

ON SUBDIFFERENTIAL CALCULUS FOR CONVEX FUNCTIONS DEFINED ON LOCALLY CONVEX SPACES

C. COMBARI, M. LAGHDIR AND L. THIBAUT

RÉSUMÉ. Nous démontrons la formule du sous-différentiel de la somme de deux fonctions convexes sous une condition de qualification très générale et nous prouvons que cette condition est impliquée par toutes les conditions de qualification utilisées jusqu'ici.

ABSTRACT. The subdifferential formula for the sum of two convex functions defined on a locally convex space is proved under a general qualification condition. It is proved that all the similar results which are already known can be derivated from the formula.

1. Introduction. The importance of the subdifferential formula for the sum of two convex functions f and g is actually well-known in convex analysis, optimization, mechanics, game theory, mathematical economics

It is also well-known that such an exact fomula requires a qualification condition. The most popular Moreau-Rockafellar qualification condition assumes that the underlying space X is locally convex and f is finite and continuous at some point where g is finite. This condition has been weakened later in some sense by Rockafellar [15] and Robinson [14] and also by Attouch and Brézis [1]. Rockafellar [15] supposed that f and g are lower semicontinuous on X , $0 \in \text{Core}(\text{dom} f - \text{dom} g)$ and X is a reflexive Banach space. Robinson [14], via his open mapping theorem, shows that the method by Rockafellar still holds for any Banach space X (see also the book by Aubin and Ekeland [2] for another independant proof). Later Attouch and Brézis proved independently and by a different method the formula for any Banach space X under the assumption that f and g are lower semicontinuous on X and $\mathbb{R}_+(\text{dom} f - \text{dom} g)$ is a closed vector subspace. This qualification condition is more general than Rockafellar and Robinson condition, since it is not difficult to see that $0 \in \text{Core}(\text{dom} f - \text{dom} g)$ if only if $X = \mathbb{R}_+(\text{dom} f - \text{dom} g)$ (note also that the Attouch-Brézis result can be derived from Robinson and Rockafellar result). However, these qualification conditions do not totally include Moreau and Rockafellar conditions which does not require the lower semicontinuity of f and g . This situation has recently led Azé [3] to introduce in any

Reçu le 12 décembre 1998 et, sous forme définitive, le 18 juin 1998.

normed vector space X the qualification that supposed there exist some real numbers $\lambda > 0, s > 0$ and $r > 0$ such that

$$s\mathbb{B}_X \subseteq [f \leq \lambda] \cap r\mathbb{B}_X - [g \leq \lambda] \cap r\mathbb{B}_X \quad (1)$$

where $[f \leq \lambda]$ denotes the λ -sublevel set of f and $\mathbb{B}_X$ the closed unit ball of X . Azé showed that all the above qualification condition ensures (1). Nevertheless, condition (1) requires X to be normed whereas the Moreau-Rockafellar condition does not need such a restriction.

The aim of this paper is to introduce, for any locally convex vector space, a qualification inspired by (1) and to prove with this condition in a unified way the subdifferential formula for sums of functions. More generally we will study the subdifferential of functions of the form $f + g \circ h$ where h is a convex operator. The considered qualification condition is so general that it allows to derive all known (exact) subdifferential formulas for sums of convex functions even the ones by Simons [19] and Zalinescu [22].

In Sections 2 and 3 we recall some definitions and establish some preliminary results. Section 4 is devoted to the case where g is nondecreasing only over an appropriate subset of this space. Finally, the relationship between the above qualification conditions is studied in Section [5].

2. Notations and definitions.

Throughout the following, let X and Y be two locally convex real topological vector spaces and X^*, Y^* respectively their dual spaces. In both cases, we denote the duality functional by $\langle \cdot, \cdot \rangle$ understanding in each case that we consider the duality (X^*, X) or (Y^*, Y) . Suppose these spaces are equipped with compatible topologies with the respect to the dual systems. Naturally, we obtain a duality between $X^* \times Y^*$ and $X \times Y$ given by

$$\langle (x^*, y^*), (x, y) \rangle = \langle x^*, x \rangle + \langle y^*, y \rangle$$

for every $(x, y) \in X \times Y$ and $(x^*, y^*) \in X^* \times Y^*$. We assume that the space Y is endowed with a partial order (symbol: $\leq_Y$) induced by a convex cone Y_+ and we denote by Y_+^* the dual positive cone of Y^* defined by

$$Y_+^* := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in Y\}.$$

The cone Y_+ is called the set of nonnegative elements of Y . If we adjoin an abstract maximal element $+\infty$ to Y with $+\infty + y = +\infty$ for all $y \in Y \cup \{+\infty\}$, a mapping $h : X \rightarrow Y \cup \{+\infty\}$ is said to be Y_+ -convex if the following inequality

$$h(\alpha x_1 + \beta x_2) \leq_Y \alpha h(x_1) + \beta h(x_2)$$

holds whenever $x_1, x_2 \in X$ and α, β are positive numbers with $\alpha + \beta = 1$. A function $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be Y_+ -nondecreasing on a subset C of Y , if for each $y_1, y_2 \in C$ satisfying $y_1 \leq_Y y_2$ we have $g(y_1) \leq g(y_2)$.

For a given function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$, one usually denotes by $\text{dom } f := \{x \in X; f(x) < +\infty\}$ its effective domain, by $\text{epi } f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its epigraph, by $f^* : X^* \rightarrow \bar{\mathbb{R}}, f^*(x^*) := \sup\{\langle x^*, x \rangle - f(x) : x \in X\}$ its conjugate

function, by $[f \leq r] := \{x \in X : f(x) \leq r\}$ its r -sublevel set and by $\partial f(\bar{x}) := \{x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x} \rangle, \forall x \in X\}$ its subdifferential at $\bar{x} \in X$. If $\text{dom} f \neq \emptyset$ and $f(X) \subset \mathbb{R} \cup \{+\infty\}$ one says that f is proper. For a nonempty convex C , one usually defines the indicator function $\delta_C : X \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise.

3. Conjugate duality.

Our purpose in this section is to establish the well-known fundamental duality theorem under a new qualification condition which we will explicit in the sequel. This result will play a crucial role in the following of the paper. Before stating the duality theorem, we establish the following result. It uses the Mackey topology and we refer the reader to [5] (for example) for this notion.

Lemma 1. *Let $p : Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be a given convex function and $Z := \text{Vect}(\text{dom } p)$, the vector space generated by $\text{dom } p$ endowed with the topology induced by Mackey topology of Y . If $p(0) \in \mathbb{R}$ and if there exists $\lambda \in \mathbb{R}$ such that the following condition*

$$(C.Q_1) : 0 \in \text{int}_Z([p \leq \lambda])$$

holds, then $\partial p(0) \neq \emptyset$. (Here int_Z denotes the topological interior in the space Z .)

Proof. The argument is classical. Let us consider $p_0 : Z \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ the restriction of p over Z . Then p_0 is convex, finite at zero and majored over some neighbourhood of zero in Z . Hence it follows from a classical convex analysis result [6], that $\partial p_0(0) \neq \emptyset$. Taking $y_0^* \in \partial p_0(0)$, any continuous linear functional y^* extending y_0^* to all Y is easily seen to be in $\partial p(0)$.

Let us consider a convex function $k : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ and let us denote by $p : Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ its corresponding marginal function given by

$$p(y) = \inf_{x \in X} k(x, y) .$$

It is known and easy to see that p is convex on Y . In the sequel we denote by $\text{proj}_Y : X \times Y \rightarrow Y$ the projection mapping given by $\text{proj}_Y(x, y) = y$ and for any nonempty subset $S \subset Y$ by $\text{Vect}(S)$ the vector subspace generated by S .

In the case $Z = Y$ the following theorem has been established in Zalinescu [23].

Theorem 1. (Duality Theorem). *Let $k : X \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function and $Z := \text{Vect}[\text{proj}_Y(\text{dom } k)]$. If there exists $\lambda \in \mathbb{R}$ such that the following condition*

$$(C.Q_2) : 0 \in \text{int}_Z(\sqcup_{x \in X} [k(x, \cdot) \leq \lambda])$$

holds (Z is endowed with the topology induced by the Mackey topology of Y), then one has

$$\inf_{x \in X} k(x, 0) = \max_{y^* \in Y^*} \{-k^*(0, y^*)\} .$$

Proof. Consider the marginal function p associated with k . Noting that $\text{dom } p = \text{proj}_Y[\text{dom } k]$ and that $p(y) \leq \lambda$ whenever $k(x, y) \leq \lambda$ for some $x \in X$, we have $Z = \text{Vect}(\text{dom } p)$ and the condition $(C.Q_2)$ ensures that

$$0 \in \text{int}_Z([p \leq \lambda]) .$$

For each $y^* \in Y^*$ we have

$$\begin{aligned}
 p^*(y^*) &= \sup_{y \in Y} \{ \langle y^*, y \rangle - p(y) \} \\
 &= \sup_{y \in Y} \{ \langle y^*, y \rangle - \inf_{x \in X} k(x, y) \} \\
 &= \sup_{(x, y) \in X \times Y} \{ \langle (0, y^*), (x, y) \rangle - k(x, y) \} \\
 &= k^*(0, y^*).
 \end{aligned}$$

Note that $p(0) \leq \lambda$ and hence either $p(0) \in \mathbb{R}$ or $p(0) = -\infty$. Assume first that

$$p(0) = \inf_{x \in X} k(x, 0) \in \mathbb{R}.$$

According to Lemma 1, we obtain $\partial p(0) \neq \emptyset$. Let $z^* \in \partial p(0)$ or equivalently $0 \in \partial p^*(z^*)$ i.e.

$$p^*(z^*) = \min_{y^* \in Y^*} p^*(y^*).$$

As $p(0) = -p^*(z^*)$ (since $z^* \in \partial p(0)$) we obtain

$$\begin{aligned}
 \inf_{x \in X} k(x, 0) &= -\min \{ k^*(0, y^*), y^* \in Y^* \} \\
 &= \max \{ -k^*(0, y^*), y^* \in Y^* \}.
 \end{aligned}$$

Consider now the case

$$p(0) = \inf_{x \in X} k(x, 0) = -\infty.$$

Then by definition of $p^*(y^*)$ we have for each $y^* \in Y^*$

$$p^*(y^*) = k^*(0, y^*) = +\infty$$

and hence

$$\inf_{x \in X} k(x, 0) = \max \{ -k^*(0, y^*), y^* \in Y^* \} = -\infty.$$

4. The main result.

Throughout this section we assume that $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex and proper, $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, proper and Y_+ -nondecreasing only over the subset $\text{im } h + Y_+$ of Y where $\text{im } h$ denotes the effective range of h , that is, $\text{im } h := h(\text{dom } h)$ and $h : X \rightarrow Y \cup \{+\infty\}$ is Y_+ -convex and proper. Let us recall [4] that the function $g \circ h : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by $(g \circ h)(x) = g(h(x))$ if $h(x) \in \text{dom } g$ and $(g \circ h)(x) = +\infty$ otherwise. Thus, this amounts to setting $g(+\infty) = +\infty$.

Lemma 2. [4] $\sqcup_{y^* \in \partial g(h(x)) \cap Y_+^*} \partial(f + y^* \circ h)(x) \subseteq \partial(f + g \circ h)(x)$, for any $x \in X$.

Remark 1. It is easy to see that the function $x \mapsto (y^* \circ h)(x)$ is convex for each $y^* \in Y_+^*$.

In what that follows, our goal is to compute the conjugate function and the sub-differential of $(f + g \circ h)$. Let us consider, for $\bar{x} \in \text{dom}(f + g \circ h)$ and $x^* \in X^*$, the following function $k : X \times Y \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$k(x, u, y) = f(x) + g(u) + \delta_{-Y_+}(y + h(x) - u) - f(\bar{x}) - g(h(\bar{x})) - \langle x^*, x - \bar{x} \rangle \quad (2)$$

If we consider its marginal function $p : Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, defined by

$$p(y) := \inf_{(x,u) \in X \times Y} k(x, u, y)$$

then we have the following results.

Lemma 3.

- (a) k is convex and proper
- (b) If $x^* \in \partial(f + g \circ h)(\bar{x})$ where $\bar{x} \in \text{dom}(f + g \circ h)$, then $p(0) = 0$
- (c) $\text{dom } p = [\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+]$.

Proof. (a) The convexity of k follows immediately from that of f, g, h and the convexity and monotonicity of the function $y \mapsto \delta_{-Y_+}$ (see [4]). Moreover $k(\bar{x}, 0, h(\bar{x}))$ is finite. (b) see [4]. (c) $y \in \text{dom } p$ if and only if there exists some $(x, u) \in X \times Y$ such that $f(x) < +\infty, g(u) < +\infty, h(x) < +\infty$ and $y \in u - h(x) - Y_+$, that is,

$$y \in \text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+ .$$

Hence

$$\text{dom } p = [\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+] .$$

Let

$$Z := \text{Vect}[\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+]$$

endowed with the topology induced by the Mackey topology of Y and let us consider the following qualification condition inspired by that of Azé [3]:

$$(C.Q_3) \left\{ \begin{array}{l} \text{for each neighborhood } V \text{ of zero in } X \text{ there exists } \lambda \in \mathbb{R} \text{ such that} \\ 0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda V) - Y_+). \end{array} \right.$$

Theorem 1. Under the condition $(C.Q_3)$, we have

- (a) $(f + g \circ h)^*(x^*) = \min_{y^* \in Y_+^*} [g^*(y^*) + (f + y^* \circ h)^*(x^*)]$, for any $x^* \in X^*$.
- (b) $\partial(f + g \circ h)(x) = \sqcup_{y^* \in \partial g(h(x)) \cap Y_+^*} \partial(f + y^* \circ h)(x)$ for any $x \in X$.

Proof. Note that $\text{dom}(f + g \circ h) \neq \emptyset$ by $(C.Q_3)$. Then let us consider any fixed $\bar{x} \in \text{dom}(f + g \circ h)$. (a) Consider any $x^* \in X^*$ and denote by V the closed convex symmetric neighbourhood of zero in X given by $V := \{x \in X : |\langle x^*, x \rangle| \leq 1\}$. By

($C.Q_3$), there exist some open neighbourhood U of zero with respect to the Mackey topology of Y such that for each $y \in U \cap Z$, there exist $z_y \in Y$ and $x_y \in V$ such that

$$\begin{cases} y = z_y - h(x_y) - Y_+ \\ g(z_y) \leq \lambda \\ f(x_y) \leq \lambda . \end{cases}$$

For the real number $\gamma := 2\lambda - f(\bar{x}) - g(h(\bar{x})) + \langle x^*, \bar{x} \rangle + 1$ and the function k given by (2), we obtain $k(x_y, z_y, y) \leq \gamma$ for all $y \in U \cap Z$ and hence

$$0 \in \text{int}_Z \left(\bigcup_{(x,u) \in X \times Y} [k(x, ., .) \leq \gamma] \right) .$$

It follows from Theorem 1 (since k is convex by Lemma 3) that

$$\inf_{(x,u) \in X \times Y} k(x, u, 0) = \max \{ -k^*(0, 0, y^*) : y^* \in Y^* \} . \quad (3)$$

As one can easily verify that

$$\inf_{(x,u) \in X \times Y} k(x, u, 0) = -(f + g \circ h)^*(x^*) + \langle x^*, \bar{x} \rangle - (f + g \circ h)(\bar{x})$$

and

$$\min_{y \in Y^*} k^*(0, 0, y^*) = \min_{y^* \in Y_+^*} \{ g^*(y^*) + (f + y^* \circ h)^*(x^*) \} - \langle x^*, \bar{x} \rangle + (f + g \circ h)(\bar{x})$$

one deduces from (3) that

$$(f + g \circ h)^*(x^*) = \min_{y^* \in Y_+^*} \{ g^*(y^*) + (f + y^* \circ h)^*(x^*) \} .$$

(b) As the reverse inclusion follows from Lemma 2, it suffices to show

$$\partial(f + g \circ h)(\bar{x}) \subseteq \bigcup_{y^* \in \partial g(h(\bar{x})) \cap Y_+^*} \partial(f + y^* \circ h)(\bar{x})$$

for all $\bar{x} \in X$. Let $x^* \in \partial(f + g \circ h)(\bar{x})$ or equivalently

$$(f + g \circ h)^*(x^*) - \langle x^*, \bar{x} \rangle + (f + g \circ h)(\bar{x}) = 0 .$$

Then by (a) there exists $z^* \in Y_+^*$ such that

$$g^*(z^*) + (f + z^* \circ h)^*(x^*) - \langle x^*, \bar{x} \rangle + (f + g \circ h)(\bar{x}) = 0$$

and hence

$$\begin{aligned} & [g^*(z^*) + g(h(\bar{x})) - \langle z^*, h(\bar{x}) \rangle] \\ & + [(f + z^* \circ h)^*(x^*) - \langle x^*, \bar{x} \rangle + (f + z^* \circ h)(\bar{x})] = 0 . \quad (4) \end{aligned}$$

By the Fenchel inequality, both terms between brackets in (4) are nonnegative and hence we deduce

$$g^*(z^*) + g(h(\bar{x})) - \langle z^*, h(\bar{x}) \rangle = 0$$

and

$$(f + z^* \circ h)^*(x^*) - \langle x^*, \bar{x} \rangle + (f + z^* \circ h)(\bar{x}) = 0$$

i.e

$$y^* \in \partial g(h(\bar{x})) \cap Y_+^* \text{ and } x^* \in \partial(f + z^* \circ h)^*(x^*) .$$

This completes the proof.

We end this section by showing how certain results (see [1,2,3,8,14,17,19]) can be derived easily from Theorem 2.

In particular, when g is nondecreasing on the whole space, we have under the following condition

$$(C.Q_4) \left\{ \begin{array}{l} \text{for each neighbourhood } V \text{ of zero in } X \text{ there exists } \lambda \in \mathbb{R} \text{ such that} \\ 0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda V)) \end{array} \right.$$

Corollary 1.

- (a) $(f + g \circ h)^*(x^*) = \min_{y^* \in Y_+^*} [g^*(y^*) + f + y^* \circ h)^*(x^*)]$, for any $x^* \in X^*$.
- (b) $\partial(f + g \circ h)(x) = \bigsqcup_{y^* \in \partial g(h(x))} \partial(f + y^* \circ h)(x)$, for any $x \in X$.

Proof. It suffices to observe that when g is nondecreasing on the whole space, we have for any $\lambda \in \mathbb{R}$ and $y \in Y : [g \leq \lambda] - Y_+ = [g \leq \lambda]$ and $\partial g(y) \subseteq Y_+^*$.

Corollary 2. Let X and Y be two locally convex topological vector spaces. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be two proper convex functions and $A : X \rightarrow Y$ be a continuous linear operator. Let us put $Z := \text{Vect}[\text{dom } g - A(\text{dom } f)]$ and assume that the following condition is satisfied

$$(C.Q_5) \left\{ \begin{array}{l} \text{for each neighbourhood } V \text{ of zero in } X \text{ there exist } \lambda \in \mathbb{R} \text{ such that} \\ 0 \in \text{int}_Z([g \leq \lambda] - A([f \leq \lambda] \cap \lambda V)) . \end{array} \right.$$

Then we have

- (a) $(f + g \circ A)^*(x^*) = \min_{y^* \in Y^*} \{g^*(y^*) + f^*(x^* - A^*y^*)\}, \forall x^* \in X^*$.
- (b) $\inf_{x \in X} \{f(x) + g(A(x))\} = \max_{y^* \in Y^*} \{g^*(y^*) - (f^*(-A^*y^*))\},$
- (c) $\partial(f + g \circ A)(\bar{x}) = \partial f(\bar{x}) + A^*(\partial g(A\bar{x})), \forall \bar{x} \in X$

(5)

Proof. It is easy to see that condition (C.Q₅) is a particular case of (C.Q₃) and hence by virtue of Theorem 2 and by taking $Y_+ = \{0\}$ we get

$$\begin{aligned} (f + g \circ A)^*(x^*) &= \min_{y^* \in Y^*} \{g^*(y^*) + (f + y^* \circ A)^*(x^*)\} , \\ \inf_{x \in X} \{f(x) + g(A(x))\} &= \max_{y^* \in Y^*} \{-g^*(y^*) - (f + y^* \circ A)^*(0)\} , \\ \partial(f + g \circ A)(\bar{x}) &= \bigsqcup_{y^* \in \partial g(A(\bar{x}))} \partial(f + y^* \circ A)(\bar{x}) , \end{aligned}$$

for any $x^* \in X^*$ and any $\bar{x} \in X$. To obtain equalities (a) and (b), it suffices to note that $(f + y^* \circ h)^*(x^*) = f^*(x^* - A^*y^*)$. Moreover, by the continuity of $y^* \circ A$, relation (5) ensures that

$$\partial(f + g \circ A)(\bar{x}) = \bigsqcup_{y^* \in \partial g(A(\bar{x}))} (\partial f + y^* \circ A)(\bar{x}) = \partial f(\bar{x}) + A^* \partial g(A\bar{x}) .$$

Corollary 3. *Let X be a locally convex topological vector space. Let $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex and proper. Let us put $Z := \text{Vect}[\text{dom } f - \text{dom } g]$ (endowed by the topology induced by the Mackey topology of X) and assume that the following condition is satisfied.*

$$(C.Q_6) \left\{ \begin{array}{l} \text{for each neighbourhood } V \text{ of zero in } X \text{ there exists } \lambda \in \mathbb{R} \text{ such that} \\ 0 \in \text{int}_Z([g \leq \lambda] - [f \leq \lambda] \cap \lambda V). \end{array} \right.$$

Then we have

- (a) $(f + g)^*(x^*) = (g^* \square f^*)(x^*)$, for any $x^* \in X^*$ and here the inf-convolution is exact.
- (b) $\inf_{x \in X} \{f(x) + g(x)\} = \max_{x^* \in X^*} \{-g^*(x^*) - f^*(-x^*)\}$.
- (c) $\partial(f + g)(x) = \partial f(x) + \partial g(x)$, for any $x \in X$.

Proof. Take in corollary 2 $A = \text{Id}_X$ (the identity mapping of X with $Y = X$).

5. Comparison with other qualification conditions.

The aim of this section is to derive from our results those by Azé [3], Rockafellar [15], Simons [19] and Zalinescu [22].

Note first that the usual Moreau-Rockafellar condition obviously implies condition $C.Q_3$. Note also that, for convex functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, the well-known qualification condition ([16])

$$ri(\text{dom } f) \cap ri(\text{dom } g) \neq \emptyset$$

(equivalent to $0 \in ri(\text{dom } f - \text{dom } g)$) also ensures condition $C.Q_3$.

In what follows, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex function, $h : X \rightarrow Y \cup \{+\infty\}$ is a Y_+ -convex and proper mapping and $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex, proper and Y_+ -nondecreasing function on Y . When X is a normed vector space, we will consider the following condition

$$(C.Q_7) \left\{ \begin{array}{l} \text{there exists } \lambda \in \mathbb{R}^+ \text{ such that} \\ 0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda \mathbb{B}_X)) , \end{array} \right.$$

where $Z := \text{Vect}[\text{dom } g - h(\text{dom } f \cap \text{dom } h)]$ is endowed with the induced topology and $\mathbb{B}_X$ is the closed unit ball of X .

The following proposition will allow us to derive the result by Azé [3].

Proposition 1. *If X is a normed vector space, then the conditions $(C.Q_3)$ and $(C.Q_7)$ are equivalent.*

Proof. Since the implication $(C.Q_3) \Rightarrow (C.Q_7)$ is obvious, it is enough to prove the converse. Let V be any circled neighbourhood of zero in X . Choose some $\alpha > 0$ such that $\alpha\mathbb{B}_X \subset V$. Since

$$0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda\mathbb{B}_X)) ,$$

putting $\beta := \max(\lambda, \lambda/\alpha)$, we get

$$0 \in \text{int}_Z([g \leq \beta] - h([f \leq \beta] \cap \text{dom } h \cap \beta V)) .$$

If we replace in condition $(C.Q_7)$ the mapping h by a continuous linear operator $A : X \rightarrow Y$ we obtain that $(C.Q_3)$ is equivalent to the Azé 's qualification condition used in Proposition 2.2 of [3].

When $X = Y$ is a locally convex space, it is easy to prove that, the condition

(a) for each neighbourhood W of zero in X there exists $\lambda \in \mathbb{R}$ such that

$$0 \in \text{int}_Z([g \leq \lambda] \cap W - [f \leq \lambda] \cap W)$$

is equivalent to

(b) for each neighbourhood W of zero in X there exists $\lambda \in \mathbb{R}$ such that

$$0 \in \text{int}_Z([g \leq \lambda] - [f \leq \lambda] \cap W)$$

(here $Z := \mathbb{R}_+[\text{dom } g - \text{dom } f]$). When $X = Y$ is a normed space and $h = \text{Id}_X$ is the identity mapping of X , the above equivalence ensures that condition $(C.Q_7)$ follows from the Azé 's qualification condition concerning two functions in Theorem 2.4 of [3] which is: there exists some $\lambda \geq 0$ such that

$$0 \in \text{int}_Z([g \leq \lambda] \cup \lambda\mathbb{B}_X - [f \leq \lambda] \cap \lambda\mathbb{B}_X).$$

Before stating the next results of this section concerning the case of metrizable locally convex spaces, let us recall the following notions. A subset C of a locally convex space is CS -complete (resp. CS -closed) if for any sequence $(x_n)_{n \in \mathbb{N}}$ in C and any sequence $(\lambda_n)_{n \in \mathbb{N}}$ of nonnegative numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$ the sequence $(\sum_{n=0}^m \lambda_n x_n)_{m \in \mathbb{N}}$ converges to some point in C whenever $(\sum_{n=0}^m \lambda_n x_n)_{m \in \mathbb{N}}$ is a Cauchy (resp. a convergent) sequence. In the same way, we say that a subset K of $X \times Y$, where Y is another metrizable locally convex space, is partially CS -complete with respect to the second variable, if for any sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in K and any sequence $(\lambda_n)_{n \in \mathbb{N}}$ of nonnegative numbers such that $\sum_{n=0}^{\infty} \lambda_n = 1$, $(\sum_{n=0}^m \lambda_n x_n)_m$ converges to $x \in X$ and $(\sum_{n=0}^m \lambda_n y_n)_m$ is a Cauchy sequence, then $(\sum_{n=0}^m \lambda_n y_n)_m$ converges to some $y \in Y$ with $(x, y) \in K$. This class of subsets of $X \times Y$ has been introduced in Zalinescy [22] as subsets satisfying the H_x -property. Let us note that any CS -complete subset of $X \times Y$ is partially CS -complete. The graph of any continuous

linear mapping from X into Y is also partially complete with respect to the second variable.

A proper function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is CS -complete (resp. CS -closed) if its epigraph is CS -complete (resp. CS -closed).

Let us also recall the following reformulation by Zalinescu [22] of a result due to Simons [19].

Proposition 2. *Let X and U be two metrizable locally convex spaces and C be a nonempty convex subset of $X \times U \times \mathbb{R}$. Let F be the multifunction from U into $\mathbb{R}$ whose graph is $\text{proj}_{U \times \mathbb{R}} C$. Assume that*

- (a) $Z := \text{Vect}(\text{proj}_U C)$ is barreled;
- (b) $0 \in \text{Core}_Z(\text{proj}_U C)$;
- (c) $[(x, u, r) \in C \text{ and } r \leq s] \Rightarrow (x, u, s) \in C$;
- (d) *for any sequence $(x_n, u_n, r_n)_{n \in \mathbb{N}}$ in C and any sequence $(\lambda_n)_{n \in \mathbb{N}}$ of non-negative numbers such that $\sum_{n=0}^{\infty} \lambda_n = 1$, $\sum_{n=0}^{\infty} \lambda_n (u_n, r_n) = (u, r)$ and $(\sum_{n=0}^m \lambda_n x_n)_{m \in \mathbb{N}}$ is a Cauchy sequence, there exists some $x \in X$ satisfying $(x, u, r) \in C$.*

Then $(F(0) \neq \emptyset)$ for any $\lambda > \inf F(0)$, and $0 \in \text{int}_Z F^{-1}(\lambda)$.

In the proposition above for a convex subset $S \subset Z$, the subset $\text{Core}_Z(S)$ denotes the Core of S relatively to Z , that is

$$\text{Core}_Z(S) := \{x \in S : \forall y \in S, \exists \epsilon > 0, x + [-\epsilon, \epsilon]y \subset S\}.$$

We can now relate the Simpson condition [19] and the Zalinescu condition [22] to condition $C.Q_6$.

Proposition 3. *Let X and Y be two metrizable locally convex spaces. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a CS -complete (resp. CS -closed) function, $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a closed (resp. CS -complete) function and $h : X \rightarrow Y \cup \{+\infty\}$ be a Y_+ -convex and proper mapping whose epigraph is partially CS -complete with respect to the second variable (resp. the first). Assume $Z := \text{Vect}[\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+]$ is barreled and $0 \in \text{Core}_Z[\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+]$. Then for any neighbourhood V of zero in X , there exists some $\lambda \in \mathbb{R}$ such that*

$$0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda V) - Y_+).$$

Proof. Fix a closed convex symmetric neighbourhood V of zero in X and put $U := Y$ and

$$\begin{aligned} C := \{ & (x, y, u, r) \in X \times Y \times U \times \mathbb{R} \\ & : r > 0, u + y \in [g \leq \lambda], (x, y) \in \text{epi } h, x \in [f \leq r] \cap rV \}. \end{aligned}$$

It is easily seen that $\text{proj}_U C = \text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+$ and that the assumptions of Proposition 2 hold. If we consider the multifunction F from $U = Y$ into $\mathbb{R}$ with

$$F(u) = \{r > 0 : u \in [g \leq r] - h([f \leq r] \cap \text{dom } h \cap rV) - Y_+\},$$

its graph is exactly $\text{proj}_{U \times \mathbb{R}} C$ and so applying Proposition 2 we get for any $\lambda > \inf F(0)$

$$0 \in \text{int}_Z F^{-1}(\lambda) = \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda V) - Y_+)$$

which completes the proof.

The corollary below has been proved by Zalinescu [22]. See also Simons [19] for the case X and Y are Fréchet spaces and both functions f and g are CS -closed.

Corollary 4. *Let X and Y be two metrizable locally convex spaces. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a CS -complete function, $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a CS -closed function and $A : X \rightarrow Y$ be a continuous operator. Assume that $Z := \text{Vect}[\text{dom } g - A(\text{dom } f)]$ is barreled and $0 \in \text{Core}_Z[\text{dom } g - A(\text{dom } f)]$. Then for any neighbourhood V of zero in X , there exists some $\lambda \in \mathbb{R}$ such that*

$$0 \in \text{int}_Z([g \leq \lambda] - A([f \leq \lambda] \cap \lambda V)) .$$

Proof. It suffices to take $Y_+ = \{0_Y\}$ in Proposition 3 and to use the fact that the graph $\text{Gr } A$ is partially CS -complete with the respect to the second variable.

We conclude with the following corollary which relates the Rockafellar condition [15], the Robinson condition [14] and the Attouch-Brézis condition [1] to condition $C.Q_6$. It is a direct consequence of Proposition 3.

Corollary 5. *Assume X and Y are two Fréchet spaces, $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous, $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuous and $h : X \rightarrow Y \cup \{+\infty\}$ is Y_+ -convex and proper and has a closed epigraph in $X \times Y$. If $Z := \text{Vect}[\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+]$ is a closed linear subspace of Y , then for any neighbourhood V of zero in X , there exists some $\lambda \in \mathbb{R}$ such that*

$$0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda V) - Y)_+ .$$

In the first version of the paper, we established only the statement of Corollary 5 and this has been done with the use of the Ursescu's version of openness Theorem [20]. Here we have used Proposition 2 and 3 in order to answer a question by a referee about Simons condition and Zalinescu condition. We must also say that references [9] and [23] have been suggested to us respectively by two referees.

Résumé substantiel en français. L'importance de la formule du sous-différentiel de la somme de deux fonctions convexes $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$ est actuellement connue en analyse convexe, optimisation, mécanique, théorie des jeux, économie mathématique, etc.

Il est également bien connu qu'une telle formule sous forme exacte exige une condition de qualification. La condition la plus populaire est celle de Moreau-Rockafellar qui suppose que f est finie et continue en un point où g est finie. Cette condition a été affaiblie plus tard dans un certain sens par Rockafellar [14] et Robinson [13] et également par Attouch et Brézis [1]. Rockafellar [14] a supposé que f et g sont semi-continues inférieurement sur X , $0 \in \text{Core}(\text{dom } f - \text{dom } g)$, et X est un espace de Banach réflexif. Robinson [13], à l'aide de son théorème d'application ouverte, a

montré que la méthode de Rockafellar reste encore valable pour tout espace de Banach (voir également le livre de Aubin et Ekeland [2] pour une autre preuve indépendante). Plus tard Attouch et Brézis ont prouvé indépendamment et par une méthode différente la formule pour tout espace de Banach X sous l'hypothèse que f et g sont semi-continues inférieurement sur X et que $\mathbb{R}_+[\text{dom } f - \text{dom } g]$ est un sous-espace vectoriel fermé. Cette condition de qualification est plus générale que celle de Rockafellar et Robinson, puisqu'il n'est pas difficile de voir que $0 \in \text{Core}(\text{dom } f - \text{dom } g)$ si et seulement si $\mathbb{R}_+[\text{dom } f - \text{dom } g] = X$ (notons également que le résultat d'Attouch-Brézis peut être obtenu à partir de celui de Robinson et Rockafellar). Cependant, ces conditions de qualification n'incluent pas totalement celle de Moreau-Rockafellar qui n'exige pas la semi-continuité inférieure de f et g . Récemment, cette situation a conduit Azé [3] à introduire dans tout espace vectoriel normé X la condition de qualification qui suppose qu'il existe des nombres réels $\lambda > 0$, $s > 0$ et $r > 0$ tels que

$$s\mathbb{B}_X \subseteq [f \leq \lambda] \cap r\mathbb{B}_X - [g \leq \lambda] \cap r\mathbb{B}_X$$

où $[f \leq \lambda]$ désigne l'ensemble sous-niveau λ de f et $\mathbb{B}_X$ la boule unité fermée de X . Azé a montré que toutes les conditions citées ci-dessus assurent (1). Néanmoins, la condition (1) exige que X soit un espace vectoriel normé alors que la condition de Moreau-Rockafellar n'a pas besoin de telle restriction.

Le but de cet article est d'introduire, dans le cadre des espaces localement convexes, une condition de qualification inspirée par (1) et qui jouera le rôle de condition unificatrice de toutes les conditions citées ci-dessus. Plus généralement nous étudierons le sous-différentiel des fonctions convexes de la forme $f + g \circ h$ où h est opérateur convexe et nous montrerons que notre condition considérée est si générale qu'elle implique la condition de Simons [18] et aussi celle de Zalinescu [21].

Soit Y un espace vectoriel topologique muni d'un préordre défini par un cône Y_+ . Nous posons $y_1 \leq_Y y_2$ dans Y si $y_2 - y_1 \in Y_+$ et nous adjoignons un élément $+\infty$ à Y . Alors une application $h : X \rightarrow Y \cup \{+\infty\}$ est dite Y_+ -convexe si pour tous $\lambda \in]0, 1[$, $x_1, x_2 \in X$ on a

$$h(\lambda x_1 + (1 - \lambda)x_2) \leq_Y \lambda h(x_1) + (1 - \lambda)h(x_2) .$$

Nous supposons dans la suite que $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ est convexe et propre, $g : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ est convexe, propre et Y_+ -croissante seulement sur l'ensemble $\text{im } h + Y_+$ où $\text{im } h$ désigne l'image de h c'est-à-dire, $\text{im } h := h(\text{dom } h)$ et $h : X \rightarrow Y \cup \{+\infty\}$ est Y_+ -convexe et propre. Posons

$$Z := \text{Vect}[\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+]$$

le sous-espace vectoriel de Y engendré par $\text{dom } g - h(\text{dom } f \cap \text{dom } h) - Y_+$ muni de la topologie de Mackey de Y . Considérons la condition de qualification notée par $(C.Q_3)$ et inspirée par celle d'Azé [3]

$$(C.Q_3) \left\{ \begin{array}{l} \text{pour tout voisinage } V \text{ de zéro dans } X \text{ il existe } \lambda \in \mathbb{R} \text{ tel que} \\ 0 \in \text{int}_Z([g \leq \lambda] - h([f \leq \lambda] \cap \text{dom } h \cap \lambda V) - Y_+) . \end{array} \right.$$

Le résultat principal de cet article est le suivant (voir Théorème 2)

Théorème. Sous la condition $(C.Q_3)$ nous avons

- (a) $(f + g \circ h)^*(x^*) = \min_{y^* \in Y_+^*} [g^*(y^*) + (f + y^* \circ h)^*(x^*)]$, pour tout $x^* \in X^*$.
 (b) $\partial(f + g \circ h)(x) = \bigsqcup_{y^* \in \partial g(h(x)) \cap Y_+^*} \partial(f + y^* \circ h)(x)$, pour tout $x \in X$.

REFERENCES

1. H. Attouch and H. Brézis, *Duality for the sum of convex functions in general Banach spaces, Aspects of Mathematics and its Applications. Edited par J. Barroso*, North Holland, Amsterdam, Elsevier Science Publishers B.V., 1986, pp. 125–133.
2. J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley, New-York, 1984.
3. D. Azé, *Duality for the sum of convex functions in general normed spaces*, Arch. Math. **62** (1994), 554–561.
4. C. Combari, M. Laghdir et L. Thibault, *Sous-différentiels de fonctions convexes composées*, Ann. Sci. Math. Québec **62** (1994), 119–148.
5. H.H. Schaefer, *Topological Vector Spaces*, Macmillan, New-York, 1966.
6. I. Ekeland and R. Teman, *Convex Analysis and Variational Problems*, North-Holland, Amsterdam, 1976.
7. E.G. Gol'shtein, *Duality theory in mathematical programming and its applications*, Nauka, Moscou, 1971.
8. M.S. Gowda and M. Teboulle, *A comparison of constraint qualification in infinite dimensional convex programming*, SIAM J. Control Optim. **28** (1990), 925–935.
9. V. Jeyakumar, *Duality and infiniter dimensional optimization*, Nonlinear Anal. Th. Meth. Appl. **15** (1990), 1111–1122.
10. S.S. Kutateladze, *Convex operator*, Uspekhi Mat. Nauk **34** (1979), 167–196.
11. B. Lemaire, *Application of a subdifferential of a convex composite functional to optimal control in variational inequalities. Nondifferentiable optimization: motivations and applications*, Proceeding Sopron Sept. 1984 Lecture Notes in Econom. and Math. Systems **225** (1985), Springer-Verlag, 103–117.
12. V.L. Levin, *On the subdifferential of a composite functional*, Soviet. Math. Dokl. **11** (1970), 1194–1195.
13. J.J. Moreau, *Fonctionnelles convexes*, Séminaire sur les équations aux dérivées partielles (1966–1967). II. (1967), Collège de France, Paris.
14. S.M. Robinson, *Regularity and stability for convex multivalued functions*, Math. Oper. Res. **1** (1976), 130–143.
15. R.T. Rockafellar, *Conjugate duality and optimisation*, SIAM Publications, Philadelphia, 1974.
16. R.T. Rockafellar, *Convex Analysis*, Princeton University J., 1970.
17. B. Rodrigues and S. Simons, *Conjugate functions and subdifferentials in nonnormed situations for operators with complete graphs*, Nonlinear Anal. Th. Meth. Appl. **12** (1988), 1069–1078.
18. A.M. Rubinov, *Sublinear operators and their applications*, Uspehi Mat. Nauk **32** (1977), 113–174.
19. S. Simons, *The occasional distributivity of $\circ$ over $+$ and the change of variable formula for conjugate functions*, Nonlinear Anal. Th. Meth. Appl. **14** (1990), 111–1120.
20. C. Ursescu, *Multifunctions with closed convex graph*, Czechoslovak Math. **25** (1975), 438–441.
21. M. Volle, *Some applications of the Attouch-Brézis condition to closedness criterion, optimization and duality*, Séminaire d'Analyse Convexe de Montpellier (1992).
22. C. Zălinescu, *On some open problems in convex analysis*, Arch. Math. (Basel) **59** (1992), 566–571.

36 On subdifferential calculus for convex functions defined on locally convex spaces

23. C. Zălinescu, *Duality for vectorial nonconvex optimization*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Sect. I a Mat. (N.S.) **29** (1983), 15–34.

C. COMBARI, M. LAGHDIR AND L. THIBAUT
UNIVERSITÉ MONTPELLIER II
LABORATOIRE D'ANALYSE CONVEXE
PLACE EUGENE BATAILLON
34095 MONTPELLIER CEDEX 5
FRANCE