

AN IDENTITY ON THRON'S DUAL CONTINUED FRACTIONS

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RÉSUMÉ. Nous construisons une famille de séries formelles en deux variables dont tout élément $f(y, t)$ admet un développement U en T-fraction, par rapport à la variable t , tel que la T-fraction duale U^* est un développement de $f(t, y^{-1})$, toujours par rapport à t . Ce résultat est établi grâce à une interprétation combinatoire du couple (U, U^*) . Nous donnons l'exemple de la série génératrice des polyominoes parallélogrammes, qui fait intervenir des q -analogues de fonctions de Bessel.

ABSTRACT. We construct a family of formal power series in two variables in which each series $f(y, t)$ has a T-fraction expansion U , with respect to the variable t , whose dual T-fraction U^* is an expansion of $f(t, y^{-1})$, also with respect to t . This is done by means of a combinatorial interpretation of the pair (U, U^*) . An example involving the generating function of staircase polyominoes and q -analogues of Bessel functions is given.

Throughout this paper, it is assumed that K is a field. We use the horizontal notation for continued fractions, where the expression $\frac{a}{b + \frac{c}{d + \frac{e}{f + \dots}}}$ denotes

$$\frac{a}{b + \frac{c}{d + \frac{e}{f + \dots}}}.$$

1. A combinatorial interpretation for T-fractions. A *Dyck path* of length $2n$ is a sequence $\omega = (s_0, \dots, s_{2n})$ of vertices in $\mathbb{N}^2$ such that $s_0 = (0, 0)$, $s_{2n} = (2n, 0)$ and such that the *steps* (s_i, s_{i+1}) , $0 \leq i \leq 2n - 1$, are raising or descending, as indicated on figure 1. The *length* of ω , denoted by $|\omega|$, is the number of its steps: if $\omega = (s_0, \dots, s_{2n})$, then $|\omega| = 2n$. The *level* of a step (s_i, s_{i+1}) is the y -coordinate of s_i , and the level of a Dyck path is the maximum level of its steps. A *peak* (*resp.* a *valley*) of a Dyck path is the last vertex of a raising (*resp.* descending) step and the first of a descending (*resp.* raising) one.

Reçu le 19 avril 1995 et, sous forme définitive, le 30 juillet 1996.

We denote by $\text{peak}(\omega)$ the number of peaks of ω , and by $\text{npeak}(\omega)$ the number of descending steps not adjacent to a peak. We have the obvious relation:

$$\text{peak}(\omega) + \text{npeak}(\omega) = \frac{|\omega|}{2}. \quad (1)$$

We say that a Dyck path is *prime* if the only vertices of level 0 are the first and the last ones; the single-vertex path $((0, 0))$ is not considered to be prime. A raising step (s_i, s_{i+1}) and a descending step (s_j, s_{j+1}) are *mates* if $i < j$ and if they can see each other on the same horizontal strip (see figure 1). Every raising step has exactly one descending mate.

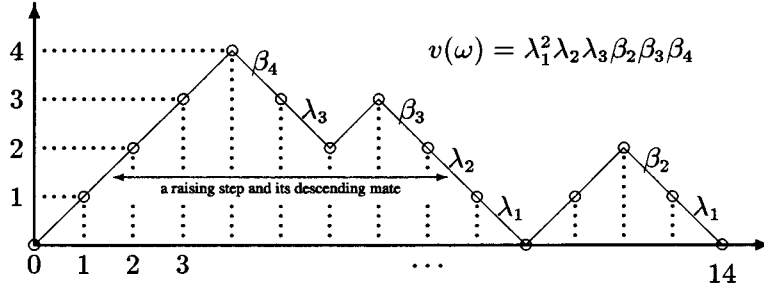


FIGURE 1. A Dyck path of level 4, length 14 and v -label $\lambda_1^2\lambda_2\lambda_3\beta_2\beta_3\beta_4$.

Let $\lambda = (\lambda_k)_{k \geq 0}$ and $\beta = (\beta_k)_{k \geq 1}$ be two sequences of elements of K . We define the v -label of a descending step (s_i, s_{i+1}) of level k by the following rules :

$$v((s_i, s_{i+1})) = \begin{cases} \beta_k & \text{if } s_i \text{ is a peak,} \\ \lambda_k & \text{otherwise.} \end{cases} \quad (2)$$

We define the v -label $v(\omega)$ of a Dyck path ω as the product of the labels of its descending steps. Labelled paths provide a combinatorial framework for interpreting various classes of continued fractions, as in [8, 14]. In our case, by labeling steps according to whether they are adjacent to a peak or not, we obtain an interpretation for the so-called Thron's continued fractions, or T-fractions for short, and their dual: see [9, 10, 16] for the connection with *two-point Padé approximants*.

From now on, we assume that $\lambda_k \neq \beta_k$ for all $k \geq 1$. Let γ and α be the sequences given by

$$\begin{aligned} \gamma_0 &= \frac{\lambda_0}{\lambda_1 - \beta_1}, \\ \gamma_k &= \frac{\lambda_k}{(\beta_k - \lambda_k)(\beta_{k+1} - \lambda_{k+1})} \quad \text{for } k \geq 1, \\ \alpha_k &= \frac{\beta_{k+1} - \lambda_{k+1} + \lambda_k}{(\beta_k - \lambda_k)(\beta_{k+1} - \lambda_{k+1})} \quad \text{for } k \geq 1. \end{aligned} \quad (3)$$

By definition, the T-fraction $U(\lambda, \beta; t)$ and its *dual* $U^*(\lambda, \beta; t)$ are the following continued fractions

$$U(\lambda, \beta; t) = \frac{\lambda_0}{1 - (\beta_1 - \lambda_1)t} - \frac{\lambda_1 t}{1 - (\beta_2 - \lambda_2)t} - \cdots - \frac{\lambda_{k-1} t}{1 - (\beta_k - \lambda_k)t} - \cdots, \quad (4)$$

$$U^*(\lambda, \beta; t) = \frac{\gamma_0 t}{1 - (\alpha_1 - \gamma_1)t} - \frac{\gamma_1 t}{1 - (\alpha_2 - \gamma_2)t} - \cdots - \frac{\gamma_{k-1} t}{1 - (\alpha_k - \gamma_k)t} - \cdots \quad (5)$$

In the sequel of this section, we first prove that each of these continued fractions *converges*, i.e. is a continued fraction expansion of a formal power series. Then we give a combinatorial interpretation of these series, and explain why U and U^* are said to be dual.

1.1. The series that corresponds to U . Let $D(\lambda, \beta; t)$ be the generating function

$$\begin{aligned} D(\lambda, \beta; t) &= \lambda_0 \sum v(\omega) t^{\frac{|\omega|}{2}}, \\ &= \lambda_0 (1 + \beta_1 t + [\beta_1^2 + \beta_2 \lambda_1] t^2 \\ &\quad + [\beta_1^3 + 2\beta_1 \beta_2 \lambda_1 + \beta_2^2 \lambda_1 + \beta_3 \lambda_2 \lambda_1] t^3 + \cdots), \end{aligned} \quad (6)$$

where the sum runs over all Dyck paths. For a non-negative integer k , we also define the generating function $D_k(\lambda, \beta; t) = \lambda_0 \sum v(\omega) t^{|\omega|/2}$ of Dyck paths of level $\leq k$. Let δ be the left shift on sequences: $\delta((a_n)_{n \geq n_0}) = (a_{n+1})_{n \geq n_0}$. We have the following partition of the set of Dyck paths:



Once carried on the corresponding generating functions, this partition is translated into the functional equation:

$$D(\lambda, \beta; t) = \lambda_0 + \beta_1 t D(\lambda, \beta; t) + \lambda_1 t \left(\frac{D(\delta\lambda, \delta\beta; t)}{\lambda_1} - 1 \right) D(\lambda, \beta; t).$$

Solving this equation with respect to $D(\lambda, \beta; t)$, and iterating the result k times leads to

$$\begin{aligned} D(\lambda, \beta; t) &= \frac{\lambda_0}{1 - (\beta_1 - \lambda_1)t - t D(\delta\lambda, \delta\beta; t)}, \\ &\vdots \\ &= \frac{\lambda_0}{1 - (\beta_1 - \lambda_1)t} - \frac{\lambda_1 t}{1 - (\beta_2 - \lambda_2)t} - \cdots \\ &\quad \cdots \frac{\lambda_k t}{1 - (\beta_{k+1} - \lambda_{k+1})t - t D(\delta^{k+1}\lambda, \delta^{k+1}\beta; t)}. \end{aligned} \quad (7)$$

On the one hand, $D_k(\lambda, \beta; t)$ is by definition obtained from $D(\lambda, \beta; t)$ by the substitutions $\lambda_n := 0$ and $\beta_n := 0$, for all $n > k$. Thanks to (7), we have subsequently

$$D_k(\lambda, \beta; t) = \frac{\lambda_0}{1 - (\beta_1 - \lambda_1)t} - \frac{\lambda_1 t}{1 - (\beta_2 - \lambda_2)t} - \cdots - \frac{\lambda_{k-1} t}{1 - \beta_k t}. \quad (8)$$

This formula shows that $D_k(\lambda, \beta; t)$ is the k -th convergent of the T-fraction (4).

On the other hand, the two series D and D_k are equal up to the coefficient of t^k , because Dyck paths of length $\leq 2k$ and level $\leq k$ form the same set as Dyck paths of length $\leq 2k$. In other words,

$$D(\lambda, \beta; t) - D_k(\lambda, \beta; t) = o(t^k) . \quad (9)$$

Therefore we have proved that $\lim D_k = D$ in $K[[t]]$ equipped with the usual distance, and that $U(\lambda, \beta; t)$ is a continued fraction expansion of the series $D(\lambda, \beta; t)$.

1.2. The series that corresponds to U^* . For a descending step (s_i, s_{i+1}) of level k , we define its v^* -label by

$$v^*((s_i, s_{i+1})) = \begin{cases} \alpha_{k-1} & \text{if } s_i \text{ is a peak and } k \geq 2, \\ \gamma_{k-1} & \text{otherwise.} \end{cases} \quad (10)$$

The label $v^*(\omega)$ of a Dyck path ω is the product of the v^* -labels of its descending steps. Let $D^*(\lambda, \beta; t)$ be the generating function

$$\begin{aligned} D^*(\lambda, \beta; t) &= \sum v^*(\omega) t^{\frac{|\omega|}{2}} , \\ &= \gamma_0 t (1 + \alpha_1 t + [\alpha_1^2 + \alpha_2 \gamma_1] t^2 \\ &\quad + [\alpha_1^3 + 2\alpha_1 \alpha_2 \gamma_1 + \alpha_2^2 \gamma_1 + \alpha_3 \gamma_2 \gamma_1] t^3 + \dots) , \end{aligned} \quad (11)$$

where the sum runs over *prime* Dyck paths. The same arguments as before prove that $U^*(\lambda, \beta; t)$ is a continued fraction expansion of $D^*(\lambda, \beta; t)$. Moreover, the generating function $D_k^*(\lambda, \beta; t)$ of prime Dyck paths of level $\leq k$, labelled by v^* , is the $(k-1)$ -th convergent of U^* :

$$D_k^*(\lambda, \beta; t) = \frac{\gamma_0 t}{1 - (\alpha_1 - \gamma_1)t -} \frac{\gamma_1 t}{1 - (\alpha_2 - \gamma_2)t -} \dots \frac{\gamma_{k-2} t}{1 - \alpha_{k-1} t} . \quad (12)$$

1.3. Taylor expansions and asymptotic expansions: the duality of T-fractions. We have carefully distinguished series (D or D^*) from T-fractions (U or U^*), because the correspondence is not one-to-one. To be precise, consider the following *inverse problem* : let f be a formal power series with $f(0) \neq 0$; does there exist a T-fraction $U(\lambda, \beta; t)$ that is an expansion of f ?

The answer is yes. Indeed, there exist infinitely many solutions to this inverse problem if $K \neq \mathbb{F}_2$ (integers modulo 2). (Recall that there exists at most one Jacobi or Stieltjes continued fraction, and exactly one P- or C-fraction, that is an expansion of a given formal power series: see [9].) In other words, we have the following result.

Lemma 1. *Let $f = \sum_{n \geq 0} c_n t^n$ be a formal power series with coefficients in K and a non-vanishing constant term. We assume that $K \neq \mathbb{F}_2$. Then there exist infinitely many pairs of sequences (λ, β) such that*

$$f(t) = D(\lambda, \beta; t) . \quad (13)$$

Proof. Equality (13) is equivalent to the following conditions on Taylor expansions: $f(t) - D_k(\lambda, \beta; t) = o(t^k)$ for all $k \geq 1$. Each of these conditions is equivalent to a (non-linear) triangular system, where the $2k$ unknowns are $\lambda_0, \dots, \lambda_{k-1}, \beta_1, \dots, \beta_k$ and the $k + 1$ equations are:

$$\begin{aligned} (E_0) \quad c_0 &= \lambda_0, \\ (E_1) \quad c_1 &= \beta_1 \lambda_0, \\ (E_n) \quad c_n &= \sum_{|\omega|=2n} v(\omega) = \beta_n \lambda_{n-1} \dots \lambda_1 \lambda_0 + \sum v(\omega), \text{ for } n = 2, \dots, k. \end{aligned} \quad (14)$$

The second sum above runs over Dyck paths ω of length $2n$ and level $\leq n-1$. Hence β_n and λ_{n-1} do not appear in its summand $v(\omega)$. From (E_0) and (E_1) , we have $\lambda_0 = c_0 \neq 0$ and $\beta_1 = c_1/c_0$. Then, for any choice of non-zero λ_i 's, it is possible to solve equations $(E_2), \dots, (E_n)$ successively, with respect to $\beta_2, \dots, \beta_k$. In other words, $k-1$ unknowns are undetermined in system (14). Since $K \neq \mathbb{F}_2$, there are infinitely many sequences $(\lambda_i)_{i \geq 1}$ in $(K \setminus \{0\})^{\mathbb{N}}$ and, therefore, infinitely many solutions to (13). $\square$

We now consider the asymptotic expansion of $D_k(\lambda, \beta; t)$. An easy induction shows that $D_k(\lambda, \beta; t)$ is the quotient of a polynomial of degree $k-1$ by a polynomial of degree k . It has therefore an asymptotic expansion beginning with $\frac{d_{1,k}}{t} + \frac{d_{2,k}}{t^2} + \dots$, for some constants $d_{1,k}, d_{2,k}, \dots \in K$, depending *a priori* on k . The importance of T-fractions lies in the following fact: these asymptotic expansions converge.

More precisely, as proved independently by Jones and Thron [9] and McCabe and Murphy [11], there exists a formal power series $g \in K[[t]]$ with $g(0) = 0$ such that

$$g(t) - D_k(\lambda, \beta; t^{-1}) = o(t^{k-1}). \quad (15)$$

In our combinatorial framework, this fact can be proved as follows. We compute $D_k(\lambda, \beta; t^{-1})$ from identity (8) to obtain

$$D_k(\lambda, \beta; t^{-1}) = \frac{\gamma_0 t}{1 - (\alpha_1 - \gamma_1)t} - \frac{\gamma_1 t}{1 - (\alpha_2 - \gamma_2)t} - \dots - \frac{\gamma_{k-2} t}{1 - \frac{t}{\beta_{k-1}}}.$$

Comparing this formula with equation (12) leads to $D_k^*(\lambda, \beta; t) - D_k(\lambda, \beta; t^{-1}) = o(t^{k-1})$. Therefore, equation (15) holds when g is the generating function $D^*(\lambda, \beta; t)$ of prime Dyck paths labelled by v^* : see formula (11). Remember that U^* is a continued fraction expansion of this series.

We may now consider a second inverse problem: let f and g two formal power series. Does there exist a pair (λ, β) of sequences such that $f(t) = D(\lambda, \beta; t)$ and $g(t) = D^*(\lambda, \beta; t)$? In other words, does there exist a T-fraction $U(\lambda, \beta; t)$ that is an expansion of $f(t)$, and whose dual $U^*(\lambda, \beta; t)$ is an expansion of $g(t)$? Such a T-fraction exists and is unique, for all f and g satisfying some conditions on Hankel determinants: see [10]. Roughly speaking, we may say that, if the $2k$ unknowns of system (14) are also submitted to condition (15), then the solution is unique.

2. A property of duality for a class of formal power series. Let y be a new variable. We assume in this section that $K = L(y)$, where L is a field, and that $\beta_i \in L$ for $i \geq 1$. Let $D(y, \beta; t)$ be the generating series of labelled Dyck paths, given by (6), where the

sequence λ is constant: $\lambda_k = y$ for all $k \geq 0$. In this case, the label of any Dyck path is the product of a power of y by a monomial in the β 's, denoted by $w(\omega)$, and we may write:

$$D(y, \beta; t) = \sum_{\omega} y^{1+n_{\text{peak}}(\omega)} w(\omega) t^{\frac{|\omega|}{2}}, \quad (16)$$

$$= y(1 + \beta_1 t + [\beta_1^2 + y\beta_2]t^2 + [\beta_1^3 + 2y\beta_1\beta_2 + y^2\beta_3]t^3 + \dots). \quad (17)$$

The dual labels (3) take the form:

$$\begin{aligned} \gamma_0 &= \frac{1}{1 - y^{-1}\beta_1}, \\ \gamma_k &= \frac{y^{-1}}{(1 - y^{-1}\beta_k)(1 - y^{-1}\beta_{k+1})} \quad \text{for } k \geq 1, \\ \alpha_k &= \frac{y^{-2}\beta_{k+1}}{(1 - y^{-1}\beta_k)(1 - y^{-1}\beta_{k+1})} \quad \text{for } k \geq 1. \end{aligned} \quad (18)$$

As explained above, the series $D(y, \beta; t)$ has infinitely many T-fraction expansions in the variable t . We already know one of them, $U(y, \beta; t)$, and we know that the dual T-fraction $U^*(y, \beta; t)$ is an expansion of the generating function $D^*(y, \beta; t)$. Keeping in mind the above remark, we do not expect $D^*(y, \beta; t)$ to be related to $D(y, \beta; t)$ in a simple way. Here is our main result.

Theorem 2. *The generating function $D(y, \beta; t)$ given by (16) has the duality property*

$$D(y, \beta; t) = U(y, \beta; t), \quad (19)$$

$$D(t, \beta; y^{-1}) = U^*(y, \beta; t), \quad (20)$$

where U and U^* are the following dual T-fractions:

$$U(y, \beta; t) = \frac{y}{1 - (\beta_1 - y)t} - \frac{yt}{1 - (\beta_2 - y)t} - \dots - \frac{yt}{1 - (\beta_k - y)t} - \dots \quad (21)$$

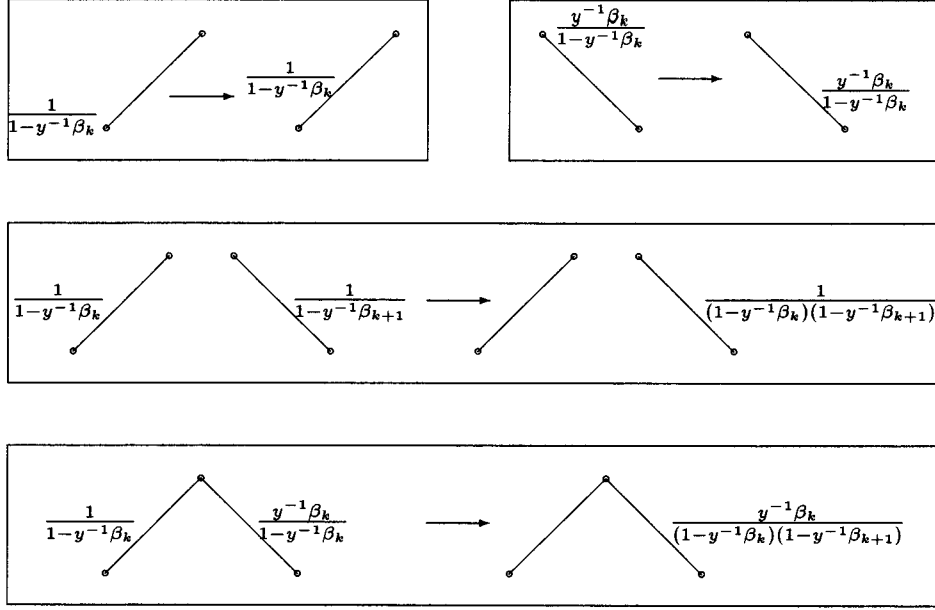
$$U^*(y, \beta; t) = \frac{\frac{t}{1 - y^{-1}\beta_1}}{1 + \frac{y^{-1}t}{1 - y^{-1}\beta_1} -} - \frac{\frac{y^{-1}t}{(1 - y^{-1}\beta_1)(1 - y^{-1}\beta_2)}}{1 + \frac{y^{-1}t}{1 - y^{-1}\beta_2} -} - \dots - \frac{\frac{y^{-1}t}{(1 - y^{-1}\beta_k)(1 - y^{-1}\beta_{k+1})}}{1 + \frac{y^{-1}t}{1 - y^{-1}\beta_{k+1}} -} - \dots \quad (22)$$

Since we already know that expansion (19) holds, and that $U^*(y, \beta; t)$ is an expansion of $D^*(y, \beta; t)$, we need only to focus on the proof of

$$D(t, \beta; y^{-1}) = D^*(y, \beta; t). \quad (23)$$

This will be done in two steps. First, we describe a bijection ϕ between Dyck paths and some set $\mathcal{M}$ of *Dyck multichains* (see [7]). Second, we explain how the labelling corresponding to the left-hand side of (23), is changed through ϕ into a labelling of the underlying paths of the multichains, which will happen to be the dual labelling v^* .

2.1. A bijection between paths and multichains. We order the vertices of any Dyck path by their x -coordinate. A *multichain* of the Dyck path $\eta = (s_0, \dots, s_n)$ is a weakly increasing non-empty sequence $M = (s_{i_1}, s_{i_2}, \dots, s_{i_p})$ of vertices of η . We say that M

FIGURE 3. The turning of h' into v^* .

(1) *We have*

$$\frac{|\eta|}{2} = n\text{peak}(\omega) + 1, \quad (24)$$

$$p = \text{peak}(\omega), \quad (25)$$

where $\text{peak}(\omega)$ and $n\text{peak}(\omega)$ are respectively the number of peaks and the number of descending steps of ω that are not adjacent to a peak;

(2) *The level of the ℓ^{th} peak of ω is the level of s_{i_ℓ} , for $1 \leq \ell \leq p$.*

2.2. Proof of theorem 2. For a vertex $s \in \mathbb{N}^2$ of level k , let $h(s)$ be the product $y^{-1}\beta_k$. When $M = (s_{i_1}, s_{i_2}, \dots, s_{i_p})$ is a Dyck multichain, we define its weight $h(M)$ by

$$h(M) = \prod_{j=i_1}^{i_p} h(s_j).$$

Remind that the valuation $v(\omega)$ of a Dyck path is the product of $y^{1+n\text{peak}(\omega)}$ by a monomial in the variables β_i , denoted by $w(\omega)$. If $M = \phi(\omega) \in \mathcal{M}$, we have

$$h(M) = y^{-\text{peak}(\omega)} w(\omega), \quad (26)$$

thanks to equality (25) and property 2 of proposition 4.

Lemma 5. *If η is a prime Dyck path, we have*

$$\sum_{\substack{M \in \mathcal{M}, \\ \eta \text{ is the underlying path of } M}} h(M) = \begin{cases} y^{\frac{|\eta|}{2}-1} v^*(\eta) & \text{if } |\eta| \geq 4, \\ y^{\frac{|\eta|}{2}-1} (v^*(\eta) - 1) & \text{if } |\eta| = 2. \end{cases} \quad (27)$$

where v^* is the labelling defined in subsection 1.2.

Proof. It is a variation on the identity $(1 - u)^{-1} = \sum_{n=0}^{\infty} u^n$. Since a multichain in $\mathcal{M}$ of underlying prime Dyck path η may contain any vertex of level ≥ 1 and must contain all the peaks of η , the left-hand side of (27) is equal to $\prod_{s \text{ vertex of } \eta} h'(s)$, where

$$h'(s) = \begin{cases} 1 & \text{if the level of } s \text{ is } 0, \\ \frac{y^{-1}\beta_k}{1-y^{-1}\beta_k} & \text{if } s \text{ is a peak of level } k, \\ \frac{1}{1-y^{-1}\beta_k} & \text{if } s \text{ is a vertex of level } k \geq 1 \text{ that is not a peak.} \end{cases}$$

We now turn this h' -labelling of the vertices into a labelling of the descending steps by the following rules, illustrated on figure 3:

- we give to any step (raising or descending) the h' -labelling of its first vertex;
- we move the labelling of any raising step to its descending mate.

By comparing the result of these rules with γ_k and α_k given by (18), one is lead to the alternative (27). $\square$

We interpret now the left-hand side of (23) through bijection ϕ , and we gather multichains according to their underlying path. Let η_2 be the only Dyck path of length 2 (it is a prime path). We obtain the following sequence of equalities, by (16), (26), (1), (24) and (27):

$$\begin{aligned} D(t, \beta; y^{-1}) &= t + \sum_{\omega \text{ non-empty Dyck path}} t^{1+n\text{peak}(\omega)} w(\omega) y^{-\frac{|\omega|}{2}}, \\ &= t + \sum_{\omega \text{ non-empty Dyck path}} t^{1+n\text{peak}(\omega)} h(\phi(\omega)) y^{\text{peak}(\omega) - \frac{|\omega|}{2}}, \\ &= t + \sum_{\omega \text{ non-empty Dyck path}} t^{1+n\text{peak}(\omega)} h(\phi(\omega)) y^{-n\text{peak}(\omega)}, \\ &= t + \sum_{\eta \text{ prime Dyck path}} t^{\frac{|\eta|}{2}} \left(\sum_{\substack{M \in \mathcal{M}, \\ \eta \text{ is the underlying path of } M}} h(M) y^{1 - \frac{|\eta|}{2}} \right), \\ &= t + [v^*(\eta_2) - 1] t + \sum_{\eta \text{ prime Dyck path, } |\eta| \geq 4} v^*(\eta) t^{\frac{|\eta|}{2}}, \\ &= D^*(y, \beta; t). \end{aligned}$$

This completes the proof of theorem 2.

2.3. The coefficient of y^n in $D(y, \beta; t)$ is a rational function. The series $D(y, \beta; t)$ given by (16) may be viewed as an element of $\mathbb{N}[y, \beta][[t]]$. The coefficient of y^n is therefore a formal power series in the variables $t, \beta_1, \beta_2, \dots$. For example, the coefficient of y is equal to $1 + \beta_1 t + \beta_1^2 t^2 + \beta_1^3 t^3 + \dots = \frac{1}{1 - \beta_1 t}$, which is a rational function. As an application of theorem 2, we consider now the general case.

Corollary 6. *Let $R_n(t, \beta)$ be the coefficient of y^n in $D(y, \beta; t)$. This formal power series is a rational function of the form*

$$R_n(t, \beta) = t^{n-1} \frac{P_n(t\beta_1, \dots, t\beta_n)}{\prod_{i=1}^n (1 - \beta_i t)^{n-i+1}}, \quad (28)$$

where P_n is a polynomial in n variables.

Proof. The formula for R_1 is already known to be true, with $P_1 = 1$. We deduce from identity (23) that $R_n(t, \beta)$ is also the coefficient of t^n in the series $D^*(y, \beta; t)$, after substitution of y^{-1} by t . In other words, using a specialization of (11), we have, for $n \geq 2$,

$$R_n(t, \beta) = \sum v^*(\omega) \Big|_{y^{-1} \rightarrow t}, \quad (29)$$

where the sum runs over the prime Dyck paths of length $2n$.

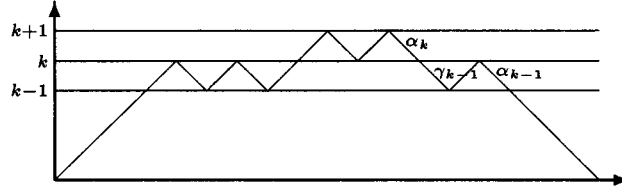


FIGURE 4. The contribution of $(1 - t\beta_k)^{n-k+1}$ for $2 \leq k \leq n-1$.

Fix an integer k with $2 \leq k \leq n-1$. Identities (18) show that the greatest power of $1 - y^{-1}\beta_k$ that appears in the denominator of $v^*(\omega)$ is $n - k + 1$, and it is achieved for prime Dyck paths of length $2n$ having all their peaks of level k or $k+1$ (remember that, for a descending step (s_i, s_{i+1}) of level k , the labelling $v^*(s_i, s_{i+1})$ is α_{k-1} or γ_{k-1} , not α_k or γ_k). One of those paths is drawn on figure 4.

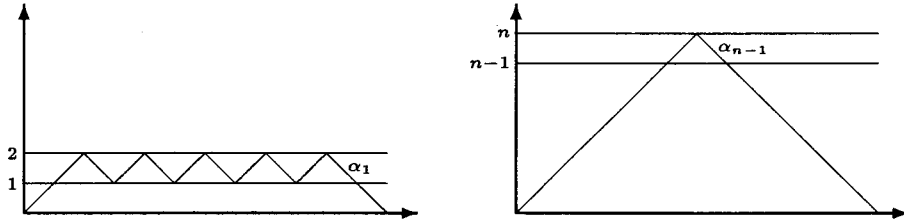


FIGURE 5. The contribution of $(1 - t\beta_1)^n$ and $(1 - t\beta_n)$.

If $k = 1$ (*resp.* $k = n$), the greatest power of $1 - y^{-1}\beta_k$ that appears in the denominator of $v^*(\omega)$ is n (*resp.* 1), and is achieved only for the path of figure 5, left (*resp.* right). This discussion proves that $\prod_{i=1}^n (1 - \beta_i t)^{n-i+1}$ is the common denominator of the sum (29), no attention being paid to possible simplifications. Since y^{-1} divides any α_k or γ_k but γ_0 , we may factor the numerator of R_n into the product of t^{n-1} by some sum of products of $(1 - t\beta_i)$ -like terms. $\square$

Here are the first values of P_n .

n	P_n
1	1
2	X_2
3	$X_3(1 - X_1) + X_2^2(1 - X_3)$
4	$X_4(1 - X_1)^2(1 - X_2) + X_3^2(1 - X_1)^2(1 - X_4)$ $+ 2X_2X_3(1 - X_1)(1 - X_3)(1 - X_4) + X_2^3(1 - X_3)^2(1 - X_4)$

3. Polyominoes, duality of T-fractions and q -analogues of Bessel functions. We illustrate theorem 2 with an example involving q -analogues of Bessel functions and generating functions of *polyominoes*. A polyomino is finite union of elementary cells of the square lattice, whose interior is a connected set.

A polyomino is *convex* if its intersection with any vertical and horizontal line is connected. It is *directed* if any cell may be reached from a special cell, called the *source*, by a path using only eastward and northward steps and remaining all the time within the polyomino. Polyominoes will be considered *up to a translation* of the square lattice, so that their enumeration makes sense. For a survey on their enumeration and related problems, see Viennot [15].

We shall be concerned here with polyominoes that are both convex and directed, and with staircase polyominoes, which form a subset of convex and directed ones. Figure 6 shows an element of each class.

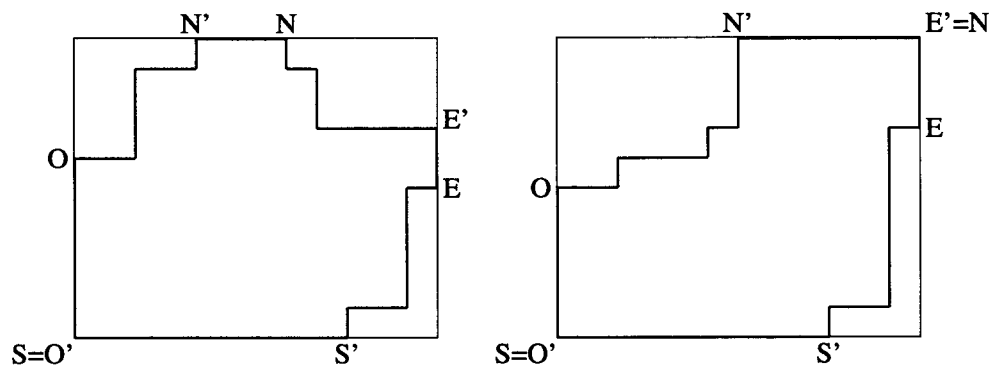


FIGURE 6. A convex and directed polyomino and a staircase polyomino.

Let P be a convex polyomino. We denote by $r(P)$ (*resp.* $c(P)$, $a(P)$ and $p(P)$) the number of its rows, (*resp.* the number of its columns, its cells and its perimeter). For any polyomino, $p(P)$ is an even integer ≥ 4 . We define the generating functions

$$\mathcal{P}(x, y, q, t) = \sum x^{c(P)} y^{r(P)} q^{a(P)} t^{\frac{p(P)-2}{2}}, \quad (30)$$

$$\mathcal{Q}(x, y, q, t) = \sum x^{c(P)} y^{r(P)} q^{a(P)} t^{\frac{p(P)-2}{2}}, \quad (31)$$

where the first (*resp.* second) sum runs over the staircase (*resp.* convex and directed) polyominoes up to a translation. Here are the first terms of these series:

$$\mathcal{P}(x, y, q, t) = xyqt + [(x^2y + xy^2)q^2]t^2 + [(x^3y + 2x^2y^2 + xy^3)q^3 + x^2y^2q^4]t^3 + \dots,$$

$$\mathcal{Q}(x, y, q, t) = xyqt + [(x^2y + xy^2)q^2]t^2 + [(x^3y + 3x^2y^2 + xy^3)q^3 + x^2y^2q^4]t^3 + \dots.$$

These generating functions are viewed as elements of $\mathbb{Q}(x, y, q)[[t]]$. In other words, for this section, K is the field of rational functions in variables x, y, q with rational coefficients.

3.1. Staircase polyominoes. The following bijection was given by Delest and Viennot [6], and is suitable for a T-fraction expansion of the series $y + \mathcal{P}(x, y, q, t)$. Let P

be a staircase polyomino with c columns and perimeter $2n + 2$. We denote by a_i for $1 \leq i \leq c$ (resp. by b_i for $1 \leq i \leq c - 1$) the number of cells in the i^{th} column (resp. the number of adjacent cells between column i and column $i + 1$). Then we specify one and only one Dyck path $\omega = \theta(P)$ of length $2n$ by the following conditions : $\theta(P)$ has c peaks of heights $a_1, \dots, a_c$ and the heights of its valleys are $b_1 - 1, \dots, b_{c-1} - 1$.

Proposition 7 (Delest, Viennot). *The mapping θ described above is a bijection between the set of staircase polyominoes of perimeter $2n + 2$ and the set of Dyck paths of length $2n$. Moreover, if $\omega = \theta(P)$, we have*

- *the number $c(P)$ of columns of P is $\text{peak}(\omega)$;*
- *for $1 \leq \ell \leq c(P)$, the number of cells of the ℓ^{th} column of P is the level of the ℓ^{th} peak of ω ;*
- *the number $r(P)$ of rows of P is $n_{\text{peak}}(\omega) + 1$.*

The series $y + \mathcal{P}(x, y, q, t)$ is therefore the generating function $D(\lambda, \beta; t)$ where the labels are given by

$$\lambda_k = y \quad \text{for } k \geq 0, \quad (32)$$

$$\beta_k = xq^k \quad \text{for } k \geq 1. \quad (33)$$

and we have the following consequence of theorem 2.

Theorem 8. *The generating function of staircase polyominoes $\mathcal{P}^*(x, y, q, t) = y + \mathcal{P}(x, y, q, t)$, counted with respect to their width (by x), height (by y), area (by q), and perimeter (by t), satisfies the duality property*

$$\begin{aligned} \mathcal{P}^*(x, y, q, t) &= U(t), \\ \mathcal{P}^*(x, t, q, y^{-1}) &= U^*(t), \end{aligned}$$

where $U(t)$ and $U^*(t)$ are the following dual T-fractions:

$$\begin{aligned} U(t) &= \frac{y}{1 + (y - xq)t -} \frac{yt}{1 + (y - xq^2)t -} \cdots \frac{yt}{1 + (y - xq^k)t -} \cdots, \\ U^*(t) &= \frac{\frac{t}{1 - xqy^{-1}}}{1 + \frac{y^{-1}t}{1 - xqy^{-1}} -} \frac{\frac{y^{-1}t}{(1 - xqy^{-1})(1 - xq^2y^{-1})}}{1 + \frac{y^{-1}t}{1 - xq^2y^{-1}} -} \cdots \frac{\frac{y^{-1}t}{(1 - xq^ky^{-1})(1 - xq^{k+1}y^{-1})}}{1 + \frac{y^{-1}t}{1 - xq^{k+1}y^{-1}} -} \cdots. \end{aligned}$$

Let ν be a complex number with $\Re(\nu) > 0$. The Bessel function of index ν is given by

$$J_\nu(x) = \sum_{n \geq 0} (-1)^n \frac{\left(\frac{x}{2}\right)^{2n+\nu}}{\Gamma(n+1)\Gamma(n+\nu+1)}. \quad (34)$$

We denote by $[n]_q! = (1+q) \cdots (1+q+\cdots+q^{n-1})$ the usual q -analogue of $n!$. When ν is an integer, we denote by $\mathcal{J}_\nu(x)$ the following q -analogue of $J_\nu(2x)$:

$$\mathcal{J}_\nu(x) = \sum_{n \geq 0} (-1)^n \frac{x^{n+\nu} q^{\binom{n+\nu}{2}}}{[n]![n+\nu]!}. \quad (35)$$

Bessel functions appear in theorem 8 in the following way: for two variables a and q , let $(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i)$ and define

$$H(x, y, q) = \sum_{n \geq 0} (-1)^n \frac{x^n q^{\binom{n+1}{2}}}{(q; q)_n (yq; q)_n} . \quad (36)$$

Using an inversion theorem in a monoid of heaps, Bousquet-Mélou and Viennot [3] have proved that $\mathcal{P}^\bullet(x, y, q, t) = y \frac{H(xqt, yt, q)}{H(xt, yt, q)}$ and obtained subsequently (see also [5]):

$$\mathcal{P}(x, 1, q, 1) = (1 - q) \frac{\mathcal{J}_1\left(\frac{xq}{(1-q)^2}\right)}{\mathcal{J}_0\left(\frac{xq}{(1-q)^2}\right)} .$$

Remark. Theorem 8 may be viewed as a T-fraction duality property for the q -analogue of the Bessel function given by (36). But we cannot formulate this theorem directly for the usual Bessel functions (34), nor for their q -analogues (35). We actually need the auxiliary variable y , which is involved in the denominator of the summand in (36).

3.2. Convex and directed polyominoes. The works by Chang and Lin [4] and by Bousquet-Mélou [1] prove that

$$\begin{aligned} \mathcal{P}(x, y, 1, t) &= \frac{1 - xt - yt - \sqrt{\Delta}}{2t} , \\ \mathcal{Q}(x, y, 1, t) &= \frac{xyt}{\sqrt{\Delta}} , \end{aligned}$$

where $\Delta = 1 - 2xt - 2yt - 2xyt^2 + x^2t^2 + y^2t^2$. The formula

$$\mathcal{Q}(x, y, 1, q) = \frac{xyt}{1 - xt - yt - 2t\mathcal{P}(x, y, 1, t)}$$

is a straightforward consequence, and may be written as follows:

$$\begin{aligned} y^{-1}\mathcal{Q}(x, y, 1, t) &= \frac{xt}{1 - (x - y)t - 2t\mathcal{P}^\bullet(x, y, 1, t)} , \\ t^{-1}\mathcal{Q}(x, t, 1, y^{-1}) &= \frac{x}{y - x + t - 2\mathcal{P}^\bullet(x, t, 1, y^{-1})} . \end{aligned}$$

We deduce from Theorem 8 the following corollary.

Corollary 9. *The generating function of convex and directed polyominoes given by $\mathcal{Q}^\bullet(x, y, 1, t) = y^{-1}\mathcal{Q}(x, y, 1, t)$, counted with respect to their width (by x), height (by y) and perimeter (by t) satisfies the duality property*

$$\begin{aligned} \mathcal{Q}^\bullet(x, y, 1, t) &= V(t) , \\ \mathcal{Q}^\bullet(x, t, 1, y^{-1}) &= V^*(t) , \end{aligned}$$

where $V(t)$ and $V^*(t)$ are the following dual T-fractions:

$$V(t) = \frac{xyt}{1 + (y-x)t-} \frac{2yt}{1 + (y-x)t-} \cdots \frac{yt}{1 + (y-x)t-} \cdots ,$$

$$V^*(t) = \frac{\frac{y^{-1}x}{(1-xy^{-1})}}{1 + \frac{y^{-1}t}{(1-xy^{-1})}-} \frac{\frac{2y^{-1}t}{(1-xy^{-1})^2}}{1 + \frac{y^{-1}t}{(1-xy^{-1})}-} \cdots \frac{\frac{y^{-1}t}{(1-xy^{-1})^2}}{1 + \frac{y^{-1}t}{(1-xy^{-1})}-} \cdots .$$

Remark. We do not know how to prove the identity $\mathcal{Q}^*(x, y, 1, t) = V(t)$ of corollary 9 in a bijective way, as we proved identity the $\mathcal{P}^*(x, y, q, t) = U(t)$ of theorem 8. A recursive bijection given by Bousquet-Mélou [2] between directed convex polyominoes and bilateral Dyck paths does not seem to provide a short explanation and leaves few hope to deal with the full series $\mathcal{Q}^*(x, y, q, t)$.

Acknowledgments. The second author thanks Mireille Bousquet-Mélou for her inspiring conversation and her careful reading of a preliminary version of this article. This work was partially supported by EC grant CHRX-CT93-0400 and the PRC Mathématique et Informatique.

Résumé substantiel en français. Philippe Flajolet [8] a expliqué comment interpréter les fractions continues de Jacobi (*resp.* de Stieltjes) comme développement de séries génératrices de *chemins de Motzkin* (*resp.* de *Dyck*) *valués*, qui sont des chemins dans le quart de plan $\mathbb{N}^2$. Les valuations ou étiquettes des pas de ces chemins ne dépendent que de leur *niveau*, c'est-à-dire de l'ordonnée de leur premier sommet.

En attribuant à un pas descendant d'un chemin de Dyck des valuations distinctes — notées β_k ou λ_k — suivant que le pas est ou non adjacent à un pic (voir la figure 1), nous obtenons une interprétation combinatoire des *fractions continues de Thron*, ou T-fractions. Plus précisément, nous définissons la valuation $v(\omega)$ d'un chemin de Dyck ω comme le produit des valuations de ses pas descendants. La série génératrice $D(\lambda, \beta; t) = \sum v(\omega)t^{|\omega|/2}$ des chemins de Dyck ainsi valués admet le développement

$$U(\lambda, \beta; t) = \frac{\lambda_0}{1 - (\beta_1 - \lambda_1)t-} \frac{\lambda_1 t}{1 - (\beta_2 - \lambda_2)t-} \cdots \frac{\lambda_{k-1} t}{1 - (\beta_k - \lambda_k)t-} \cdots .$$

Contrairement aux cas des fractions continues de Jacobi ou de Stieltjes, une série formelle f à coefficients dans un corps commutatif K n'admet pas *au plus* un développement en T-fraction, mais *une infinité* (à l'exception du cas où $K = \mathbb{F}_2$). En d'autres termes, le problème inverse consistant à représenter une série formelle par une T-fraction est indéterminé.

L'étude de cette question a conduit McCabe et Murphy [11] et Jones et Thron [9] à la notion de paires de T-fractions *duales*. La duale de la T-fraction ci-dessus, notée U^* , est définie par

$$U^*(\lambda, \beta; t) = \frac{\gamma_0 t}{1 - (\alpha_1 - \gamma_1)t-} \frac{\gamma_1 t}{1 - (\alpha_2 - \gamma_2)t-} \cdots \frac{\gamma_{k-1} t}{1 - (\alpha_k - \gamma_k)t-} \cdots ,$$

où les suites α et γ sont données par les formules (3).

En tant que quotient d'un polynôme de degré $k - 1$ par un polynôme de degré k , la réduite d'ordre k de U admet un développement asymptotique de la forme

$$\frac{d_{1,k}}{t} + \frac{d_{2,k}}{t^2} + \dots,$$

pour des éléments $d_{1,k}, d_{2,k}, \dots$ qui dépendent *a priori* de k . Le couple de T-fractions $(U(\lambda, \beta; t), U^*(\lambda, \beta; t))$ possède les propriétés suivantes:

- (1) les développements asymptotiques des réduites de $U(\lambda, \beta; t)$ convergent formellement : il existe une suite $(d_n)_{n \geq 1}$ telle que pour tout $n \geq 1$, on a $d_{n,k} = d_n$ si $k > n$;
- (2) la série $\sum_{n \geq 1} d_n t^n$ admet $U^*(\lambda, \beta; t)$ comme développement en T-fraction.

La section 1 est consacrée à une présentation combinatoire de ces résultats. Nous définissons en particulier une série génératrice $D^*(\lambda, \beta; t)$ de chemins de Dyck dits *premiers*, valués par la valuation v^* définie par la formule (10) et la ligne suivante.

Le second problème inverse, consistant à représenter un couple de séries formelles (f, g) par un couple de T-fraction duales $(U(\lambda, \beta; t), U^*(\lambda, \beta; t))$, admet une et une seule solution, en dehors d'une hypersurface de $K[[t]]^2$ donnée par l'annulation de certains déterminants de Hankel [10]. Ceci montre que la série g n'est pas une fonction de la série f , lorsque f admet le développement en T-fraction U et g admet le développement dual U^* .

L'interprétation combinatoire que nous donnons, tant de U que de U^* , nous permet de démontrer une propriété de dualité pour une famille de séries formelles. Supposons que la suite $(\lambda_k)_{k \geq 0}$ soit constante et égale à une indéterminée y . Nous savons alors que, pour toute suite β , la T-fraction $U(y, \beta; t)$ est un développement de la série génératrice $D(y, \beta; t)$, et que la T-fraction duale $U^*(y, \beta; t)$ est un développement de la série génératrice $D^*(y, \beta; t)$.

Notre résultat principal (théorème 2) établit que $U^*(y, \beta; t)$ est liée à la série D elle-même par l'égalité

$$D(t, \beta; y^{-1}) = D^*(y, \beta; t).$$

On passe du membre de gauche au membre de droite de cette égalité au moyen d'une bijection ϕ entre les chemins de Dyck et certaines *multichaînes* portées par les chemins de Dyck premiers. On transporte à travers ϕ la valuation v des chemins de Dyck pour obtenir une valuation sur les multichaînes, et cette dernière est ensuite transformée en une valuation sur les chemins de Dyck premiers sous-jacents : c'est l'objet de la section 2. Nous en déduisons le corollaire 9 sur la forme du coefficient de y^n dans $D(y, \beta; t)$.

En section 3, nous présentons une application de ce résultat à une série génératrice de polyominos parallélogrammes (voir figure 6, à droite), comptés suivant leur hauteur, par la variable y , et leur périmètre par la variable t (ainsi que suivant leur largeur, par x , et leur aire, par q). C'est le théorème 8. Nous établissons le lien avec certains q -analogues de fonctions de Bessel, et donnons, comme conséquence de résultats de Chang et Lin [4], une propriété de dualité pour une autre série génératrice : celle des polyominos convexes dirigés.

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