

ON THE CONVERGENCE AND SINGULARITIES OF ANALYTIC FUNCTIONS DEFINED BY E-DIRICHLETIAN ELEMENTS

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RÉSUMÉ. On se propose d'étudier la convergence et les singularités de certaines séries d'exponentielles dont les coefficients sont des fonctions entières d'indices bornés, au sens de B. Lepson. Utilisant des résultats dus à Blambert-Berland-Parvatham, on montre quelques théorèmes sur la convergence absolue et uniforme. Dans la deuxième partie, on considère une sous-classe dont les coefficients sont des fonctions entières d'indices bornés n'ayant qu'un nombre fini de zéros. En appliquant un théorème de Gross-Shah et une inégalité de Mandelbrojt, sous certaines conditions, on localise les singularités des fonctions analytiques définies par ces séries.

ABSTRACT. In this work, we consider a certain class of exponential series whose coefficients are entire functions of bounded index (after B. Lepson) and study the properties of convergence and singularities of such series. This paper consists of two chapters. In the first, we prove certain theorems on absolute convergence of these series, using results of Blambert-Berland-Parvatham in the case of exponential series with polynomial coefficients. In the second chapter, we consider a subclass of this class whose coefficients are functions with at most a finite number of zeros. Using a theorem of Gross-Shah and a general Mandelbrojt-type inequality for the general term of the series of Blambert-Parvatham, we locate, under certain conditions, the singularities of analytic functions defined by these series.

1. Let $(f_n)_1^\infty$ be a sequence of entire functions f_n . We have:

$$\forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} : f_n(s) = \sum_{i=0}^{\infty} a_{n,i}(s_0)(s - s_0)^i$$

where

$$a_{n,i}(s_0) = \frac{f_n^{(i)}(s_0)}{i!} \text{ and } s_0 \text{ of } \mathbf{C}$$

(in which $f_n^{(i)}(s_0)$ is the value at s_0 of the i^{th} -derivative of the function f_n with $f_n^{(0)}(s_0) = f_n(s_0)$). Now

$$\lim_{i \rightarrow \infty} \{a_{n,i}(s_0)\} = 0$$

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then, we put:

$$\begin{aligned} \forall n \in \mathbf{N} \setminus \{0\} : \quad A_n(s_0) &= \max \{ |a_{n,i}(s_0)| \mid i \in \mathbf{N} \} > 0 \\ \nu_n(s_0) &= \max \{ i \in \mathbf{N} \mid |a_{n,i}(s_0)| = A_n(s_0) \} \\ \nu_n &= \sup \{ \nu_n(s_0) \in \mathbf{N} \mid s_0 \in \mathbf{C} \} . \end{aligned}$$

Definition. (B. Lepson [13]) An entire function φ is said to be of bounded index if there exists a non negative integer ν such that:

$$\forall i \in \mathbf{N}, \forall s \in \mathbf{C} : \max \left\{ \frac{|\varphi^{(k)}(s)|}{k!} \mid k \in \{0, 1, \dots, \nu\} \right\} \geq \frac{|\varphi^{(i)}(s)|}{i!} .$$

The least such integer ν is called the index of φ .

We have (W. K. Hayman [12]):

$$\forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} :$$

$$|f_n(s)| \leq \max \left\{ \frac{|f_n^{(k)}(0)|}{(\nu_n + 1)^k} \mid k \in \{0, 1, \dots, \nu_n\} \right\} \exp [(\nu_n + 1)|s|] .$$

It is trivial that:

$$\forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} : |f_n(s)| \leq A_n \exp [(\nu_n + 1)|s|]$$

where

$$A_n = \max \left\{ \frac{|f_n^{(k)}(0)|}{k!} \mid k \in \{0, 1, \dots, \nu_n\} \right\} .$$

Let $(\lambda_n)_1^\infty$ be a sequence of complex numbers such that $(|\lambda_n|)_1^\infty$ is a D -sequence ($\Leftrightarrow 0 < |\lambda_n| < |\lambda_{n+1}| \lim_{n \rightarrow \infty} \{|\lambda_n|\} = \infty$) and the element:

$$\{f\} : \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n)$$

where $(f_n)_1^\infty$ is the sequence of entire functions f_n of bounded index ν_n . By definition $\{f\}$ is the “ E -dirichletian element”. Let $\{f_A\}$ be the associated element:

$$\sum_{n=1}^{\infty} A_n \exp [(\nu_n + 1)|s| - s\lambda_n]$$

and $\text{Arg } \lambda_n$ the main argument of λ_n ($\Leftrightarrow \text{Arg } \lambda_n \in]-\pi, \pi]$). Let:

$$\forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} : d(n, s) = \frac{-\log |A_n \exp [(\nu_n + 1)|s| - s\lambda_n]|}{|\lambda_n|} .$$

We have:

$$\begin{aligned} |A_n \exp [(\nu_n + 1)|s| - s\lambda_n]| &= \exp [-d(n, s)|\lambda_n|] \\ &\Updownarrow \\ d(n, s) &= -\frac{\log A_n + (\nu_n + 1)|s|}{|\lambda_n|} + \operatorname{Re} (s \exp(i \operatorname{Arg}(\lambda_n))) . \end{aligned}$$

It is trivial that:

$$\begin{aligned} \forall n \in \mathbf{N} \setminus \{0\}, \forall (s, s') \in \mathbf{C} \times \mathbf{C} : \\ |d(n, s) - d(n, s')| &\leq |s - s'| + \frac{\nu_n + 1}{|\lambda_n|} ||s| - |s'|| \\ &\leq \left(1 + \frac{\nu_n + 1}{|\lambda_n|}\right) |s - s'| . \end{aligned}$$

Let:

$$\begin{aligned} L &= \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{|\lambda_n|} \right\} \\ \beta &= \limsup_{n \rightarrow \infty} \left\{ \frac{\nu_n + 1}{|\lambda_n|} \right\} \\ \forall s \in \mathbf{C} : d_*(s) &= \liminf_{n \rightarrow \infty} \{d(n, s)\} \\ \forall \alpha \in \mathbf{R} : D_{*,\alpha} &= \{s \in \mathbf{C} \mid d_*(s) > \alpha\} . \end{aligned}$$

Remark 1.1. The function $s \mapsto d(n, s)$ is lipschitzian.

Remark 1.2. If $\beta < \infty$, and if there exists a s_0 of $\mathbf{C}$ such that $|d_*(s_0)| < \infty$, then the function d_* is k -lipschitzian on $\mathbf{C}$ with $k = 1 + \beta$ and d_* is continuous on $\mathbf{C}$, which implies that $D_{*,\alpha}$ is an open of $\mathbf{C}$.

Let K be any compact subset of $\mathbf{C}$.

Theorem 1.1. When $\beta < \infty$ and $L < \infty$, the E -dirichletian element $\{f\}$ converges absolutely in $D_{*,L}$ and uniformly on any compact K subset of $D_{*,L}$.

Proof. Let us suppose that $D_{*,L}$ is non empty. Otherwise the theorem is trivial. We have:

$$\forall s \in D_{*,L}, \exists \varepsilon_s > 0 :$$

$$d_*(s) > L + \varepsilon_s ,$$

$$\exists n_s \in \mathbf{N} \setminus \{0\}, \forall n \geq n_s :$$

$$d(n, s) > L + \varepsilon_s ,$$

$$\begin{aligned} \log |f_n(s) \exp(-s\lambda_n)| &\leq \log |A_n \exp [(\nu_n + 1)|s| - s\lambda_n]| \\ &< -(L + \varepsilon_s)|\lambda_n| \end{aligned}$$

hence

$$\sum_{n=n_s}^{\infty} |f_n(s) \exp(-s\lambda_n)| < \sum_{n=n_s}^{\infty} \exp [-(L + \varepsilon_s)|\lambda_n|]$$

and the series on the right hand side is convergent which proves that converges absolutely on $D_{*,L}$.

Also, we have:

$$\forall \varepsilon > 0, \exists n' \in \mathbf{N} \setminus \{0\}, \forall n \geq n' :$$

$$\frac{\nu_n + 1}{|\lambda_n|} < \beta + \varepsilon.$$

$\exists s \in D_{*,L}$ and $s \notin K$ (since $D_{*,L}$ is an open on $\mathbf{C}$ and K be a compact subset of $D_{*,L}$). Hence $\exists \alpha > L, \forall s \in K: d_*(s) \geq \alpha$ (M. Berland [2]),

$$\forall \varepsilon' \in]0, \frac{\alpha - L}{3}[: K \subset D_{*,\alpha-\varepsilon'}, \exists \varepsilon_1 > 0 \text{ such as } \varepsilon_1 = \frac{\varepsilon'}{1 + \beta + \varepsilon}.$$

We have :

$$D_{*,\alpha-\varepsilon'} \subset D_{*,\alpha-2\varepsilon'} \subset D_{*,L+\varepsilon'}$$

and

$$\forall s \in K, \exists n_s \in \mathbf{N} \setminus \{0\}, \forall n \geq n_s : d(n, s) > \alpha - \varepsilon'.$$

Let $d_{(s,\varepsilon_1)}$ be the open disc centred at s and of radius ε_1 . We have:

$$\forall s \in K, \forall s' \in d_{(s,\varepsilon_1)}, \forall n \geq n_s :$$

$$\begin{aligned} d(n, s') &\geq d(n, s) - |s - s'| \left(1 + \frac{\nu_n + 1}{|\lambda_n|} \right) \\ &> \alpha - \varepsilon' - \varepsilon_1 \left(1 + \frac{\nu_n + 1}{|\lambda_n|} \right). \end{aligned}$$

Therefore:

$$\forall s \in K, \forall n \geq \max\{n_s, n'\}, s' \in d_{(s,\varepsilon_1)}:$$

$$d(n, s') > \alpha - 2\varepsilon' > L + \varepsilon'$$

where:

$$|A_n \exp[(\nu_n + 1)|s'| - s'\lambda_n]| < \exp[-(L + \varepsilon')|\lambda_n|] .$$

The set of open discs $d_{(s,\varepsilon_1)}$ indexed by s on K is an open covering of K . Hence we have a finite subcovering:

$$\exists \{s_1, s_2, \dots, s_k\} \subset K, \bigcup_{i=1}^k d_{(s_i, \varepsilon_1)} \supset K .$$

Let $s \in K, \exists j \in \{1, 2, \dots, k\} : s \in d_{(s_j, \varepsilon_1)}$.

Hence $\forall n \geq \max\{n', n_1\}$ where $n_1 = \max \{n_{s_j} \mid j \in \{1, 2, \dots, k\}\}$ (n_1 is independent of s), $\forall s \in K$:

$$\begin{aligned} |f_n(s) \exp(-s\lambda_n)| &\leq |A_n \exp[(\nu_n + 1)|s| - s\lambda_n]| \\ &< \exp[-(L + \varepsilon')|\lambda_n|]. \end{aligned}$$

Hence

$$K \subset D_{*,L}, \exists \alpha > L, \forall \varepsilon' \in]0, \frac{\alpha - L}{3}[, \exists n_2 = \max\{n', n_1\} ,$$

$$\forall s \in K : \sum_{n=n_2}^{\infty} |f_n(s) \exp(-s\lambda_n)| < \sum_{n=n_2}^{\infty} \exp [- (L + \varepsilon') |\lambda_n|]$$

and the series on the right hand side is convergent which proves that

$$\{f\} : \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n)$$

converges absolutely and uniformly on K . Since K is any arbitrary compact subset of $D_{*,L}$ E -dirichletian element $\{f\}$ converges uniformly on any compact subset of $D_{*,L}$ and absolutely on $D_{*,L}$. $\square$

Theorem 1.2. *When $\beta < \infty$, the $\{f_A\}$ diverges on $\mathbf{C} \setminus \overline{D}_{*,0}$ (where $\overline{D}_{*,0}$ is the closure of $D_{*,0}$).*

Proof. If $s \in \mathbf{C} \setminus \overline{D}_{*,0}$, then $d_*(s) < 0$ and $\exists \alpha > 0 : d_*(s) < -\alpha$ (since $\overline{D}_{*,0} = \{s \in \mathbf{C} \mid d_*(s) \geq 0\}$). Hence:

$$\forall s \in \mathbf{C} \setminus \overline{D}_{*,0}, \exists \alpha > 0, \exists (n_j)_1^\infty : d(n_j, s) < -\alpha$$

where $(n_j)_1^\infty$ is an infinite subsequence of $\mathbf{N} \setminus \{0\}$.

Therefore:

$$|A_{n_j} \exp [(\nu_{n_j} + 1)|s| - s\lambda_{n_j}]| > \exp(\alpha|\lambda_{n_j}|) > 1$$

and which shows that $\{f_A\}$ diverges on $\mathbf{C} \setminus \overline{D}_{*,0}$. $\square$

Remark 1.3. Let

$$D = \limsup_{n \rightarrow \infty} \left\{ \frac{\sum_{i=1}^n (\nu_i + 1)}{|\lambda_n|} \right\}.$$

If $D < \infty$ ($\implies \beta < \infty$ and $L = 0$) $\{f\}$ converges absolutely on $D_{*,0}$ and uniformly on any compact subset of $D_{*,0}$.

2. Theorem A. (F. Gross [11]) *An entire function φ with at most a finite number of zeros is of bounded index if and only if it is of the form:*

$$\forall s \in \mathbf{C} : \varphi(s) = P(s) \exp(\alpha s)$$

where P is polynomial and α is a complex constant.

Theorem B. (S. M. Shah [17]) *Let $\varphi(s) = P(s) \exp(\alpha s)$ where α is any complex number and P is a polynomial of degree less than n . Then φ is of bounded index and the index $\nu \leq p$ where p is any integer such that $p \geq n-1$ and*

$$\frac{n|\alpha|}{p+1} + \left(\frac{n(n-1)}{2!} |\alpha|^2 \right) \frac{1}{p(p+1)} + \cdots + \frac{|\alpha|^n}{(p-n+2) \cdots (p+1)} \leq 1.$$

Let us consider the following element

$$\{f\} : \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n) \quad (E\text{-dirichletian element})$$

where $(f_n)_1^\infty$ is a sequence of entire functions f_n with (at most) a finite number m_n ($\neq 0$) of zeros and of bounded index ν_n and $(\lambda_n)_1^\infty$ is a sequence of complex numbers λ_n such that $(|\lambda_n|)_1^\infty$ is a D -sequence. Using theorems A-B, we have:

$$\forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} : f_n(s) = P_n(s) \exp(\alpha_n s)$$

where P_n is a polynomial of degree m_n ($\neq 0$) and α_n is a complex constant. Let $\mathcal{E}_n$ be the set of points of $\mathbf{C}$ which are zeros of P_n and $\mathcal{E} = \bigcup_n \mathcal{E}_n$. Let us denote by $\mathcal{E}^d$ the derived set of $\mathcal{E}$ and

$$\mathcal{E}_\infty = \{s \in \mathbf{C} \mid \exists (n_j)_1^\infty, P_{n_j}(s) = 0\}$$

where $(n_j)_1^\infty$ is an infinite subsequence of $\mathbf{N} \setminus \{0\}$ depending on s . Let $\mathcal{E}^* = \mathcal{E}^d \cup \mathcal{E}_\infty$. $\mathcal{E}^*$ is a closed set. Let us suppose that $\mathbf{C} \setminus \mathcal{E}^*$ is non empty. We put:

$$\mathcal{D} = \limsup_{n \rightarrow \infty} \left\{ \frac{\sum_{i=1}^n (m_i + 1)}{|\mu_n|} \right\}$$

where $\mu_n = \lambda_n - \alpha_n$

$$L^* = \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{|\mu_n|} \right\}$$

$$\beta^* = \limsup_{n \rightarrow \infty} \left\{ \frac{m_n}{|\mu_n|} \right\}$$

Let us suppose that:

- 1) $m_n \sim \nu_n$ ($\iff \lim_{n \rightarrow \infty} \left\{ \frac{m_n}{\nu_n} \right\} = 1$),
- 2) $|\mu_n| \sim |\lambda_n|$ ($\iff \lim_{n \rightarrow \infty} \left\{ \left| \frac{\mu_n}{\lambda_n} \right| \right\} = 1$),
- 3) $(|\mu_n|)_1^\infty$ is a D -sequence.

$\{f\}$ is a LC -dirichletian element (Berland [1], Blambert and Parvatham [8, 9]):

$$\{f\} : \sum_{n=1}^{\infty} P_n(s) \exp(-s\mu_n).$$

We put:

$$\forall s \in \mathbf{C} \setminus \mathcal{E}^* : \delta(n, s) = -\frac{\log |P_n(s) \exp(-s\mu_n)|}{|\mu_n|}$$

for sufficiently large n ,

$$\delta_*(s) = \liminf_{n \rightarrow \infty} \{\delta(n, s)\}$$

$$\forall \alpha \in \mathbf{R} : \mathcal{D}_{*,\alpha} = \left\{ s \in \mathbf{C} \setminus \mathcal{E}^* \mid \delta_*(s) > \alpha \right\}.$$

Theorem 2.1. *When:*

$$m_n \sim \nu_n, |\mu_n| \sim |\lambda_n|, \beta \left(= \limsup_{n \rightarrow \infty} \left\{ \frac{\nu_n + 1}{|\lambda_n|} \right\} \right) < \infty,$$

and

$$L \left(= \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{|\lambda_n|} \right\} \right) < \infty,$$

the E -dirichletian element

$$\{f\} : \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n)$$

converges absolutely on $\mathcal{D}_{*,L+\beta}$ and uniformly on any compact subset of $\mathcal{D}_{*,L+\beta}$.

Proof. Let us state a theorem due to M. Blambert and R. Parvatham [8] when $L^* < \infty$, $\beta^* < \infty$, the LC -dirichletian element

$$\{f\} : \sum_{n=1}^{\infty} P_n(s) \exp(-s\mu_n)$$

converges absolutely on $\mathcal{D}_{*,L^*+\beta^*}$. Now $m_n \sim \nu_n$ and $|\mu_n| \sim |\lambda_n|$, hence

$$\begin{aligned} \beta^* &= \limsup_{n \rightarrow \infty} \left\{ \frac{m_n}{|\mu_n|} \right\} \\ &= \limsup_{n \rightarrow \infty} \left\{ \frac{\nu_n + 1}{|\lambda_n|} \right\} \left(\lim_{n \rightarrow \infty} \left\{ \frac{m_n}{(\nu_n + 1)} \right\} \right) \left(\lim_{n \rightarrow \infty} \left\{ \left| \frac{\lambda_n}{\mu_n} \right| \right\} \right). \end{aligned}$$

Then: $\beta^* = \beta$.

On the one hand

$$\begin{aligned} L^* &= \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{|\mu_n|} \right\} \\ &= \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{|\lambda_n|} \right\} \left(\lim_{n \rightarrow \infty} \left\{ \left| \frac{\lambda_n}{\mu_n} \right| \right\} \right) \\ L^* &= L. \end{aligned}$$

Thus proving the theorem. $\square$

Remark 2.1. If $s \in (\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}_{*,0}}$ (where $\overline{\mathcal{D}_{*,0}}$ is the closure of $\mathcal{D}_{*,0}$) then

$$\delta_*(s) < 0 \text{ and } \exists \alpha > 0 : \delta_*(s) < -\alpha.$$

Hence $\forall s \in (\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}$, $\exists \alpha > 0$, $\exists (n_j)_1^\infty$:

$$\delta(n_j, s) < -\alpha$$

where $(n_j)_1^\infty$ is an infinite subsequence of $\mathbf{N} \setminus \{0\}$. Therefore:

$$|P_{n_j}(s) \exp(-s\mu_{n_j})| > \exp(\alpha|\mu_{n_j}|) > 1$$

and which shows that $\{f\}$ diverges on $(\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}$.

When $L = 0$, we have convergence of the series $\{f\}$ in $\mathcal{D}_{*,0}$ ($\subset \mathbf{C} \setminus \mathcal{E}^*$) and divergence in $(\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}$.

Theorem 2.2. *Suppose that:*

$$m_n \sim \nu_n, |\mu_n| \sim |\lambda_n|, \exists (\theta_n)_1^\infty, \theta_n > 0, \lim_{n \rightarrow \infty} \{\theta_n\} = \infty$$

and

$$\exists n' \in \mathbf{N} \setminus \{0\}, \forall n \geq n' : |\mu_{n+1}| > (1 + \theta_n)|\mu_n|,$$

then each point of $\text{Fr}(\mathcal{D}_{*,0}) \cap (\mathbf{C} \setminus \mathcal{E}^*)$, is a singular point for f . In particular if $\text{Fr}(\mathcal{D}_{*,0}) \subset (\mathbf{C} \setminus \mathcal{E}^*)$ then $\text{Fr}(\mathcal{D}_{*,0})$ is a natural boundary for the analytic function defined by E -dirichletian element $\{f\}$.

Proof. Let us suppose that the theorem is false. Then there exists a point s_0 of $\text{Fr}(\mathcal{D}_{*,0}) \cap (\mathbf{C} \setminus \mathcal{E}^*)$ and a disc $d_{(s_0, \varepsilon)}$ centred at s_0 and of radius $\varepsilon > 0$ on which f is holomorphic. As a result of Blambert-Parvatham [8] the sequence $(S_n)_1^\infty$ converges on $d_{(s_0, \varepsilon)}$ where

$$\forall s \in d_{(s_0, \varepsilon)} : S_n(s) = \sum_{i=1}^n P_i(s) \exp(-s\mu_i).$$

From remark 2.1, $\{f\}$ diverges on $(\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}$. There exists necessarily point common to $(\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}$ and $d_{(s_0, \varepsilon)}$. For these points there is a contradiction which establishes the theorem. $\square$

Let g be an integral function of growth not exceeding mean type T ($< \infty$) of Borel-order 1 ($\iff g \in B(1, T)$) having a zero of order $m_n + 1$ at μ_n , $n \in \mathbf{N} \setminus \{0\}$ and $\mathcal{C}$ the reflection in the imaginary axis of the indicator diagram of g . Suppose that $\mathcal{C} \subset \mathcal{D}_{*,\beta^*}$.

Remark 2.2. $\mathcal{C}$ be any compact subset of $\mathcal{D}_{*,\beta^*}$. Let us put:

$$\forall n \in \mathbf{N} \setminus \{0\} : M_n = \max \left\{ \frac{|\gamma_{n,j}|}{j!} \mid j \in \{0, 1, \dots, m_n\} \right\}$$

where $\gamma_{n,j}$ is the value at $s = \mu_n$ of the j^{th} -derivative of the function: $s \mapsto \frac{(s - \mu_n)^{m_n+1}}{g(s)}$ which is defined by continuity at the point μ_n ,

$$\gamma = \limsup_{n \rightarrow \infty} \left\{ \frac{\log M_n}{|\mu_n|} \right\}.$$

We have (M. Blambert and R. Parvatham [9]):

$$-T \leq \gamma \leq \infty .$$

We put:

$$\forall \varepsilon > 0, \forall s \in \mathbf{C} \setminus \mathcal{E}^*:$$

$$\delta_*^\varepsilon(s) = \liminf_{n \rightarrow \infty} \left\{ \frac{-\log |P_n(s) \exp(-s\mu_n)|}{|\mu_n|^{1+\varepsilon}} \right\}$$

$$\forall \alpha \in \mathbf{R} : \mathcal{D}_{*,\alpha}^\varepsilon = \{s \in \mathbf{C} \setminus \mathcal{E}^* \mid \delta_*^\varepsilon(s) > \alpha\} .$$

It is said that: $\mathcal{D} < \infty$, then we have:

$$\forall \varepsilon > 0 : \limsup_{n \rightarrow \infty} \left\{ \frac{\sum_{j=1}^n (m_j + 1)}{|\mu_n|^{1+\varepsilon}} \right\} = 0 ;$$

this implies that:

$$\limsup_{n \rightarrow \infty} \left\{ \frac{n}{|\mu_n|^{1+\varepsilon}} \right\} = 0, \quad \limsup_{n \rightarrow \infty} \left\{ \frac{m_n}{|\mu_n|^{1+\varepsilon}} \right\} = 0 ,$$

and

$$\mathcal{D}_{*,0} = \mathcal{D}_{*,0}^\varepsilon .$$

Theorem 2.3. *Suppose that*

$$m_n \sim \nu_n, \quad |\mu_n| \sim |\lambda_n|, \quad \beta < \infty, \quad \mathcal{D} < \infty \text{ and } \gamma < \infty$$

$$(\implies |\gamma| < \infty \implies \forall \varepsilon > 0 : \limsup_{n \rightarrow \infty} \left\{ \frac{\log M_n}{|\mu_n|^{1+\varepsilon}} \right\} = 0) ,$$

then we cannot obtain a holomorphic extension by means of $\mathcal{C}$ (in the sense of Mandelbrojt [15]) of function:

$$\mathcal{D}_{*,\beta} \ni s \longmapsto f(s) = \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n)$$

from $\{s_1\} + \mathcal{C}$ ($= \{s \in \mathbf{C} \setminus \mathcal{E}^* \mid s = s_1 + s', \ s' \in \mathcal{C}\} \subset \mathcal{D}_{*,\beta}$, to $\{s_2\} + \mathcal{C}$ where $s_2 \in (\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}^\varepsilon$.

Proof. Let us suppose that the theorem is false. Then there exists a point s_2 of $(\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}^\varepsilon$ on which

$$\mathcal{D}_{*,\beta} (= \mathcal{D}_{*,\beta^*}) \ni s \longmapsto f(s) = \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n)$$

$$(m_n \sim \nu_n \text{ and } |\mu_n| \sim |\lambda_n| \implies \beta = \beta^*, \ \mathcal{D}_{*,\beta} = \mathcal{D}_{*,\beta^*})$$

a holomorphic extension by means of $\mathcal{C}$ from $\{s_1\} + \mathcal{C} \subset \mathcal{D}_{*,\beta}$ to $\{s_2\} + \mathcal{C}$ along of the rectifiable arc $\mathcal{L}$ joining s_1 to s_2 . Using the general Mandelbrojt type inequality due to Blambert-Parvatham [9] (claim that their method is simpler than that of M.M. Dzrbasjan [10]), we have

$\exists n_1 \in \mathbf{N} \setminus \{0\}, \forall n \geq n_1$:

$$|P_n(s_2) \exp(-s_2 \mu_n)| \leq \frac{km(f) |\mu_n|^{m_n} 2^{m_n+1} \omega_n(m_n+1)!}{\varepsilon_1^{m_n+1}}$$

where k is constant depending on ε_1 of $]0, 1[$ and the function g ,

$$m(f) = \max \{|f(s)| \mid s \in \{s_2\} + \mathcal{C} + \varepsilon_1 E_0\}$$

(E_0 denotes the closed unit disc) and

$$\omega_n = \sum_{j=0}^{m_n} |\mu_n|^{j-m_n} \frac{|\gamma_{n,j}|}{j!}.$$

Where for n such as $|\mu_n| \geq 1$, we have:

$$\omega_n \leq m_n M_n$$

(because $\forall j \in \{0, 1, \dots, m_n\} : |\mu_n|^{j-m_n} \leq 1$ and $\sum_{j=0}^{m_n} |\mu_n|^{j-m_n} \leq m_n$).

ε is arbitrary fixed, we have:

$\forall n \geq n_1$:

$$\frac{\log |P_n(s_2) \exp(-s_2 \mu_n)|}{|\mu_n|^{1+\varepsilon}} \leq \frac{1}{|\mu_n|^{1+\varepsilon}} \log \left(\frac{km(f) |\mu_n|^{m_n} 2^{m_n+1} \omega_n(m_n+1)!}{\varepsilon_1^{m_n+1}} \right).$$

Hence, under the condition $\mathcal{D} < \infty$,

$$\limsup_{n \rightarrow \infty} \left\{ \frac{\log |P_n(s_2) \exp(-s_2 \mu_n)|}{|\mu_n|^{1+\varepsilon}} \right\} \leq \limsup_{n \rightarrow \infty} \left\{ \frac{\log M_n}{|\mu_n|^{1+\varepsilon}} \right\} = 0.$$

Now:

$$\delta_*^\varepsilon(s_2) \geq 0 \implies s_2 \in \overline{\mathcal{D}}_{*,0}^\varepsilon$$

and

$$s_2 \in (\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}^\varepsilon.$$

Hence a contradiction establishes the truth of the theorem. $\square$

Remark 2.3. Using the Mandelbrojt-type inequality due to M. Blambert and R. Parvatham [9], we get:

$\exists n_1 \in \mathbf{N} \setminus \{0\}, \forall n \geq n_1, \forall t \in \{1, 2, \dots, m_n+1\}$
 $(\implies t \in \{1, \dots, T_0+1\} \text{ where } T_0 = \inf\{m_n \mid n \in \mathbf{N} \setminus \{0\}\})$

$$\frac{|P_n^{(t-1)}(s_2) \exp(-s_2 \mu_n)|}{(t-1)!} \leq \frac{km(f) |\mu_n|^{m_n} 2^{m_n+1} \omega_{n,t}(m_n+1)!}{\varepsilon_1^{m_n+1}}$$

(the $P_n^{(t-1)}(s_2)$ is the value at s_2 of the $(t-1)^{\text{th}}$ -derivative of P_n) where

$$\omega_{n,t} = \frac{1}{(t-1)!} \sum_{j=0}^{m_n-t+1} |\mu_n|^{j+t-m_n-1} \frac{|\gamma_{n,j}|}{j!}.$$

Suppose that $t = 1$, we have therefore (with $P_n^{(0)}(s_2) = P_n(s_2)$ and $0! = 1$)

$$\omega_{n,1} = \omega_n$$

and

$$|P_n(s_2)| \exp(-s_2 \mu_n) \leq \frac{km(f) |\mu_n|^{m_n} 2^{m_n+1} \omega_n (m_n + 1)!}{\varepsilon_1^{m_n+1}}.$$

Thus proving the above inequality.

Résumé substantiel en français. Dans ce travail, on considère une certaine classe de fonctions définies par des séries d'exponentielles complexes dont les coefficients sont des fonctions entières d'indices bornés. La définition de l'indice borné d'une fonction entière φ est due à B. Lepson (1966) :

φ est d'indice borné ν sur $\mathbf{C} \iff \forall j \in \mathbf{N}, \forall s \in \mathbf{C} :$

$$\max \left\{ \frac{|\varphi^{(k)}(s)|}{k!} \mid k \in \{0, 1, \dots, \nu\} \right\} \geq \frac{|\varphi^{(j)}(s)|}{j!}$$

(où $\varphi^{(k)}(s)$ est la valeur au point s de la fonction dérivée d'ordre k de φ avec $\varphi^{(0)}(s) = \varphi(s)$).

D'après une inégalité due à W. K. Hayman (1973), on a :

$$\forall s \in \mathbf{C} : |\varphi(s)| \leq \max \left\{ \frac{|\varphi^{(k)}(0)|}{(\nu+1)^k} \mid k \in \{0, 1, \dots, \nu\} \right\} \exp [(\nu+1)|s|].$$

On considère une suite $(\lambda_n)_1^\infty$ de constantes complexes λ_n telle que $(|\lambda_n|)_1^\infty$ soit une D -suite ($\iff 0 < |\lambda_n| < |\lambda_{n+1}|$ et $\lim_{n \rightarrow \infty} |\lambda_n| = \infty$) et l'élément suivant

$$\{f\} : \sum_{n=1}^{\infty} f_n(s) \exp(-s \lambda_n)$$

où $(f_n)_1^\infty$ est une suite de fonctions entières f_n d'indices bornés ν_n sur $\mathbf{C}$. Par définition, $\{f\}$ est un élément E -dirichélien.

Dans la première section, utilisant une méthode due à M. Blambert-M. Berland (1975) et R. Parvatham (1978) dans le cas des séries à coefficients polynomiaux, on considère l'élément :

$$\{f_A\} : \sum_{n=1}^{\infty} A_n \exp [(\nu_n + 1)|s| - s \lambda_n]$$

avec

$$A_n = \max \left\{ \frac{|f_n^{(k)}(0)|}{k!} \mid k \in \{0, 1, \dots, \nu_n\} \right\}.$$

On pose

$$\begin{aligned} \forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} : d(n, s) &= \frac{-\log |A_n \exp[(\nu_n + 1)|s| - s\lambda_n]|}{|\lambda_n|} \\ d_*(s) &= \liminf_{n \rightarrow \infty} \{d(n, s)\} \\ \forall \alpha \in \mathbf{R} : D_{*,\alpha} &= \{s \in \mathbf{C} \mid d_*(s) > \alpha\} \\ \beta &= \limsup_{n \rightarrow \infty} \left\{ \frac{\nu_n + 1}{|\lambda_n|} \right\}. \end{aligned}$$

Théorème 1.1. *Sous les conditions suivantes :*

$$\beta < \infty, L = \limsup_{n \rightarrow \infty} \left\{ \frac{\log n}{|\lambda_n|} \right\} < \infty$$

alors $\{f\}$ converge absolument sur $D_{*,L}$ et uniformément sur tout compact inclus dans $D_{*,L}$.

Théorème 1.2. *Sous la condition $\beta < \infty$, alors $\{f_A\}$ diverge sur $\mathbf{C} \setminus \overline{D}_{*,0}$ (où $\overline{D}_{*,0}$ est l'adhérence de $D_{*,0}$).*

Dans la seconde section, on considère la suite $(f_n)_1^\infty$ de fonctions entières f_n n'ayant qu'un nombre fini $m_n (\neq 0)$ de zéros et d'indices bornés ν_n sur $\mathbf{C}$. D'après un théorème de F. Gross (1967), S. M. Shah (1968) on a :

$$\forall n \in \mathbf{N} \setminus \{0\}, \forall s \in \mathbf{C} : f_n(s) = P_n(s) \exp(\alpha_n s)$$

où P_n est un polynôme taylorien et α_n est une constante complexe.

L'auteur de cet article a démontré ce théorème (1983) par une méthode fondamentalement distincte de F. Gross, en utilisant une propriété due à Hadamard, sur le produit canonique. On a donc un élément E -dirichlétien à savoir :

$$\{f\} : \sum_{n=1}^{\infty} P_n(s) \exp[-s(\lambda_n - \alpha_n)].$$

On pose :

$$\forall n \in \mathbf{N} \setminus \{0\} : \mu_n = \lambda_n - \alpha_n$$

et on suppose que $(|\mu_n|)_1^\infty$ est une D -suite. Cet élément est donc un élément LC -dirichlétien. On donne, ici, quelques applications de certains théorèmes dus à M. Blambert-R. Parvatham (1983) liés aux prolongements des fonctions analytiques définies par les éléments LC -dirichlétiens, en utilisant une inégalité de ces auteurs plus simple que celle de M. M. Dzrbasjan (1973). On dit que $\mathcal{E}_n$ est l'ensemble des zéros de P_n et

$\mathcal{E} = \bigcup_n \mathcal{E}_n$. On pose $\mathcal{E}^* = \mathcal{E}^d \cup \mathcal{E}_\infty$ où $\mathcal{E}^d$ est le dérivé de $\mathcal{E}$, $\mathcal{E}_\infty = \{s \in \mathbf{C} \mid \exists (n_j)_1^\infty : P_{n_j}(s) = 0\}$. $\mathcal{E}^*$ est fermé de $\mathbf{C}$. On pose :

$$\begin{aligned} \forall s \in \mathbf{C} \setminus \mathcal{E}^* \neq \emptyset : \delta_*(s) &= \liminf_{n \rightarrow \infty} \left\{ \frac{-\log |P_n(s) \exp(-s\mu_n)|}{|\mu_n|} \right\} \\ \forall \alpha \in \mathbf{R} : \mathcal{D}_{*,\alpha} &= \{s \in \mathbf{C} \setminus \mathcal{E}^* \mid \delta_*(s) > \alpha\} \\ \beta^* &= \limsup_{n \rightarrow \infty} \left\{ \frac{m_n}{|\mu_n|} \right\}. \end{aligned}$$

Théorème 2.1. *Sous les conditions suivantes :*

$$m_n \sim \nu_n, \quad |\mu_n| \sim |\lambda_n|, \quad (\implies \beta^* =) \beta < \infty, \quad L < \infty,$$

alors $\{f\}$ converge absolument sur $\mathcal{D}_{*,L+\beta}$ et uniformément sur tout compact inclus dans $\mathcal{D}_{*,L+\beta}$.

Théorème 2.2. *Sous les conditions suivantes :*

$$m_n \sim \nu_n, \quad |\mu_n| \sim |\lambda_n|, \quad \exists (\theta_n)_1^\infty, \quad \theta_n > 0, \quad \lim_{n \rightarrow \infty} \{\theta_n\} = \infty$$

$$\exists n' \in \mathbf{N} \setminus \{0\}, \quad \forall n \geq n' : |\mu_{n+1}| > (1 + \theta_n)|\mu_n|,$$

alors chaque point de $\text{Fr}(\mathcal{D}_{*,0}) \cap (\mathbf{C} \setminus \mathcal{E}^*)$ est singulier pour la fonction définie par $\{f\}$.

On considère une fonction entière g satisfaisant aux conditions : g admet, pour chaque n de $\mathbf{N} \setminus \{0\}$, μ_n pour zéro à l'ordre $m_n + 1$, g est au plus de l'ordre borélien 1 et au plus du type fini T de leur ordre si cet ordre est égal à 1, et enfin, désignant par $\mathcal{C}$ l'image par rapport à l'axe imaginaire du diagramme indicateur de g , on a $\mathcal{C} \subset \mathcal{D}_{*,\beta^*}$. On pose :

$$\forall n \in \mathbf{N} \setminus \{0\} : M_n = \max \left\{ \frac{|\gamma_{n,j}|}{j!} \mid j \in \{0, 1, \dots, m_n\} \right\},$$

où $\gamma_{n,j}$ est la valeur au point μ_n de la fonction dérivée d'ordre j de la fonction :
 $s \mapsto \frac{(s - \mu_n)^{m_n+1}}{g(s)}$ qui est définie par continuité au point μ_n :

$$\gamma = \limsup_{n \rightarrow \infty} \left\{ \frac{\log M_n}{|\mu_n|} \right\}.$$

On sait que :

$$-T \leq \gamma \leq \infty.$$

Théorème 2.3. *Sous les conditions suivantes :*

$$m_n \sim \nu_n, \quad |\mu_n| \sim |\lambda_n|, \quad \beta < \infty, \quad \mathcal{D} < \infty, \quad \gamma < \infty,$$

alors il n'existe pas d'extension holomorphe (au sens de S. Mandelbrojt (1962)) au moyen de $\mathcal{C}$ de la fonction :

$$\mathcal{D}_{*,\beta} \ni s \longmapsto f(s) = \sum_{n=1}^{\infty} f_n(s) \exp(-s\lambda_n)$$

de $\{s_1\} + \mathcal{C}$ ($= \{s \in \mathbf{C} \mid s = s_1 + s', s' \in \mathcal{C}\}$) $\subset \mathcal{D}_{*,\beta}$, à $\{s_2\} + \mathcal{C}$ avec $s_2 \in (\mathbf{C} \setminus \mathcal{E}^*) \setminus \overline{\mathcal{D}}_{*,0}^\varepsilon$ où

$$\mathcal{D} = \limsup_{n \rightarrow \infty} \left\{ \frac{\sum_{i=1}^n (m_i + 1)}{|\mu_n|} \right\}$$

et

$$\forall \varepsilon > 0 : \mathcal{D}_{*,0}^\varepsilon = \left\{ s \in \mathbf{C} \setminus \mathcal{E}^* \mid \liminf_{n \rightarrow \infty} \left\{ \frac{-\log |P_n(s) \exp(-s\mu_n)|}{|\mu_n|^{1+\varepsilon}} \right\} > 0 \right\}.$$

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