.2) is essential in the proof of the convergence of the generalized fundamental matrix. But if the function h is nonnegative, nonincreasing and continuous on the right, the opposite inequality holds (see [12] Theorem I.4.29). It results that if the system matrix A is not continuous on the left, we cannot prove the convergence of the matrix series (3.8).

In this paragraph we propose an algorithm for the calculus of the fundamental matrix for a particular subclass of $BV^{n \times n}$; this algorithm works even for system matrices that are not continuous on the left.

Theorem 4.1. *Let us assume that $A \in BV^{n \times n}$ has the expression $A(t) = \int_a^t B(s) ds$ for some matrix $B \in \mathbb{R}^{n \times n}$ which is continuous on $[a, b]$ with the exception of a finite number of points $t_i \in (a, b)$, $i = 1, \dots, k$, $t_i < t_{i+1}$.*

If the conditions (2.8) and (2.9) hold, then the fundamental matrix $U(t, s)$ can be calculated by the following algorithm:

Step 1 *Calculate the fundamental matrix $\Phi(t, s)$ corresponding to the system of ordinary differential equations $\dot{x}(t) = B(t)x(t)$, for $t, s \in [a, t_1)$, $t, s \in (t_i, t_{i+1})$,*

$i = 1, \dots, k-1$ and $t, s \in (t_k, b]$ (using for example the “classical” Peano-Baker formula, see for instance [2]).

Step 2 Put $U(t, a) = \Phi(t, a)$ for $t \in [a, t_1]$.

Step 3 If $U(t, a)$ is determined for $t \in [a, t_i]$ then calculate

$$\begin{aligned} U(t_i, a) &= [I - \Delta^- A(t_i)]^{-1} U(t_i-, a), \\ U(t_i+, a) &= [I + \Delta^+ A(t_i)] U(t_i, a), \\ U(t, a) &= \Phi(t, t_i) U(t_i+, a) \text{ for } t \in (t_i, t_{i+1}). \end{aligned}$$

Do Step 3 for $i = 1, \dots, k$ (where $t_{k+1} = b$).

Step 4 Calculate $U(t, s)$ for $t, s \in [a, b]$ by the formula

$$U(t, s) = U(t, a) U(s, a)^{-1}.$$

Proof. The uniqueness of the fundamental matrix as well as the formulae (2.11)–(2.15) prove that the result of this algorithm is indeed the fundamental matrix $U(t, s)$.

5. Concluding remarks.

The two classes of system matrices analysed in paragraphs 3 and 4 are those which are natural in the study of GLDE (see [14], §2). These methods for the calculus of the fundamental matrix are useful in the study of generalized dynamical and acausal systems, of the systems with the boundary condition containing a Perron-Stieltjes integral, 2D generalized systems, etc.

Résumé substantiel en français.

La théorie des équations différentielles linéaires généralisées sur l'espace des fonctions à variation bornée a été développée par Kurzweil, Halanay et Moro, Schwabik, Tvrdý et Vejvoda sur la base de l'intégrale Perron-Stieltjes. Cette théorie a été utilisée par Tvrdý et Prepeliță dans l'étude des systèmes dynamiques généralisés et des systèmes commandés aux limites. Un instrument important dans cette théorie est la matrice fondamentale (généralisée).

Dans cet article sont établies deux méthodes pour le calcul effectif de la matrice fondamentale des systèmes sur l'espace de Banach des fonctions à variation bornée.

L'intégrale de Perron-Stieltjes est introduite (Définition 2.1) et quelques propriétés de cette intégrale sont présentées (Proposition 2.2 et 2.8, Corollaires 2.3 et 2.4 et les Théorèmes 2.5, 2.6 et 2.7). Ces propriétés sont nécessaires pour la démonstration de la convergence uniforme de la série de Peano-Baker.

On définit les équations différentielles linéaires généralisées (GLDE) sur l'espace de Banach des fonctions à variation bornée, ainsi que la solution d'une GLDE (Définition 2.9, équations (2.3) et (2.4)).

Le Théorème 2.11 contient les conditions suffisantes pour l'existence et l'unicité de la matrice fondamentale d'une GLDE (conditions (2.8) et (2.9)), ainsi que les principales propriétés de cette matrice (formules (2.11)–(2.15)). La formule de variation des constantes (2.16) est donnée dans le Théorème 2.12.

Le résultat principal de l'article est le Théorème 3.1; on montre que si la matrice de la GLDE est continue à gauche et vérifie l'hypothèse (3.1) alors la suite des matrices

à variation bornée $M_k(t, t_0)$ définie par les formules (3.2) et (3.3) est uniformément convergente sur l'intervalle $[t_0, b]$. On montre aussi que la limite de cette suite est la matrice fondamentale. Il en résulte une formule de type Peano-Baker dans ce cadre généralisé (Corollaire (3.2)).

Quand les conditions suffisantes mentionnées ne sont pas remplies, on propose une autre méthode qui utilise (par morceaux) la matrice fondamentale classique; cet algorithme est donné par le Théorème 4.1.

Ces méthodes pour le calcul de la matrice fondamentale d'une GLDE sont utiles dans l'étude des systèmes dynamiques généralisés, des systèmes commandés aux limites contenant une intégrale de Perron-Stieltjes, des systèmes 2D généralisés, etc.

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