

MORE ON THE ROOTS OF EULER-FROBENIUS POLYNOMIALS

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RÉSUMÉ. Une propriété importante de croissance de la racine principale des polynômes d'Euler-Frobénius $P_n(x)$ est obtenue à partir d'une méthode élémentaire d'analyse des propriétés d'entrelacement des racines de $P_n(x)$ and $P_{n+2}(x)$. Une borne sur la somme du carré des inverses des racines transformées non-nulles est également présentée.

ABSTRACT. An important increasing property of the main root of Euler-Frobenius polynomials $P_n(x)$ is obtained from an elementary method used to analyse interlacing properties of the roots of $P_n(x)$ and $P_{n+2}(x)$. A bound for the sum of the square of the inverse of transformed non-zero roots is obtained.

1. Introduction. The Euler-Frobenius polynomials $p_n(\lambda, \cdot)$ can be defined [2, 6, 7] by

$$p_n(\lambda, z) = \frac{(1-z)^{n+1}}{z} \left(z \frac{\partial}{\partial z} - \lambda \right)^n \left(\frac{z}{1-z} \right)$$

(note that $p_n(\lambda, z) = z^n H_n(\lambda, 1/z) = H_n(1-\lambda, z)$, where $H_n(\lambda, z)$ is the form used in [5, 8, 10]).

It is well known [2, 6, 7] that the roots $z_j^{(n)}(\lambda)$ of $p_n(\lambda, \cdot)$, for $\lambda \in [0, 1]$, are simple and

$$-\infty \leq z_1^{(n)}(\lambda) < z_2^{(n)}(\lambda) < \cdots < z_{n-1}^{(n)}(\lambda) < z_n^{(n)}(\lambda) \leq 0.$$

Moreover $z_j^{(n)}(\lambda)$ is an increasing function for $\lambda \in [0, 1]$, and

$$z_r^{(2r+1)}(0) < -1 < z_{r+1}^{(2r+1)}(0),$$

$$z_r^{(2r)}\left(\frac{1}{2}\right) < -1 < z_{r+1}^{(2r)}\left(\frac{1}{2}\right).$$

The main root of $p_n(\lambda, \cdot)$, denoted $\zeta_n(\lambda)$, is the greatest root of $p_n(\lambda, \cdot)$ which does not exceed -1 . Hence

$$\begin{cases} \zeta_n(0) = z_r^{(n)}(0) & \text{if } n = 2r + 1, r \in \mathbb{N}; \\ \zeta_n\left(\frac{1}{2}\right) = z_r^{(n)}\left(\frac{1}{2}\right) & \text{if } n = 2r, r \in \mathbb{N}. \end{cases}$$

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The Euler-Frobenius polynomials are related to the spline interpolation problem at uniformly shifted nodes. The shifted interpolating cardinal Lagrange spline series associated to this problem is

$$\sum_{j \in \mathbb{Z}} y_j l_j(x - \lambda) \quad x \in \mathbb{R}, y_j \in \mathbb{R}, \lambda \in [0, 1],$$

where $l_j(\cdot)$ denotes the Lagrange spline of degree $n \in \mathbb{N}$ with respect to the integer grid and node j . Following [9, 10], the series converges/diverges if its radius of convergence ρ verifies

$$\rho = \overline{\lim}_{|j| \rightarrow \infty} |y_j|^{1/|j|} \leq \min\{|\zeta_n(\lambda)|, |\zeta_n(1 - \lambda)|\}.$$

Since $p_n(\lambda, z) = z^n p_n(1 - \lambda, 1/z)$, it follows that

$$\min\{|\zeta_n(\lambda)|, |\zeta_n(1 - \lambda)|\} \leq \begin{cases} |\zeta_n(0)| & \text{for } n = 2r + 1, \\ |\zeta_n(\frac{1}{2})| & \text{for } n = 2r, \end{cases}$$

and the greatest possible radius of convergence is obtained for $\lambda = 0$ if n is odd and for $\lambda = \frac{1}{2}$ if n is even.

The object of this paper is twofold. After preliminaries in sections 2 and 3, we first prove the following relation

$$\zeta_2(\frac{1}{2}) < \zeta_3(0) < \zeta_4(\frac{1}{2}) < \zeta_5(0) < \dots < -1 \quad (1)$$

in section 4. It was proved in [11] that

$$\zeta_3(0) < \zeta_5(0) < \dots < -1, \quad (2)$$

and the method has been extended in [3] to obtain

$$\zeta_2(\frac{1}{2}) < \zeta_4(\frac{1}{2}) < \dots < -1. \quad (3)$$

The relation (1) was suggested by direct computations for low values of n and by the following asymptotic results

$$\zeta_{2r+1}(0) = -\{1 + \frac{\pi^2}{4r+4} + O(\frac{1}{r^2})\}, \quad (4a)$$

$$\zeta_{2r}(\frac{1}{2}) = -\{1 + \frac{\pi^2}{4r+2} + O(\frac{1}{r^2})\}, \quad (4b)$$

proved respectively in [11] and [8]. The result (1) indicates that the radius of convergence decreases as the degree of the interpolating spline increases.

The second result, obtained in section 5, is an upper bound for the expression

$$J_n(\lambda) = \sum_{i \in \mathcal{I}_n(\lambda)} \left[\frac{z_i^{(n)}(\lambda) - 1}{z_i^{(n)}(\lambda) + 1} \right]^2,$$

where $\mathcal{I}_n(\lambda) = \{i \mid z_i^{(n)}(\lambda) \neq -1 \text{ and } -\infty < z_i^{(n)}(\lambda) < 0\}$. This bound is useful for calculating some constants related to error bounds for the shifted cardinal spline interpolation problem [4]. We are particularly interested by the cases:

(i) $n = 2r + 1$ and $\lambda = 0$, for which we obtain

$$J_{2r+1}(0) \leq \frac{4}{\pi^2}(2r+2)(2r+3) \sim \frac{16r^2}{\pi^2} \quad r \rightarrow \infty; \quad (5a)$$

(ii) $n = 2r$ and $\lambda = \frac{1}{2}$, for which we obtain

$$J_{2r}(\frac{1}{2}) \leq \frac{16}{\pi^3}(2r+1)(2r+2) \sim \frac{64r^2}{\pi^3} \quad r \rightarrow \infty. \quad (5b)$$

Let us observe that, using (4a) and (4b), the term corresponding to the main root in this summation (in fact there are two such terms) has almost the same order of magnitude than the upper bound

$$\left[\frac{\zeta_{2r+1}(0) - 1}{\zeta_{2r+1}(0) + 1} \right]^2 \quad \text{and} \quad \left[\frac{\zeta_{2r}(\frac{1}{2}) - 1}{\zeta_{2r}(\frac{1}{2}) + 1} \right]^2 \sim \left(\frac{8r}{\pi^2} \right)^2 \quad r \rightarrow \infty.$$

2. The transformation. Using the substitution $z = (x+1)/(x-1)$ which preserves interlacing properties of the roots, and following [11], we define for $\lambda = 0$

$$P_{n-1}(x) = (x-1)^{n-1} p_n(0, z) \quad (6)$$

for $n \geq 1$. Then $P_0(x) = 1$, and

$$P_{n+1}(x) = 2xP_n(x) - (1-x^2)DP_n(x)$$

for $n \geq 0$ where $D = \frac{d}{dx}$. For example

$$P_1(x) = 2x$$

$$P_2(x) = 2(3x^2 - 1)$$

$$P_3(x) = 8(3x^3 - 2x)$$

$$P_4(x) = 8(15x^4 - 15x^2 + 2)$$

$$P_5(x) = 16(45x^5 - 60x^3 + 17x)$$

$$P_6(x) = 16(315x^6 - 525x^4 + 231x^2 - 17).$$

For $\lambda = \frac{1}{2}$, we define

$$Q_n(x) = (x-1)^n p_n(\frac{1}{2}, z). \quad (7)$$

Then $Q_0(x) = 1$, and

$$Q_{n+1}(x) = xQ_n(x) - (1-x^2)DQ_n(x)$$

for $n \geq 0$. For example

$$\begin{aligned} Q_1(x) &= x \\ Q_2(x) &= 2x^2 - 1 \\ Q_3(x) &= 6x^3 - 5x \\ Q_4(x) &= 24x^4 - 28x^2 + 5 \\ Q_5(x) &= 120x^5 - 180x^3 + 61x \\ Q_6(x) &= 720x^6 - 1320x^4 + 662x^2 - 61. \end{aligned}$$

Consequently, the Euler-Frobenius polynomials are examples of polynomials generated by the following general scheme

$$(G) \quad \left\{ \begin{array}{l} \text{(i) } P_0(x) = 1 ; \\ \text{(ii) for } n \geq 0 \\ \qquad c_{n+1}P_{n+1}(x) = 2r_n x P_n(x) - (1 - x^2)DP_n(x) \\ \text{where } \{r_n\}_{n=0}^{\infty} \text{ is a sequence of strictly positive real numbers and} \\ \qquad \{c_n\}_{n=1}^{\infty} \text{ is a sequence of non-zero real numbers.} \end{array} \right.$$

3. Intermediate results. Let $P(x)$ be any polynomial of degree n such that $P(-x) = (-1)^m P(x)$, and let us define the operator T_r for $r > 0$ by the formula

$$T_r(P)(x) = 2rxP(x) - (1 - x^2)DP(x).$$

We note that

$$T_r(P)(0) = -DP(0), \quad T_r(P)(1) = 2rP(1),$$

and

$$T_r(P)(-x) = (-1)^{m+1} T_r(P)(x).$$

We also have the following properties (proofs of results of this section are given in [3]).

Property 1. If $W_r(x) = (1 - x^2)^r P(x)$, then $DW_r(x) = -(1 - x^2)^{r-1} T_r(P)(x)$.

Property 2. If $Q(x) = c(1 - x^2)P(x)$, then $T_r(Q)(x) = c(1 - x^2)T_{r+1}(P)(x)$.

From these properties we conclude that the roots of $DW_r(x)$ and the roots of $T_r(P)(x)$ are the same on $(-1, 1)$.

In the sequel we assume that all the non-zero roots of $P(x)$ are simple and on the interval $[-1, 1]$.

Definition 3. Let l be any non-negative integer, and

$$P(x) = x^l \prod_{i=1}^k (x^2 - \alpha_i^2) \quad \text{and} \quad P^*(x) = x^l \prod_{i=1}^k (x^2 - (\alpha_i^*)^2)$$

be two polynomials of degree $n = 2k + l$ where $0 = \alpha_0 < \alpha_1 < \dots < \alpha_k \leq \alpha_{k+1} = 1$ and $0 = \alpha_0^* \leq \alpha_1^* < \dots < \alpha_k^* \leq \alpha_{k+1}^* = 1$. We shall say that the roots of $P^*(x)$ precede the roots of $P(x)$ on $[-1, 1]$, and write $\text{Rts}\{P^*(x)\} < \text{Rts}\{P(x)\}$, provided

$$\alpha_i^* < \alpha_i \text{ for } i = 1, \dots, k-1,$$

and

$$\alpha_k^* < \alpha_k \leq 1 \text{ or } \alpha_k^* = \alpha_k = 1.$$

The following three lemmas are direct consequences of this definition.

Lemma 4. *If $\text{Rts}\{P^*(x)\} < \text{Rts}\{P(x)\}$, then for any non-negative integer j*

$$\text{Rts}\{x^j P^*(x)\} < \text{Rts}\{x^j P(x)\}.$$

Lemma 5. *If $\text{Rts}\{P^*(x)\} < \text{Rts}\{P(x)\}$ and $\alpha_k < 1$, then*

$$\text{Rts}\{(1 - x^2)P^*(x)\} < \text{Rts}\{(1 - x^2)P(x)\}.$$

Lemma 6. *If $\text{Rts}\{P^{**}(x)\} < \text{Rts}\{P^*(x)\}$ and $\text{Rts}\{P^*(x)\} < \text{Rts}\{P(x)\}$, then*

$$\text{Rts}\{P^{**}(x)\} < \text{Rts}\{P(x)\}.$$

The next lemma localizes the roots of $T_r(P)(x)$ with respect to the roots of $P(x)$.

Lemma 7. *Let $r > 0$,*

$$L = \begin{cases} 1 & \text{if } l > 0, \\ 2 & \text{if } l = 0; \end{cases} \quad \text{and} \quad U = \begin{cases} k & \text{if } \alpha_k = 1, \\ k+1 & \text{if } \alpha_k < 1. \end{cases}$$

Then

- (a) $\beta_0 = 0$ is a root of $T_r(P)(x)$ of multiplicity 1 if $l = 0$ or of multiplicity $l - 1$ if $l > 0$;
- (b) there exists a sequence $\{\beta_i\}_{i=L}^U$ such that $\alpha_{i-1} < \beta_i < \alpha_i$ ($i = L, \dots, U$) and β_i is a simple root of $T_r(P)(x)$;
- (c) if $\alpha_k = 1$ then $\beta_{k+1} = 1$ is a simple root of $T_r(P)(x)$.

In the next results we analyse the dependence of the roots of $T_r(P)(x)$ on the roots of $P(x)$, and the parameter r . The results are based on the following lemmas.

Lemma 8. *Let $r > 0$, j fixed such that $1 \leq j \leq k$, and assume*

$$0 \leq \alpha_1^* < \alpha_1 \text{ if } j = 1, \quad \alpha_{j-1} < \alpha_j^* < \alpha_j \text{ if } j > 1.$$

Let $P^*(x) = P(x)\psi(x)$ where $\psi(x) = \frac{x^2 - (\alpha_j^*)^2}{x^2 - \alpha_j^2}$. Then

$$\text{Rts}\{T_r(P^*)(x)\} < \text{Rts}\{T_r(P)(x)\}.$$

Lemma 9. *Let $r > 0$. If $\text{Rts}\{P^*(x)\} < \text{Rts}\{P(x)\}$, then*

$$\text{Rts}\{T_r(P^*)(x)\} < \text{Rts}\{T_r(P)(x)\}.$$

Lemma 10. *Let $s > r > 0$. Then*

$$\text{Rts}\{T_s(P)(x)\} < \text{Rts}\{T_r(P)(x)\}.$$

4. Increasing property of the main root. Let $\{P_n\}_{n=0}^\infty$ be the sequence of polynomials generated by (G). From the definition, we have $P_n(-x) = (-1)^n P_n(x)$. Moreover, from Lemma 7 it follows that the roots of $P_n(x)$ are simple and all in the interval $(-1, 1)$. Let the roots of $P_n(x)$ be

$$\begin{aligned} \alpha_0^{(n)} &= 0 \quad \text{of multiplicity } n - 2\eta_n, \\ \pm \alpha_i^{(n)} &\quad \text{for } i = 1, \dots, \eta_n, \end{aligned}$$

where $\eta_n = \lfloor \frac{n}{2} \rfloor$ is the integral part of $\frac{n}{2}$, and

$$0 \leq \alpha_0^{(n)} < \alpha_1^{(n)} < \dots < \alpha_{\eta_n}^{(n)} < 1.$$

Then there exist constants C_n such that

$$P_n(x) = C_n x^{n-2\eta_n} \prod_{i=1}^{2\eta_n} (x^2 - (\alpha_i^{(n)})^2). \quad (8)$$

The first result of this section corresponds to the standard interlacing property

$$\alpha_{\eta_{n+1}-i}^{(n+1)} < \alpha_{\eta_n+1-i}^{(n)} < \alpha_{\eta_{n+1}+1-i}^{(n+1)} \quad (i = 1, \dots, \eta_n),$$

and it follows directly from Lemma 7.

Theorem 11. *Let $\{P_n\}_{n=0}^\infty$ be the sequence generated by (G). Then*

$$\text{Rts}\{xP_{n+1}(x)\} < \text{Rts}\{(1-x^2)P_n(x)\}$$

$$\text{Rts}\{xP_n(x)\} < \text{Rts}\{P_{n+1}(x)\}$$

for any $n \geq 0$.

The second result corresponds to the following interlacing property

$$\alpha_{\eta_n+1-i}^{(n+2)} < \alpha_{\eta_n+1-i}^{(n)} < \alpha_{\eta_n+2-i}^{(n+2)} \quad (i = 1, \dots, \eta_n),$$

and it is used to obtain (2) and (3). Its proof is similar to the proof of Theorem 13.

Theorem 12. Let $\{P_n\}_{n=0}^{\infty}$ be the sequence generated by (G). Then

$$\text{Rts}\{x^2 P_n(x)\} < \text{Rts}\{P_{n+2}(x)\}.$$

Moreover, if $r_{n+2} > \max\{0, r_n - 1\}$ for all $n \geq 0$, then

$$\text{Rts}\{P_{n+2}(x)\} < \text{Rts}\{(1 - x^2)P_n(x)\}$$

for any $n \geq 0$.

The last result is used to compare the roots of two sequences of polynomials generated by (G). It is a key result to obtain (1).

Theorem 13. Let $\{P_n\}_{n=0}^{\infty}$ and $\{Q_n\}_{n=0}^{\infty}$ be two sequences generated by (G) for $\{r_n\}_{n=0}^{\infty}$ and $\{s_n\}_{n=0}^{\infty}$ respectively. If

$$s_n < r_n \text{ for } n \geq 1, \quad \text{and} \quad r_{n-2} < s_n + 1 \text{ for } n \geq 2,$$

then

$$\text{Rts}\{P_n(x)\} < \text{Rts}\{Q_n(x)\} < \text{Rts}\{(1 - x^2)P_{n-2}(x)\} \quad (9)$$

for $n \geq 2$.

Proof. We can consider that $P_1(x) = x = Q_1(x)$, $P_2(x) = 1 - (1 + 2r_1)x^2$, $Q_2(x) = 1 - (1 + 2s_1)x^2$, and the result holds for $n = 2$. If (9) holds, then using Lemma 9,

$$\text{Rts}\{T_{s_n}(P_n)(x)\} < \text{Rts}\{T_{s_n}(Q_n)(x)\} < \text{Rts}\{T_{s_n}((1 - x^2)P_{n-2})(x)\}.$$

But, from Lemma 10,

$$\text{Rts}\{T_{r_n}(P_n)(x)\} < \text{Rts}\{T_{s_n}(P_n)(x)\}$$

since $r_n > s_n$. Also

$$T_{s_n}((1 - x^2)P_{n-2})(x) = (1 - x^2)T_{s_n+1}(P_{n-2})(x)$$

from Property 2. Again, from Lemma 10,

$$\text{Rts}\{T_{s_n+1}(P_{n-2})(x)\} < \text{Rts}\{T_{r_{n-2}}(P_{n-2})(x)\}$$

because $s_n + 1 > r_{n-2}$. Finally, from the definition of $P_{n-1}(x)$ and $P_{n+1}(x)$, we have

$$\text{Rts}\{P_{n+1}(x)\} < \text{Rts}\{Q_{n+1}(x)\} < \text{Rts}\{(1 - x^2)P_{n-1}(x)\}.$$

The result follows by induction. $\square$

Example 14. For the Generalized Euler-Frobenius polynomials we have $P_0(x) = 1 = Q_0(x)$,

$$\begin{aligned} r_n &= 1 & \text{and} & & P_{n+1}(x) &= T_1(P_n)(x) & \text{for } n \geq 0, \\ s_n &= \frac{1}{2} & \text{and} & & Q_{n+1}(x) &= T_{\frac{1}{2}}(Q_n)(x) & \text{for } n \geq 0, \end{aligned}$$

and

$$\text{Rts}\{P_n(x)\} < \text{Rts}\{Q_n(x)\} < \text{Rts}\{(1 - x^2)P_{n-2}(x)\}$$

for $n \geq 2$. In particular, this result for n even is used to obtain (1). TABLE I indicates the first eight values of $\zeta_m(\lambda)$ which is the main root for the shifted interpolating cardinal Lagrange spline series of degree m .

TABLE I. SOME VALUES OF $\zeta_m(\lambda)$.

n	r_n	m	λ	$\zeta_m(\lambda)$
2	$\frac{1}{2}$	2	$\frac{1}{2}$	-5.82847
2	1	3	0	-3.73205
4	$\frac{1}{2}$	4	$\frac{1}{2}$	-2.76746
4	1	5	0	-2.32247
6	$\frac{1}{2}$	6	$\frac{1}{2}$	-2.04794
6	1	7	0	-1.86818
8	$\frac{1}{2}$	8	$\frac{1}{2}$	-1.74008
8	1	9	0	-1.64474

5. Bound for the sum of roots. To measure the closedness of the *non-zero* roots of $P_n(x)$ from 0 we can consider the following quantity

$$I_n = \sum_{i=1}^{\eta_n} \frac{1}{(\alpha_i^{(n)})^2}.$$

From (8), we have

$$I_n = \begin{cases} -\frac{D^2 P_n(0)}{2P_n(0)} & \text{if } n \text{ is even;} \\ -\frac{D^3 P_n(0)}{6DP_n(0)} & \text{if } n \text{ is odd.} \end{cases}$$

Using (G), it follows

$$I_n = \begin{cases} -r_n - \frac{c_{n+1}c_{n+2}P_{n+2}(0)}{2P_n(0)} & \text{if } n \text{ is even;} \\ -\frac{1+2r_n+r_{n+1}}{3} - \frac{c_{n+2}c_{n+3}P_{n+3}(0)}{6P_{n+1}(0)} & \text{if } n \text{ is odd.} \end{cases}$$

Application 15. For the Generalized Euler-Frobenius polynomials, we consider the roots $z_i^{(n)}(\lambda) < 0$ and $z_i^n(\lambda) \neq -1$, and we define the set

$$\mathcal{I}_n(\lambda) = \left\{ i \mid \begin{array}{l} -\infty < z_i^{(n)}(\lambda) < 0, \\ \text{and } z_i^{(n)}(\lambda) \neq -1 \end{array} \right\}.$$

For $i \in \mathcal{I}_n(\lambda)$, let

$$\alpha_i^{(n)}(\lambda) = \frac{z_i^{(n)}(\lambda) + 1}{z_i^{(n)}(\lambda) - 1},$$

and

$$J_n(\lambda) = \sum_{i \in \mathcal{I}_n(\lambda)} \frac{1}{(\alpha_i^{(n)}(\lambda))^2}.$$

Since (see [2])

$$p_n(\lambda, -1) = (-1)^n 2^n E_n(\lambda)$$

where $E_n(\lambda)$ is the Euler polynomial of degree n [1], we obtain the following expressions:

(i) if $\lambda = 0$, $r_n = 1$ and $c_n = 1$ for all n , using (6) we have

$$J_n(0) = 2I_{n-1}, \quad (10)$$

and

$$I_n = \begin{cases} -1 - \frac{p_{n+3}(0, -1)}{2p_{n+1}(0, -1)} = -1 - \frac{2E_{n+3}(0)}{E_{n+1}(0)} & \text{if } n \text{ is even,} \\ -\frac{4}{3} - \frac{p_{n+4}(0, -1)}{6p_{n+2}(0, -1)} = -\frac{4}{3} - \frac{2E_{n+4}(0)}{3E_{n+2}(0)} & \text{if } n \text{ is odd;} \end{cases}$$

(ii) if $\lambda = \frac{1}{2}$, $s_n = \frac{1}{2}$ and $c_n = 1$ for all n , using (7) we have

$$J_n(\frac{1}{2}) = 2I_n, \quad (11)$$

and

$$I_n = \begin{cases} -\frac{1}{2} - \frac{p_{n+2}(\frac{1}{2}, -1)}{2p_n(\frac{1}{2}, -1)} = -\frac{1}{2} - \frac{2E_{n+2}(\frac{1}{2})}{E_n(\frac{1}{2})} & \text{if } n \text{ is even,} \\ -\frac{5}{6} - \frac{p_{n+3}(\frac{1}{2}, -1)}{6p_{n+1}(\frac{1}{2}, -1)} = -\frac{5}{6} - \frac{2E_{n+3}(\frac{1}{2})}{3E_{n+1}(\frac{1}{2})} & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, expressions in terms of the n th Euler number E_n and the n th Bernoulli number B_n can be obtained since

$$E_n(\frac{1}{2}) = 2^{-n} E_n \quad \text{and} \quad E_n(0) = -2^{\frac{2^{n+1}-1}{n+1}} B_{n+1}.$$

An upper bound for I_n can be obtained from the expressions of $E_n(\cdot)$. From [1, p. 805], we have

$$E_{2l-1}(0) = (-1)^l 4^l \frac{(2l-1)!}{\pi^{2l}} \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^{2l}}$$

and

$$E_{2l}(\frac{1}{2}) = (-1)^l 4^l \frac{(2l)!}{\pi^{2l+1}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)^{2l+1}}.$$

Then

(i) if $\lambda = 0$, $r_n = 1$, and $c_n = 1$, we have

$$I_n \leq \begin{cases} 2^{\frac{(n+3)(n+2)}{\pi^2}} & \text{if } n \text{ is even,} \\ 2^{\frac{(n+4)(n+3)}{3\pi^2}} & \text{if } n \text{ is odd,} \end{cases} \quad (12)$$

since

$$\left| \frac{E_{2l+1}(0)}{E_{2l-1}(0)} \right| \leq \frac{(2l+1)(2l)}{\pi^2} \quad l = 1, 2, 3, \dots;$$

(ii) if $\lambda = \frac{1}{2}$, $s_n = \frac{1}{2}$, and $c_n = 1$, we have

$$I_n \leq \begin{cases} 8 \frac{(n+2)(n+1)}{\pi^3} & \text{if } n \text{ is even,} \\ 8 \frac{(n+3)(n+2)}{3\pi^3} & \text{if } n \text{ is odd,} \end{cases} \quad (13)$$

since

$$\left| \frac{E_{2l+2}(\frac{1}{2})}{E_{2l}(\frac{1}{2})} \right| \leq \frac{4(2l+2)(2l+1)}{\pi^3} \quad l = 1, 2, 3, \dots$$

follows from the increasing property of

$$\phi(p) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)^p}$$

for $p \in [1, \infty)$, $\phi(1) = \frac{\pi}{4}$, and $\lim_{p \rightarrow +\infty} \phi(p) = 1$.

Finally, (5a), respectively (5b), follows from (10) and the first inequality in (12), respectively from (11) and the first inequality in (13).

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Résumé substantiel en français. La série

$$\sum_{j \in \mathbb{Z}} y_j l_j(x - \lambda) \quad x \in \mathbb{R}, y_j \in \mathbb{R}, \lambda \in [0, 1],$$

où $l_j(\cdot)$ est la spline de Lagrange de degré $n \in \mathbb{N}$ définie à l'aide de la partition $\mathbb{Z}$ de $\mathbb{R}$, converge/diverge si

$$\overline{\lim}_{|j| \rightarrow \infty} |y_j|^{\frac{1}{|j|}} \leq \min\{|\zeta_n(\lambda)|, |\zeta_n(1 - \lambda)|\},$$

où $\zeta_n(\lambda)$ est la racine principale du polynôme d'Euler-Frobenius

$$p_n(\lambda, z) = \frac{(1-z)^{n+1}}{z} \left(z \frac{\partial}{\partial z} - \lambda \right)^n \left(\frac{z}{1-z} \right).$$

Il est connu que les racines $z_j^{(n)}(\lambda)$ ($j = 1, \dots, n$) de $p_n(\lambda, \cdot)$ sont réelles, ont la propriété suivante

$$-\infty \leq z_1^{(n)}(\lambda) < z_2^{(n)}(\lambda) < \dots < z_{n-1}^{(n)}(\lambda) < z_n^{(n)}(\lambda) \leq 0,$$

et sont des fonctions croissantes pour $\lambda \in [0, 1]$. La racine principale de $p_n(\lambda, z)$ est sa plus grande racine qui ne dépasse pas -1. Nous avons que

$$\begin{cases} \zeta_n(0) = z_r^{(n)}(0) & \text{if } n = 2r + 1, r \in \mathbb{N}; \\ \zeta_n(\frac{1}{2}) = z_r^{(n)}(\frac{1}{2}) & \text{if } n = 2r, r \in \mathbb{N} \end{cases}$$

et que le plus grand rayon de convergence de la série est obtenue pour n impair avec $\lambda = 0$, et pour n pair avec $\lambda = \frac{1}{2}$.

Dans cet article nous démontrons la relation suivante

$$\zeta_2(\frac{1}{2}) < \zeta_3(0) < \zeta_4(\frac{1}{2}) < \zeta_5(0) < \cdots < -1. \quad (1)$$

En utilisant la transformation $z = (x + 1)/(x - 1)$ qui préserve les propriétés d'entrelacements des racines, nous identifions les polynômes d'Euler-Frobénius $p_n(\lambda, z)$ pour $\lambda = 0$ et $\lambda = \frac{1}{2}$ à des polynômes $P_n(x)$ générés par le schéma

$$(G) \quad \begin{cases} \text{(i) } P_0(x) = 1 ; \\ \text{(ii) pour } n \geq 0 \\ \quad c_{n+1}P_{n+1}(x) = 2r_n x P_n(x) - (1 - x^2)DP_n(x) \\ \quad \text{où } \{r_n\}_{n=0}^{\infty} \text{ est une suite de nombres réels strictement positifs et} \\ \quad \{c_n\}_{n=1}^{\infty} \text{ est une suite de nombres réels non nuls.} \end{cases}$$

L'analyse des propriétés d'entrelacement des racines de $P_n(x)$ et $P_{n+1}(x)$, et des racines de $P_n(x)$ et $P_{n+2}(x)$ nous permet d'obtenir une démonstration élémentaire de (1). On complète ainsi des résultats établis en [11] et [3].

Un deuxième résultat de cet article est une majoration de l'expression suivante

$$J_n(\lambda) = \sum_{i \in \mathcal{I}_n(\lambda)} \left[\frac{z_i^{(n)}(\lambda) - 1}{z_i^{(n)}(\lambda) + 1} \right]^2$$

où $\mathcal{I}_n(\lambda) = \{i \mid z_i^{(n)}(\lambda) \neq -1 \text{ et } -\infty < z_i^{(n)}(\lambda) < 0\}$ et qui est utilisée pour le calcul des meilleurs bornes des erreurs d'interpolation [4].

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