

MEASURABLE TRANSFORMATIONS ON PROJECTIVE LIMITS OF TOPOLOGICAL SPACES

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RÉSUMÉ. Soit M_1, M_2 deux ensembles mesurés σ -finis. Nous démontrons un résultat très général concernant les homomorphismes de σ -algèbres entre M_1 et M_2 qui sont induits par une transformation ponctuelle mesurable. Nous démontrons également que si M_1 et M_2 sont des limites projectives d'espaces de Souslin, alors sous certaines conditions, tout isomorphisme de σ -algèbres entre M_1 et M_2 est induit par une transformation ponctuelle mesurable et *invertible*. Nous montrons que ce dernier résultat contient et améliore les plus importants théorèmes obtenus dans le domaine des isomorphismes de σ -algèbres induits par des transformations ponctuelles mesurables.

ABSTRACT. Let M_1, M_2 be two finite (or even σ -finite) measure spaces. We prove a very general theorem concerning set homomorphisms from M_1 to M_2 which can be induced by a point homomorphism. We also prove that when M_1, M_2 are two projective limits of Souslin spaces, then under certain general conditions, any *invertible* set homomorphism can be induced by an *invertible* point homomorphism. We then show that this last result contains and even improves all of the most important theorems obtained on this subject.

1. Introduction. A problem of interest in measure theory is whether a given (measure algebra) set homomorphism φ on a σ -finite measure space $M = (X, \mathcal{B}, \mu)$ can be induced by a (completion) point homomorphism $T : X \rightarrow X$ in the sense that $T^{-1}E = \varphi E$ a.e. for all $E \in \mathcal{B}$. An even more interesting problem is whether T can be chosen to be invertible whenever φ is. It is well known that the second problem cannot always be solved (see [18], [2, p. 97], and [3, Theorem 0]) and this is so even though $\mathcal{B}$ has a separating sequence of generators (see [7, Lemma 12]).

One of the first mathematician to be interested in our second problem was von Neumann. In [20], he proved that any set automorphism φ of the finite measure algebra of a Polish space with its Baire (=Borel) σ -algebra is induced by a point automorphism. This result was generalized by Dorothy Maharam [15] to an arbitrary product of such measure algebras. Inspired by the techniques of Maharam used in her paper, Choksi [2] improved this result by showing that the conclusion remains true if the measure on the product of the measurable spaces $(X_\alpha, \mathcal{B}_\alpha)$ (where X_α is Polish and $\mathcal{B}_\alpha$ = the Borel

Reçu le 13 avril 1995 et, sous forme définitive, le 17 juin 1996.

σ -algebra on X_α) is *any* finite (or even σ -finite) Baire measure and not just a product measure. In [3], it was shown that any (Baire) set automorphism of a compact group is induced by a (Baire) point automorphism. To prove his result, Choksi adapted the techniques used in [2] to projective limits which constitute a generalization of products. In [4], the same author proved the same kind of result for an arbitrary locally compact homogeneous space acted upon by a locally compact σ -compact group.

One of our main objectives in this paper is to present a synthesis of all the results discussed in the previous paragraph. This is done by proving a theorem which is stated in a very general setting: Projective limits of Souslin spaces. (We should note that our interest in Souslin spaces is motivated by the important role they play in the theory of modern integration.) Not only does our theorem contain the well-known historical results but it improves each of them as well. For example, we generalize the main theorems in [2, 15, 20], by proving that whenever 1) $X = \pi\{X_\alpha : \alpha \in A\}$ and $Y = \pi\{Y_\alpha : \alpha \in A\}$ are two products of (c.r.=completely regular) Souslin spaces such that X, Y are uncountable, and 2) μ and λ are respectively σ -finite Baire measures on X and on Y , then any set isomorphism from X to Y is induced by a completion point isomorphism, the word completion being unnecessary if each factor is Polish. In particular, when μ and λ are two set isomorphic probability Baire measures on the same product X of c.r. Souslin spaces, then μ and λ are also completion point isomorphic, provided X is uncountable. We also show that our main theorem contains the following new result: If 1) M is an arbitrary product of locally compact homogeneous spaces $\{M_\alpha\}_{\alpha \in A}$, where M_α is acted upon by a locally compact σ -compact group G_α , 2) $\mathcal{B}$ is the Baire σ -algebra on M , and 3) μ is an arbitrary σ -finite measure on M , then any set automorphism on $(M, \mathcal{B}, \mu)$ is induced by a completion point automorphism. This result generalizes the main theorems in [3] and [4] and simplifies their proofs as well. Indeed, when the context described above is seen as being a particular example of our more general abstract setting, then to achieve our goal, the number of arguments specific to groups is minimized and the remaining arguments are easily settled in the more general theorem.

In our paper, we also prove a very general result concerning (not necessarily invertible) set homomorphisms which can be induced by a (not necessarily invertible) point homomorphism. This result is stated in the general abstract setting of projective limits of completely regular Hausdorff spaces. Note that there exist examples of σ -finite measure spaces $(X, \mathcal{B}, \mu)$ for which it is possible to construct a set homomorphism on X which cannot be induced by a completion point homomorphism $T : X \rightarrow X$ (see [7, Lemma 12]).

2. Notations. $\mathbb{F}$ is always either the field $\mathbb{R}$ or $\mathbb{C}$. When $(X, \mathcal{F}, \mu)$ is a measure space, we use the symbol $\sim$ over any relation between functions or sets, whenever this relation is satisfied almost everywhere. For example, $f \lesssim 0$ means $f \geq 0$ a.e. and $f \gtrsim 0$ means $f(x) \geq 0$ for every $x \in X$. As usual, the symbols $L^p(X, \mu, \mathbb{R})$ and $L^p(X, \mu, \mathbb{C})$ respectively designate the real and complex L^p -spaces associated with $(X, \mathcal{F}, \mu)$, and $(X, \mathcal{F}^*, \mu)$ designates the completion of $(X, \mathcal{F}, \mu)$. $M(X, \mathcal{F}, \mathbb{F})$ represents the set of all $\mathbb{F}$ -valued $\mathcal{F}$ -measurable functions on X and $M^\infty(X, \mathcal{F}, \mathbb{F})$, the subset consisting of all its bounded functions. Finally, when $E \in \mathcal{F}$, the element of the measure algebra of

$(X, \mathcal{F}, \mu)$ containing E is denoted by $\{E\}$.

We define a collection $\mathcal{C}$ of $\mathbb{F}$ -valued functions to be 1) a *lattice* if $f^+ = f \vee 0$, $f^- = (-f) \vee 0$ belong to $\mathcal{C}$ whenever $f \in \mathcal{C}$ is real-valued, and 2) *self-adjoint* if for every $f \in \mathcal{C}$, the real and imaginary part of f also belong to $\mathcal{C}$. (Note that when $\mathbb{F} = \mathbb{R}$, $\mathcal{C}$ is automatically self-adjoint, provided $0 \in \mathcal{C}$.)

The definitions of Polish, Lusin and Souslin spaces agree with those given in [19, pp. 92–96]. Furthermore, if X is a topological space and $C_b(X, \mathbb{F}) = \{\text{bounded continuous } \mathbb{F}\text{-valued functions on } X\}$ separates points of X , X is said to be *strongly Hausdorff*.

We introduce some special terminology about projective limits. Let $\{X_\alpha, \pi_{x_{\alpha_2, \alpha_1}} : \alpha, \alpha_1, \alpha_2 \in A \text{ and } \alpha_1 \leq \alpha_2\}$ be a projective system of topological spaces such as defined in [8, pp. 93–94], except that we do not assume that A is directed nor do we assume that each $\pi_{x_{\alpha_2, \alpha_1}}$ is surjective. For every α , let $\mathcal{B}_\alpha$ be either the Baire or the Borel σ -algebra on X_α . We denote by $X = \text{projlim}_{\alpha \in A} X_\alpha$ the projective limit of our system and by $\mathcal{B} = \text{projlim}_{\alpha \in A} \mathcal{B}_\alpha$ the σ -algebra on X induced by the projection maps $\pi_{x_\alpha} : X \rightarrow X_\alpha$ ($\alpha \in A$). $B \subseteq A$ is said to be *complete* if $\alpha \leq \beta$ and $\beta \in B$ imply $\alpha \in B$. For all $B \subseteq A$, $\mathcal{C}B$ designates the smallest complete set containing B i.e. $\mathcal{C}B = \{\alpha \in A : \alpha \leq \beta \text{ for some } \beta \in B\}$. If $B \subseteq A$, we put $X(B) = \text{projlim}_{\alpha \in \mathcal{C}B} X_\alpha$ and $\mathcal{B}(B) = \text{projlim}_{\alpha \in \mathcal{C}B} \mathcal{B}_\alpha$, and we denote by X^B the σ -algebra of cylinders in X of the form $\pi_{x_B}^{-1}(E)$, where $E \in \mathcal{B}(B)$. If μ is any measure on X , then, via X^B and π_{x_B} , μ induces measures μ_B on each $X(B)$. The measure algebras of X , X^B and $X(B)$ are denoted respectively by E_X , E_X^B and $E_X(B)$. Note that π_{x_B} induces a canonical isomorphism between E_X^B and $E_X(B)$. Finally, when $\alpha \in A$ and $B \subseteq A$, we denote by B_α the set $\{\beta \in B : \beta \leq \alpha\}$.

3. Preliminary results on projective systems of Souslin spaces. Throughout the present and following sections, we will make heavy use of the notations introduced in section 2 without further notice.

Proposition 3.1. *Let i) $X = \text{projlim}_{\alpha \in A} X_\alpha$, where each X_α is Souslin, and ii) $\mathcal{B} = \text{projlim}_{\alpha \in A} \mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ is the Borel σ -algebra on X_α .*

If A contains a countable cofinal subset, then X is Souslin and $\mathcal{B}$ is the Borel σ -algebra on X .

Proof. Let $\{\alpha_n\} \subseteq A$ be cofinal. Then, clearly, X is homeomorphic to a closed subset S of $Y = \prod\{X_{\alpha_n} : n \geq 1\}$ via a natural projection map. But Y is Souslin and $\mathcal{F} = \prod\{\mathcal{B}_{\alpha_n} : n \geq 1\}$ is the Borel σ -algebra on Y (see [19, p. 108]). Hence, the fact that S is closed in Y implies the desired conclusions. $\square$

Proposition 3.2. *Let $(X, \mathcal{B})$ be as in Proposition 3.1 and suppose furthermore that each X_α is strongly Hausdorff.*

Then if $S \subseteq X$ is Souslin, there exists $\{\alpha_n\} \subseteq A$ such that if $x, y \in S$ are such that $x_{\alpha_n} = y_{\alpha_n}$ for all $n \geq 1$, then $x = y$.

Furthermore, $\mathcal{B}_S = \{S \cap E : E \in \mathcal{B}\}$ coincides with the Borel σ -algebra on S and also with the σ -algebra generated by the cylinders of the form $\bigcap\{\pi_{x_{\alpha_n}}^{-1}(E_n) \cap S : n \geq 1\}$ (where E_n is a Borel subset of X_{α_n}).

Proof. For each $\alpha \in A$, denote by πs_α the restriction of πx_α on S , and put $\mathcal{C} = \{f \circ \pi s_\alpha : \alpha \in A \text{ and } f \in C_b(X_\alpha, \mathbb{F})\}$. Then $\mathcal{C}$ separates points of S . By [19, p. 105], there exists $\{f_n \circ \pi s_{\alpha_n}\} \subseteq \mathcal{C}$ which separates points of S . Consequently, by [19, p. 108], $\{f_n \circ \pi s_{\alpha_n}\}$ generates the Borel σ -algebra on S and so, we obtain the second part of our conclusion.

Now, take $x, y \in S$ such that $x_{\alpha_n} = y_{\alpha_n}$ for all $n \geq 1$, and let $\alpha \in A$. Then, we now know that $\pi s_\alpha^{-1}(x_\alpha)$ belongs to the σ -algebra generated by the cylinders of the form $\bigcap \{\pi s_{\alpha_n}^{-1}(E_n) \cap S : n \geq 1\}$ (where E_n is a Borel subset of X_{α_n}). Since $x \in \pi s_\alpha^{-1}(x_\alpha)$ and $x_{\alpha_n} = y_{\alpha_n}$ for all $n \geq 1$, we therefore deduce that $y \in \pi s_\alpha^{-1}(x_\alpha)$, which proves that $y_\alpha = x_\alpha$. Hence, $x = y$ and the proof of Proposition 3.2 is complete. $\square$

Remark 3.3. The proof of Proposition 3.2 contains the non surprising fact that for a strongly Hausdorff Souslin space X , the Borel and Baire σ -algebra coincide on X .

4. Set homomorphisms induced by point homomorphisms. Let us set up the terminology which will be used in this section. First, $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ will denote arbitrary σ -finite measure spaces, unless further assumptions on these spaces are specified. Furthermore, $\mathcal{V}$ and $\mathcal{W}$ will respectively be subsets of $M(X, \mathcal{B}, \mathbb{F})$ and $M(Y, \mathcal{F}, \mathbb{F})$. In addition, we will suppose that $\mathcal{V}$ is a self-adjoint lattice which contains the class of all integrable simple functions. Finally, we adopt the word e-linear which stands for essentially linear. The same remark applies for e-positive, e-multiplicative and e-support separating.

Theorem 4.2. *Let i) $X = \text{projlim}_{\alpha \in A} X_\alpha$ be a projective limit of strongly Hausdorff spaces, where A is an indexed set such that for all $\alpha \in A$ and all complete $B \subseteq A$, $B_\alpha = \{\beta \in B : \beta \leq \alpha\}$ has a countable cofinal subset, ii) $\mathcal{B} = \text{projlim}_{\alpha \in A} \mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ is the Baire σ -algebra on X_α , iii) μ be a finite measure on $(X, \mathcal{B})$ such that μ_α is tight for all $\alpha \in A$ (see [21, pp. 33–55] for the definition and properties of tight measures), and iv) φ be a set homomorphism from $(X, \mathcal{B}, \mu)$ to $(Y, \mathcal{F}, \lambda)$.*

Then there exists a completion measurable transformation $T : Y \rightarrow X$ such that for all $E \in \mathcal{B}$, we have

$$T^{-1}E \cap \varphi X \cong \varphi E.$$

Proof. Since in the following argument, we can replace $(Y, \mathcal{F}, \lambda)$ with $(Y' = \varphi X, \mathcal{F}' = \varphi X \cap \mathcal{F}, \lambda' = \lambda|_{\mathcal{F}'})$ and T with $T' = T|_{\varphi X}$, suppose without loss of generality that $Y = \varphi X$. Denote by $\mathcal{V}$ (respectively $\mathcal{W}$) the set of all measurable simple functions on $(X, \mathcal{B})$ (respectively $(Y, \mathcal{F})$) and define an operator $U : \mathcal{V} \rightarrow \mathcal{W}$ by $U(\chi_E) = \chi_{\varphi E}$. Then U is e-linear, e-positive and e-multiplicative. Furthermore, U satisfies the following property: $s_n \downarrow 0 \Rightarrow U(s_n)(x) \rightarrow 0$ a.e. (see [13, Corollary 3.6]). Hence, the operator $U' : M^\infty(X, \mathcal{B}, \mathbb{F}) \rightarrow M^\infty(Y, \mathcal{F}, \mathbb{F})$ defined by $U'f \cong \lim_n U(s_n)$, where $\{s_n\} \subseteq \mathcal{V}$ and $0 \leq s_n \uparrow f$, is well defined, e-linear, e-positive and e-multiplicative (see, for example, [13, step 1 of the proof of Theorem 4.5]). By [9], there exists a *lifting* ρ of $(Y, \mathcal{F}^*, \lambda)$. Put $\bar{U} = \rho \circ U'$. Then $\bar{U} : M^\infty(X, \mathcal{B}, \mathbb{F}) \rightarrow M(Y, \mathcal{F}^*, \mathbb{F})$ is (exactly) positive, linear and multiplicative and furthermore, $\bar{U}s \cong Us$ for all $s \in \mathcal{V}$.

Now, denote by $\mathcal{S}$ the family of all couples (B_ω, T_ω) , where i) $B_\omega \subseteq A$, and ii) $T_\omega : Y \rightarrow X(B_\omega)$ is a completion measurable transformation such that 1) $U(f \circ \pi x_{B_\omega}) \cong f \circ T_\omega$ for all $f \in \mathcal{V}_{B_\omega} = \{f \in M(X(B_\omega), \mathcal{B}(B_\omega), \mathbb{F}) : f \circ \pi x_{B_\omega} \in \mathcal{V}\}$, and 2)

for all $\alpha \in B_\omega$, there is a set $S_\alpha \cong Y$ with $\overline{U}(f \circ \pi x_\alpha) = f \circ \pi x_{B_\omega, \alpha} \circ T_\omega$ on S_α , for all $f \in C_b(X_\alpha, \mathbb{F})$. By [13, Theorem 4.5 and step 2 of its proof], we have that for all $\alpha \in A$, there exists a completion measurable transformation $T_\alpha : Y \rightarrow X_\alpha$ such that $(\{\alpha\}, T_\alpha) \in \mathcal{S}$. Consequently, $\mathcal{S} \neq \emptyset$. We say that $(B_\omega, T_\omega) \leq (B_\nu, T_\nu)$ if $B_\omega \subseteq B_\nu$ and T_ν extends T_ω in the sense that $\pi x_{B_\nu, B_\omega} \circ T_\nu = T_\omega$. Then $\leq$ is a partial ordering on $\mathcal{S}$. Furthermore, let $\{(B_\nu, T_\nu) : \nu \in M\}$ be a linearly ordered subfamily and put $B' = \bigcup B_\nu$. Given $y \in Y$ and $\alpha \in B'$, choose any ν such that $\alpha \in B_\nu$ and define $T'y \subseteq X(B')$ by $\pi x_{B', \alpha}(T'y) = \pi x_{B_\nu, \alpha}(T_\nu y)$. Then one easily verifies that $(B', T') \in \mathcal{S}$.

By Zorn's lemma, there exists a maximal member (B, T) of $\mathcal{S} \neq \emptyset$, for which B is necessarily complete. Let us now prove that $B = A$. Suppose not and take $\alpha \in A \setminus B$. Then there exists a completion measurable transformation $T_\alpha : Y \rightarrow X_\alpha$ such that

$$U(f \circ \pi x_\alpha) \cong f \circ T_\alpha \quad \text{for all } f \in \mathcal{V}_\alpha$$

and

$$\overline{U}(f \circ \pi x_\alpha) = f \circ T_\alpha \text{ on } S_\alpha \cong Y, \text{ for all } f \in C_b(X_\alpha, \mathbb{F}). \quad (*)$$

Consider the (complete) subset $B_\alpha \subseteq B$ and suppose that $B \neq \emptyset$. (If $B = \emptyset$, the arguments developed in the remainder of this paragraph are irrelevant.) Put $T' = \pi x_{B, B_\alpha} \circ T$ and let $\{\beta_n\}$ be a countable cofinal subset of B_α . For each n , there exists $S_n \cong Y$ such that

$$\overline{U}(f \circ \pi x_{\beta_n}) = f \circ \pi x_{B, \beta_n} \circ T \text{ on } S, \text{ for all } f \in C_b(X_{\beta_n}, \mathbb{F}). \quad (**)$$

Put $S = S_\alpha \cup (\bigcup S_n)$. Then $S \cong Y$. Take $s \in S$ and put $x = (\pi x_{\alpha, B_\alpha} \circ T_\alpha)(s)$, $y = T's$. Suppose that x, y (which belong to $X(B_\alpha)$) are such that $x \neq y$. Then, since $\{\beta_n\}$ is cofinal in B_α , there exists n with $x_{\beta_n} \neq y_{\beta_n}$. X_{β_n} being strongly Hausdorff implies that there exists $f \in C_b(X_{\beta_n}, \mathbb{F})$ such that $f(x_{\beta_n}) \neq f(y_{\beta_n})$ i.e.

$$(f \circ \pi x_{B_\alpha, \beta_n} \circ \pi x_{\alpha, B_\alpha} \circ T_\alpha)(s) \neq (f \circ \pi x_{B_\alpha, \beta_n} \circ T')(s).$$

But, by (*),

$$\begin{aligned} (f \circ \pi x_{B_\alpha, \beta_n} \circ \pi x_{\alpha, B_\alpha} \circ T_\alpha)(s) &= \overline{U}(f \circ \pi x_{B_\alpha, \beta_n} \circ \pi x_{\alpha, B_\alpha} \circ \pi x_\alpha)(s) \\ &= \overline{U}(f \circ \pi x_{\beta_n})(s) \end{aligned}$$

and by (**),

$$\begin{aligned} (f \circ \pi x_{B_\alpha, \beta_n} \circ T')(s) &= (f \circ \pi x_{B, \beta_n} \circ T)(s) \\ &= \overline{U}(f \circ \pi x_{\beta_n})(s). \end{aligned}$$

We have thus reached a contradiction and hence, we must have $\pi x_{\alpha, B_\alpha} \circ T_\alpha = T'$ on S . Redefine T_α on $Y \setminus S$ so that $\pi x_{\alpha, B_\alpha} \circ T_\alpha = T'$ on Y .

Put $B_1 = B \cup A_\alpha$ and define $T_1 : Y \rightarrow X(B_1)$ by $\pi x_{B_1, \beta}(T_1 y) = \pi x_{B, \beta}(T y)$ if $\beta \in B$ and $\pi x_{B_1, \beta}(T_1 y) = \pi x_{\alpha, \beta}(T_\alpha y)$ if $\beta \in A_\alpha$. Then, it is easy to verify that in both cases $B_\alpha = \emptyset$ and $B_\alpha \neq \emptyset$, we have $(B_1, T_1) \in \mathcal{S}$. Since $(B, T) < (B_1, T_1)$, we contradict the fact that (B, T) is maximal and so, we must have $B = A$ and the proof is complete. $\square$

Remarks 4.3. a) Theorem 4.2 remains true if, more generally, the indexed set A satisfies the property that for all $\alpha \in A$ and all complete $B \subseteq A$ for which $B_\alpha \neq \emptyset$, there exists $\{\alpha_n\} \subseteq B_\alpha$ such that: $(x, y \in \pi x_{B, B_\alpha}(X(B)) \text{ and } x_{\alpha_n} = y_{\alpha_n} \text{ for all } n \geq 1) \Rightarrow x = y$. The theorem remains also valid if, instead of assuming that μ_α is tight, we only suppose that μ_α is equivalent to a tight finite measure ($\alpha \in A$).

b) Obviously, the assumption that each X_α is strongly Hausdorff is unnecessary (in Theorem 4.2) if the ordering $\leq$ on A is such that $B_\alpha = \emptyset$ whenever B is a complete subset of A not containing α . This happens only when $\leq$ is the identity relation on A and in this case, $\text{projlim}_{\alpha \in A} X_\alpha$ and $\text{projlim}_{\alpha \in A} \mathcal{B}_\alpha$ respectively coincide with $\Pi\{X_\alpha : \alpha \in A\}$ and $\Pi\{\mathcal{B}_\alpha : \alpha \in A\}$. Hence, Theorem 4.2 contains the following result:

Let i) $X = \Pi\{X_\alpha : \alpha \in A\}$ be an arbitrary product of topological spaces, ii) $\mathcal{B} = \Pi\{\mathcal{B}_\alpha : \alpha \in A\}$, where $\mathcal{B}_\alpha$ is the Baire σ -algebra on X_α , iii) μ be a finite measure on $(X, \mathcal{B})$ such that μ_α , the contraction of μ on X_α , is tight ($\alpha \in A$), and iv) φ be a set homomorphism from $(X, \mathcal{B}, \mu)$ to $(Y, \mathcal{F}, \lambda)$. Then the conclusion of Theorem 4.2 holds for φ .

It should be noted that this result on products can be proved using more elementary techniques than those used in the proof of Theorem 4.2. (See [14, Theorem 3.2].)

We now obtain an interesting application of Theorem 4.2 to projective limits of Souslin spaces.

Corollary 4.4. *Let i) $X = \text{projlim}_{\alpha \in A} X_\alpha$ be an arbitrary projective limit of strongly Hausdorff Souslin spaces such that for all $\alpha \in A$ and all complete $B \subseteq A$ with $B \neq \emptyset$, we have $\pi x_{B, B_\alpha}(X(B)) \subseteq \pi x_{\alpha, B_\alpha}(X_\alpha)$, ii) $\mathcal{B} = \text{projlim}_{\alpha \in A} \mathcal{B}_\alpha$, where $\mathcal{B}_\alpha$ is the Borel (=Baire) σ -algebra on X_α , iii) μ be a finite measure on $(X, \mathcal{B})$, and iv) φ be a set homomorphism from $(X, \mathcal{B}, \mu)$ to $(Y, \mathcal{F}, \lambda)$.*

Then the conclusion of Theorem 4.2 holds for φ .

Proof. First, we know that for all $\alpha \in A$, μ_α is necessarily tight (see [21, p. 52] or [19, p. 122]). Now, let $\alpha \in A$ and B be a complete subset of A such that $B_\alpha \neq \emptyset$. Then $S = \pi x_{\alpha, B_\alpha}(X_\alpha)$ is clearly a strongly Hausdorff Souslin space contained in $X(B_\alpha) = \text{projlim}_{\beta \in B_\alpha} X_\beta$. By Proposition 3.2, there exists $\{\alpha_n\} \subseteq B_\alpha$ such that if $x, y \in S$ and $x_{\alpha_n} = y_{\alpha_n}$ for all $n \geq 1$, then $x = y$. Hence, the desired conclusion is a consequence of remarks 4.3 a) and the fact that $\pi x_{B, B_\alpha}(X(B)) \subseteq \pi x_{\alpha, B_\alpha}(X_\alpha)$. $\square$

From Theorem 4.2, we also obtain the following very general result:

Theorem 4.5. *Let i) $(X, \mathcal{B})$ be as in Theorem 4.2 or, more generally, as in remarks 4.3 a), ii) μ be a σ -finite measure on $(X, \mathcal{B})$ such that for all α , μ_α is equivalent to a tight finite measure on $(X_\alpha, \mathcal{B}_\alpha)$, and iii) $U : \mathcal{V} \rightarrow \mathcal{W}$ be an e -linear e -support separating operator satisfying the conditions:*

- a) $f \cong 0 \Rightarrow Uf \cong 0$
- b) $f_n \downarrow 0 \Rightarrow$ for all $\epsilon > 0$, there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$U(f_{n_k})(x) \rightarrow 0 \text{ except possibly on a set of measure less than } \epsilon.$$

(By Corollary 4.3 in [13], condition b) is equivalent to: $f_n \downarrow 0 \Rightarrow (Uf_n)(x) \rightarrow 0$ a.e.)

Then, there exists 1) a function $h \in M(Y, \mathcal{F}, \mathbb{F})$ which is uniquely determined a.e. and 2) a completion measurable transformation $T : Y \rightarrow X$, such that

- i) $\mu(E) = 0 \Rightarrow \lambda(T^{-1}E \cap W) = 0$ (where W denotes the support of h)
- ii) $Uf \cong h(f \circ T)$ for all $f \in \mathcal{V}$.

Furthermore, the above conclusions hold even when $\mathcal{V}$ contains functions in $M(X, \mathcal{B}^*, \mathbb{F})$ which are not in $M(X, \mathcal{B}, \mathbb{F})$.

Proof. By [13, Proposition 4.2 and step 1 of the proof of Theorem 4.5], there exists $h \in M(Y, \mathcal{F}, \mathbb{F})$ and an e-positive e-linear e-multiplicative operator $\bar{U} : M^\infty(X, \mathcal{B}, \mathbb{F}) \rightarrow M^\infty(Y, \mathcal{F}, \mathbb{F})$ such that $h \cdot \bar{U}f \cong Uf$ for all $f \in \mathcal{V} \cap M^\infty(X, \mathcal{B}, \mathbb{F})$. For all $E \in \mathcal{B}$, put $\varphi\{E\} = \{S\}$, where $\chi_S = \bar{U}(\chi_E)$. Since U satisfies the property: $f_n \downarrow 0 \Rightarrow (Uf_n)(x) \rightarrow 0$ a.e. and μ is σ -finite, we have that φ is a set homomorphism from $(X, \mathcal{B}, \mu)$ to $(Y, \mathcal{F}, \lambda)$. The proof is therefore completed by an easy application of Theorem 4.2. $\square$

The following new result requires some preliminary comments: Let $\phi : [0, \infty[\rightarrow [0, \infty[$ be a continuous strictly increasing function such that $\phi(0) = 0$ and let $(M, \mathcal{M}, m)$ be any measure space. As in [12], we designate by 1) $\mathcal{L}_\phi(M, m, \mathbb{F})$ the space of all measurable $\mathbb{F}$ -valued functions f on M , which satisfy

$$I_\phi(f) = \int_M \phi(|f|) dm < \infty,$$

and 2) $L_\phi(M, m, \mathbb{F})$ the set $\mathcal{L}_\phi(M, m, \mathbb{F})$ modulo the relation $\cong$. When $\phi(t) = t^p$ ($0 < p < \infty$), $L_\phi(M, m, \mathbb{F}) = L^p(M, m, \mathbb{F})$ and when $\phi(t) = t/(1+t)$, $L_\phi(M, m, \mathbb{F})$ is the space S or s in [1, pp. 9–10]. In general, this space $L_\phi(M, m, \mathbb{F})$ need not be a vector space as shown by many trivial examples.

Corollary 4.6. Let i) $(X, \mathcal{B}, \mu)$ be as in Theorem 4.5, ii) ϕ_1, ϕ_2 , be two continuous strictly increasing functions from $[0, \infty[$ to $[0, \infty[$, such that $\phi_1(0) = \phi_2(0) = 0$, and iii) $U : L_{\phi_1}(X, \mu, \mathbb{F}) \rightarrow L_{\phi_2}(Y, \lambda, \mathbb{F})$ be a linear operator (in the sense that $U(\alpha f + \beta g) = \alpha Uf + \beta Ug$ whenever $\alpha f + \beta g \in L_{\phi_1}(X, \mu, \mathbb{F})$) which is support separating and bounded (i.e. there exists $M > 0$ for which $I_{\phi_2}(Uf) \leq M \cdot I_{\phi_1}(f)$ for all $f \in L_{\phi_1}(X, \mu, \mathbb{F})$).

Then there exists 1) a function $h \in M(Y, \mathcal{F}, \mathbb{F})$ which is uniquely determined a.e., and 2) a completion measurable transformation $T : Y \rightarrow X$, such that

- i) there is $N > 0$ with

$$\int_{T^{-1}E} \phi_2(|h|) d\lambda \leq N \cdot \mu(E) \text{ for all } E \in \mathcal{B}$$

- ii) $Uf \cong h(f \circ T)$ for all $f \in L_{\phi_1}(X, \mu, \mathbb{F})$.

Proof. Clearly, $\mathcal{L}_{\phi_1}(X, \mu, \mathbb{F})$ is a self-adjoint lattice which contains the class of all integrable simple functions. Further, it is obvious that $\phi(t_n) \rightarrow 0$ implies $t_n \rightarrow 0$ and so, if $f_n \downarrow 0$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $U(f_{n_k})(x) \rightarrow 0$ a.e. Hence, we get ii) by applying Theorem 4.5. Now, put $\alpha_1 = \phi_1(1) > 0$ and

$N = M \cdot \alpha_1$. Then, if $f = \chi_E$ ($E \in \mathcal{B}$), we have $M \cdot I_{\phi_1}(f) = N \cdot \mu(E)$ and $I_{\phi_2}(Uf) = \int_{T^{-1}E} \phi_2(|h|)d\lambda$. Since $I_{\phi_2}(Uf) \leq M \cdot I_{\phi_1}(f)$, we obtain i) and the proof is complete. $\square$

Remark 4.7. We can obtain versions of Theorem 4.5 and Corollary 4.6 in the same way we obtained Corollary 4.4 from Theorem 4.2.

5. Set isomorphisms induced by point isomorphisms. The terminology used in this section is the following: $\{(X_\alpha, \mathcal{B}_\alpha)\}_{\alpha \in A}$ and $\{(Y_\alpha, \mathcal{F}_\alpha)\}_{\alpha \in A}$ will be arbitrary (non-empty) collections of strongly Hausdorff Souslin spaces with their Borel (=Baire) σ -algebras, and we will put $X = \text{projlim}_{\alpha \in A} X_\alpha$ and $Y = \text{projlim}_{\alpha \in A} Y_\alpha$. Furthermore μ and λ will be probability measures defined respectively on $\mathcal{B} = \text{projlim}_{\alpha \in A} \mathcal{B}_\alpha$ and $\mathcal{F} = \text{projlim}_{\alpha \in A} \mathcal{F}_\alpha$. Unless X and Y are normal or A is countable, it is not known whether $\mathcal{B}$ and $\mathcal{F}$ respectively coincide with the Baire σ -algebra on X and the Baire σ -algebra on Y .

For the sake of completeness and to ease up the proof of our main theorem, we now give the statement of a useful result proved in [14, Theorem 4.4].

Theorem 5.1. *Let i) $(S_1, \mathcal{B}_{01}, m_1)$, $(S_2, \mathcal{B}_{02}, m_2)$ be two σ -finite Borel measure spaces of Souslin spaces S_1 and S_2 and ii) $\mathcal{V} \subseteq M(S_1, \mathcal{B}_{01}, \mathbb{F})$ and $\mathcal{W} \subseteq M(S_2, \mathcal{B}_{02}, \mathbb{F})$ be self-adjoint lattices containing the class of all integrable simple functions, and iii) $U : \mathcal{V} \rightarrow \mathcal{W}$ be an ε -linear support separating operator satisfying conditions a) and b) of Theorem 4.5 and which is invertible in the sense that there exists $U^{-1} : \mathcal{W} \rightarrow \mathcal{V}$ possessing the same properties as U and satisfying $U(U^{-1}g) \cong g$ and $U^{-1}(Uf) \cong f$ for all $f \in \mathcal{V}$ and all $g \in \mathcal{W}$.*

Then there exists 1) a measurable function $h : S_2 \rightarrow \mathbb{F}$ having support equal to S_2 and 2) a point isomorphism mod 0 $T : S_2 \rightarrow S_1$ (i.e. a map $T : S_2 \rightarrow S_1$ for which there exists null sets $N_{S_1} \subseteq S_1$, $N_{S_2} \subseteq S_2$, such that T is a point isomorphism from $\Omega_{S_2} = S_2 \setminus N_{S_2}$ to $\Omega_{S_1} = S_1 \setminus N_{S_1}$), such that

$$Uf \cong h(f \circ T) \quad \text{for all } f \in \mathcal{V}$$

and

$$U^{-1}g \cong (1/h \circ T^{-1}) \cdot (g \circ T^{-1}) \quad \text{for all } g \in \mathcal{W}.$$

Furthermore, if S_1 and S_2 are both uncountable, then T can be chosen to be bijective. (In particular, T becomes a completion point isomorphism.)

Finally, when i) $(S_1, \mathcal{B}_{01}, m_1) = (S_2, \mathcal{B}_{02}, m_2)$ or ii) S_1 and S_2 are both uncountable Polish spaces or, more generally, uncountable Lusin spaces, then T can be chosen to be a point isomorphism.

We also need the following definitions:

Definition 5.2. Let φ be a set isomorphism from X to Y and (whenever $\text{cardinal}(X) = \text{cardinal}(Y)$), let $T : Y \rightarrow X$ be a completion point isomorphism. Then a set $B \subseteq A$ is called

- a) φ -invariant if $\varphi(E_X^B) = E_Y^B$.
- b) T -invariant if for all $E \in X^B$ and all $F \in Y^B$, $T^{-1}(E)$ and $T(F)$ are based on B (i.e. there exists $E_1 \in \mathcal{F}(B)^*$ and $F_1 \in \mathcal{B}(B)^*$ such that $E_1 = \pi y_B^{-1}(T^{-1}(E))$ and $F_1 = \pi x_B^{-1}(T(F))$).

Definition 5.3. A completion point isomorphism $T : Y \rightarrow X$ is said to be σ -invariant if the image (and reciprocal image) of a measurable set based on a countable number of coordinates is also based on countably many coordinates.

Remarks 5.4. a) When φ is a point isomorphism from X to Y and $B \subseteq A$ is φ -invariant, then φ induces a set isomorphism from $X(B)$ to $Y(B)$ which we will from now on denote by $\varphi|_B$.

b) Any point isomorphism $T : Y \rightarrow X$ is necessarily σ -invariant. In [14, Example 5.4], an example is given of a completion point isomorphism which is not σ -invariant and for which there is a $\alpha \in A$ such that $\{\alpha\}$ is not contained in any countable T -invariant set. Thus, Lemma 5.6 below is false without the additional assumption of σ -invariance.

Using the preceding definitions, we prove a lemma which is a generalization of Lemma 4 in [3].

Lemma 5.5. Let i) φ be a set isomorphism from X to Y , ii) $T : Y \rightarrow X$ be a completion point isomorphism which is σ -invariant, and iii) B, C be countable subsets of A .

Then there exists countable sets $\overline{B}, \overline{C} \subseteq A$ such that 1) $B \subseteq \overline{B}$, 2) $C \subseteq \overline{C}$, 3) $\overline{B}$ is φ -invariant, and 3) $\overline{C}$ is T -invariant.

Proof. By Proposition 3.1, $X(B)$ is (strongly Hausdorff) Souslin and $\mathcal{B}(B)$ is the (countably generated) Borel σ -algebra on $X(B)$. The construction of $\overline{B}$ now follows as in [14, Lemma 5.5] and the construction of $\overline{C}$ is similar to that of $\overline{B}$ by using the fact that T is σ -invariant together with remark 5.4 b). $\square$

To facilitate the material exposed in the remainder of this paper, we adopt the following terminology: 1) A set $B \subseteq A$ is called 1) *hereditary* if π_{x_B} and π_{y_B} are both surjective. 2) If $B \subseteq C$ and the projections $\pi_{x_{C,B}}$ and $\pi_{y_{C,B}}$ are respectively point isomorphisms from $(X(C), \mathcal{B}(C), \mu_C)$ to $(X(B), \mathcal{B}(B), \mu_B)$ and from $(Y(C), \mathcal{F}(C), \lambda_C)$ to $(Y(B), \mathcal{F}(B), \lambda_B)$, then we say that C is an *equivalent* set containing B .

From now on and until the remainder of this section, P and P' will respectively designate a certain property relative to the class of all (non-empty) subsets of A and a certain property relative to the class of all couples of (non-empty) countable subsets of A . We say that $B \subseteq A$ is a P -set if B satisfies property P and that $(B, C) \subseteq A \times A$ is a P' -couple if (B, C) satisfies property P' . A P -maximal set B is a complete P -set such that whenever B is contained in an equivalent P -set C , then $B = C$. In addition to those definitions, we give 6 conditions that P and P' may satisfy:

The hereditary condition (abbreviated HC).

A is a P -set and each P -set is hereditary.

The totally ordered union condition (abbreviated TOUC).

- 1) The union of a totally ordered collection of P -sets is a P -set.
- 2) The union of a countable totally ordered collection of P' -couples is a P' -couple.

The existence condition (abbreviated EC).

- 1) If B is P -maximal and C is a countable subset of B , then C is contained in a countable P -set D which is contained in B .

- 2) Any couple (C, D) of countable subsets of A is contained in a P' -couple (C_1, D_1) . Furthermore, if C is contained in a P -maximal set B , then C_1 can be chosen so that $C_1 \subseteq B$.

The union condition (abbreviated UC).

If B is P -maximal and C is a countable P -set, then there exists a countable set $C \subseteq D \subseteq A$ such that $B \cup D$ is a P -set.

The completion countable extension condition (abbreviated CCEC).

Whenever 1) B, C are countable P -sets such that $B \subseteq C$ and (B, C) is a P' -couple, and 2) $T : Y(B) \rightarrow X(B)$ is a completion point isomorphism such that the set isomorphism $\varphi : E_X(B) \rightarrow E_Y(B)$ induced by T (which is defined by $\varphi(\{E\}) = \{T^{-1}E\}$) can be extended to a set isomorphism $\varphi' : E_X(C) \rightarrow E_Y(C)$ in the sense that $\pi y_{C,B}^{-1}(\varphi E) \cong \varphi'(\pi x_{C,B}^{-1}(E))$ ($E \in \mathcal{B}(B)$), then T can be extended to a completion point isomorphism $T' : Y(C) \rightarrow X(C)$ such that $\pi x_{C,B} \circ T' = T \circ \pi y_{C,B}$.

The countable extension condition (abbreviated CEC).

Same statement as CCEC but with T and T' being point isomorphisms instead of completion point isomorphisms.

The conditions listed above seem at first very abstract but as we will see, they allow a great deal of flexibility when working in specific contexts. Later on, we will show that there exists a great variety of examples for $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ where strategic properties P and P' can be chosen so to satisfy the first five conditions. In some examples, P and P' can even be chosen so to satisfy all the conditions. Now, the following lemma, whose proof is obvious, is vital to the main theorem of this section.

Lemma 5.6. *Suppose P and P' satisfy TOUC, EC and UC. Furthermore, let i) φ be a set isomorphism from X to Y , ii) B be a P -maximal φ -invariant set, iii) $T : Y(B) \rightarrow X(B)$ be a σ -invariant completion point isomorphism, and iv) $C, D \subseteq A$ be countable sets such that $C \subseteq B$.*

Then there exists countable sets $\overline{C}, \overline{D} \subseteq A$ such that 1) $\overline{D}$ is a φ -invariant P -set containing D for which $B \cup \overline{D}$ is a P -set, 2) $\overline{C}$ is a T -invariant P -set such that $C \subseteq \overline{C} \subseteq B$, and 3) $(\overline{C}, \overline{D})$ is a P' -couple.

We are now well prepared to prove the main result of this section.

Theorem 5.7. *Suppose P and P' satisfy HC, TOUC, EC and UC.*

a) *If $\text{cardinal}(X(B)) = \text{cardinal}(Y(B)) \neq \aleph_0$ for all countable P -sets $B \subseteq A$, and P, P' satisfy CCEC, then any set isomorphism φ from X to Y is induced by a σ -invariant completion point isomorphism $T : Y \rightarrow X$. This T can be chosen to be a point isomorphism if, in addition, each X_α and Y_α is Polish and P, P' satisfy CEC instead of CCEC.*

b) *If $(X, \mathcal{B}, \mu) = (Y, \mathcal{F}, \lambda)$, the results of a) relative to the properties CCEC and CEC remain valid even though we do not assume that 1) $\text{cardinal}(X(B)) \neq \aleph_0$ for all countable P -sets $B \subseteq A$, or 2) X_α is Polish for all $\alpha \in A$.*

Proof. Consider the family $\mathcal{S}$ of ordered couples (B_ω, T_ω) , where i) B_ω is a φ -invariant P -set, ii) $T_\omega : Y(B_\omega) \rightarrow X(B_\omega)$ is a σ -invariant completion point isomorphism and iii) T_ω induces $\varphi|_{B_\omega}$. Observe that $\mathcal{S}$ is not empty because of Lemma 5.6, Proposition 3.1,

and Theorem 5.1 combined with the fact that $\text{cardinal}(X(B)) = \text{cardinal}(Y(B)) \neq \aleph_0$ for all countable P -sets $B \subseteq A$. We say that $(B_\omega, T_\omega) \leq (B_\nu, T_\nu)$ if $B_\omega \subseteq B_\nu$ and T_ν extends T_ω in the sense that $\pi x_{B_\nu, B_\omega} \circ T_\nu = T_\omega \circ \pi y_{B_\nu, B_\omega}$. Then $\leq$ is a partial ordering. Furthermore, let $\{(B_\nu, T_\nu) : \nu \in M\}$ be a linearly ordered subfamily, and put $B = \bigcup B_\nu$. Then, clearly, B is φ -invariant and by TOUC, B is also a P -set. Given $x \in Y(B)$ and $\alpha \in B$, choose any ν such that $\alpha \in B_\nu$ and put $(Tx)_\alpha = (T_\nu(\pi y_{B, B_\nu}(x)))_\alpha$. One therefore easily verifies that (B, T) is a member of $\mathcal{S}$ and $(B_\nu, T_\nu) \leq (B, T)$ for all $\nu \in M$.

By Zorn's lemma, there exists a maximal member (B, T) of $\mathcal{S} \neq \emptyset$, for which B is necessarily complete. This B is also P -maximal. Indeed, let C be an equivalent P -set containing B . Then one easily verifies that C is φ -invariant and that $T' : Y(C) \rightarrow X(C)$ defined by $T' = \pi x_{C, B}^{-1} \circ T \circ \pi y_{C, B}$ is a σ -invariant completion point isomorphism which induces $\varphi|_C$. Consequently $(C, T') \in \mathcal{S}$ and $(B, T) \leq (C, T')$. Since (B, T) is maximal, we must have $C = B$. B is therefore P -maximal.

Let us now prove that $B = A$. To achieve this, suppose not and take $\alpha \in A \setminus B$. We now construct countable P -sets C, D such that

- 1) $C \subseteq B$
- 2) $C \cup \{\alpha\} \subseteq D$
- 3) C is T -invariant
- 4) D is φ -invariant
- 5) $B \cup D$ is a P -set
- 6) (C, D) is a P' -couple
- 7) Whenever $x, y \in \pi x_{B, B \cap C \cup D}(X(B))$ are such that $x_\beta = y_\beta$ for all $\beta \in C$, then $x = y$.

In the process of constructing these sets, we will freely apply Proposition 3.1, Proposition 3.2 and Lemma 5.6 without notice.

First, put $D_1 = \{\alpha\}$, $C_1 = \{\alpha'\}$ (where α' is any element in B), $E_1 = C_1 \cup D_1$, and $B_1 = \{\beta \in B : \beta \leq \lambda \text{ for some } \lambda \in E_1\} \neq \emptyset$. Then $\pi x_{E_1, B_1}(X(E_1))$ is strongly Hausdorff and Souslin. Thus, there exists $\{\alpha_{1,n}\} \subseteq B_1$ such that $(x, y \in \pi x_{E_1, B_1}(X(E_1)) \text{ and } x_{\alpha_{1,n}} = y_{\alpha_{1,n}} \text{ for all } n \geq 1) \text{ implies } x = y$. Choose a countable T -invariant P -set C_2 and a countable φ -invariant P -set D_2 such that $C_1 \cup \{\alpha_{1,n}\} \subseteq C_2 \subseteq B$, $E_1 \cup \{\alpha_{1,n}\} \subseteq D_2$, $B \cup D_2$ is a P -set, and (C_2, D_2) is a P' -couple. Put $E_2 = C_2 \cup D_2$ and $B_2 = \{\beta \in B : \beta \leq \lambda \text{ for some } \lambda \in E_2\}$. Then $\pi x_{E_2, B_2}(X(E_2))$ is strongly Hausdorff Souslin and so, there exists $\{\alpha_{2,n}\} \subseteq B_2$ such that $(x, y \in \pi x_{E_2, B_2}(X(E_2)) \text{ and } x_{\alpha_{2,n}} = y_{\alpha_{2,n}} \text{ for all } n \geq 1) \text{ implies } x = y$. Choose a countable T -invariant P -set C_3 and a countable φ -invariant P -set D_3 for which $C_2 \cup \{\alpha_{2,n}\} \subseteq C_3 \subseteq B$, $E_2 \cup \{\alpha_{2,n}\} \subseteq D_3$, $B \cup D_3$ is a P -set, and (C_3, D_3) is a P' -couple. Continuing this process, we define two increasing sequences of sets $\{C_n\}$ and $\{D_n\}$ such that if we put $C = \bigcup C_n$ and $D = \bigcup D_n$, then C and D are (countable) P -sets (by TOUC) clearly satisfying properties 1) to 4). Now, $B \cup D = \bigcup \{B \cup D_n : n \geq 2\}$ and $(C, D) = \bigcup \{(C_n, D_n) : n \geq 2\}$ and so, by TOUC, $B \cup D$ is a P -set and (C, D) is a P' -couple. It remains to show that C and D also satisfy 7).

Let $\{B_n\}$ and $\{\alpha_{n,m}\}_{n,m \geq 1}$ be defined as in the construction of C and D , and let $x, y \in \pi x_{B, B \cap C \cup D}(X(B))$ such that $x_\beta = y_\beta$ for all $\beta \in C$. If $\lambda \in B \cap C \cup D$, then there

exists n for which $\lambda \leq \beta$ for some $\beta \in D_n$. But $\lambda \in B$ and so $\lambda \in B_n \subseteq B \cap CD$. Put $x' = \pi x_{B,B_n}(x)$, $y' = \pi x_{B,B_n}(y)$. Since B is a P -set which is hereditary by HC, then $x', y' \in \pi x_{D_n,B_n}(X(D_n))$. Furthermore, $\{\alpha_{n,m}\} \subseteq C$ and so, $x'_{\alpha_{n,m}} = y'_{\alpha_{n,m}}$ for all $m \geq 1$. Hence, by definition of $\{\alpha_{n,m}\}_{m \geq 1}$, we must have $x' = y'$ and so, in particular, $x_\lambda = y_\lambda$. λ being arbitrary in $B \cap CD$, this proves that $x = y$. Property 7) is therefore satisfied and consequently C and D satisfy all the desired properties.

Now, $Y(C)$ is strongly Hausdorff Souslin and $\mathcal{F}(C)$ is the (countably generated) Borel σ -algebra on $Y(C)$. Thus, there exists a separating sequence $\{E_n\} \subseteq \mathcal{F}(C)$. For each n , there is a set $E'_n \subseteq \mathcal{B}(C)^*$ such that $T(\pi y_{B,C}^{-1}(E_n)) = \pi x_{B,C}^{-1}(E'_n)$. Hence, for all $y \in Y(C)$, there exists $E \in \mathcal{B}(C)^*$ such that $T(\pi y_{B,C}^{-1}(y)) = \pi x_{B,C}^{-1}(E)$ i.e. $\pi y_{B,C}^{-1}(y) = T^{-1}(\pi x_{B,C}^{-1}(E))$. (Note that $\pi y_{B,C}^{-1}(y) \neq \emptyset$ since C is hereditary.) Thus, using the fact that T is bijective and C is T -invariant, it follows that $E = \{x\}$ for a certain $x \in X(C)$. Put $T_C y = x$. Then, obviously, $T_C : Y(C) \rightarrow X(C)$ is a completion point isomorphism which induces $\varphi|_C$ and furthermore, T extends T_C .

Now, the countable P -set D is φ -invariant and so, by Theorem 5.1, there exists a completion point isomorphism $T'_D : Y(D) \rightarrow X(D)$ which induces $\varphi|_D$. (The fact that $\text{cardinal}(X(D)) = \text{cardinal}(Y(D)) \neq \aleph_0$ was used here.) Consider the two transformations $\pi x_{D,C} \circ T'_D$, $T_C \circ \pi y_{D,C}$. Since $\mathcal{B}(C)$ is countably generated, it is easy to prove that there exists a set $Z \in \mathcal{F}(D)$ such that $\lambda_D(Z) = 0$ and $\pi x_{D,C} \circ T'_D = T_C \circ \pi y_{D,C}$ on $Y(D) \setminus Z$. Furthermore, $\varphi|_D$ is an extension of $\varphi|_C$ which is induced by T_C and so, the fact that CCEC is satisfied implies that there exists a completion point isomorphism $T''_D : Y(D) \rightarrow X(D)$ which extends T_C . Define $T_D : Y(D) \rightarrow X(D)$ by $T_D = T''_D$ on Z and $T_D = T'_D$ on $Y(D) \setminus Z$. Then, clearly, T_D extends T_C and T_D induces $\varphi|_D$. Put $B' = B \cup D$ (which is a P -set) and define $T' : Y(B') \rightarrow X(B')$ by $\pi x_{B',\alpha}(T'y) = \pi x_{B,\alpha}(T(\pi y_{B',By}))$ if $\alpha \in B$ and by $\pi x_{B',\alpha}(T'y) = \pi x_{D,\alpha}(T_D(\pi y_{B',Dy}))$ if $\alpha \in CD$.

We now prove that T' is well-defined. To achieve this, we must show that if $E = B \cap CD$ and $y \in Y(B')$, then $\pi x_{B,E}(T(\pi y_{B',By})) = \pi x_{D,E}(T_D(\pi y_{B',Dy}))$. Put $x = \pi x_{B,E}(T(\pi y_{B',By}))$ and $x' = \pi x_{D,E}(T_D(\pi y_{B',Dy}))$. Since D is hereditary, both x and x' belong to $\pi x_{B,E}(X(B))$. Furthermore, for all $\beta \in C \subseteq E$, we have

$$\begin{aligned} x_\beta &= \pi x_{B,\beta}(T(\pi y_{B',By})) \\ &= \pi x_{C,\beta}(T_C(\pi y_{B',Cy})) \quad (\text{since } T \text{ extends } T_C) \\ &= \pi x_{D,\beta}(T_D(\pi y_{B',Dy})) \quad (\text{since } T_D \text{ extends } T_C) \\ &= x'_\beta. \end{aligned}$$

Hence, by an important property satisfied by C and D , we must have $x = x'$. Consequently T' is well-defined. Furthermore, B' is φ -invariant and T' is a σ -invariant completion point isomorphism which induces $\varphi|_{B'}$. Since B' is a P -set, $(B', T') \in \mathcal{S}$. But $(B, T) < (B', T')$ and this contradicts the fact that (B, T) is maximal in $\mathcal{S}$ and so, we must have $B = A$. This completes the proof of the first part of a). The other part is proved similarly by making an adequate use of Theorem 5.1 when needed. $\square$

We define a new condition that P and P' may satisfy, which implies CCEC and which is satisfied in some important examples. It is called the *fiber condition*. Note that

in the statement of this condition, we use the fact that any uncountable Souslin space has cardinality $\mathfrak{c}$ (See [11, p. 437].)

The fiber condition (abbreviated FC).

If B, C are countable P -sets such that $B \subseteq C$ and (C, D) is a P' -couple, then $(X(C), \mathcal{B}(C), \mu_C) = (Y(C), \mathcal{F}(C), \lambda_C)$ or $\text{cardinal}(X(C)) = \text{cardinal}(Y(C)) \neq \aleph_0$, and all fibers of the form $\pi x_{C,B}^{-1}(x_1)$ and $\pi y_{C,B}^{-1}(x_2)$ (where $x_1 \in X(B)$ and $x_2 \in Y(B)$) have same cardinality $= \ell$. Furthermore, if $\ell = \aleph_0$ or $\ell = \mathfrak{c}$, then there exist $E \in \mathcal{B}(C)$ and $F \in \mathcal{F}(C)$ such that $\mu_C(E) = \lambda_C(F) = 0$ and all fibers $E \cap \pi x_{C,B}^{-1}(x_1)$ and $F \cap \pi y_{C,B}^{-1}(x_2)$ ($x_1 \in X(B)$, $x_2 \in Y(B)$) have cardinality $= \ell$.

Theorem 5.8. *FC implies CCEC and so Theorem 5.7 remains true with FC in place of CCEC.*

Proof. Let B, C be countable P -sets such that $B \subseteq C$ and (B, C) is a P' -couple. Furthermore, let $T : Y(B) \rightarrow X(B)$ be a completion point isomorphism such that the set isomorphism $\varphi : E_X(B) \rightarrow E_Y(B)$ induced by T can be extended to a set isomorphism $\varphi' : E_X(C) \rightarrow E_Y(C)$. Assuming that FC is satisfied, we can first apply Theorem 5.1 to get a completion point isomorphism $T' : Y(C) \rightarrow X(C)$ which induces φ' . $\mathcal{B}(C)$ being countably generated, we have that $\pi x_{C,B} \circ T' = T \circ \pi y_{C,B}$ on $Y(C) \setminus Z$, where $\lambda_C(Z_Y) = 0$. We also know that all fibers of the form $\pi x_{C,B}^{-1}(x_1)$ and $\pi y_{C,B}^{-1}(x_2)$ ($x_1 \in X(B)$, $x_2 \in Y(B)$) have same cardinality $= \ell$. There are now two cases to consider (which are given in FC):

1. ℓ is finite.

Put $Z_X = T'(Z_Y) \subseteq X(C)$. Then $\mu_C(Z_X) = 0$ and $T'(Z_Y) = Z_X$. Let $y \in Y(C)$. Since $\pi x_{C,B} \circ T' = T \circ \pi y_{C,B}$ on $Y(C) \setminus Z_Y$, T' must map $(Y(C) \setminus Z_Y) \cap \pi y_{C,B}^{-1}(y)$ bijectively onto $(X(C) \setminus Z_X) \cap \pi x_{C,B}^{-1}(Ty)$. But $\text{cardinal}(\pi y_{C,B}^{-1}(y)) = \text{cardinal}(\pi x_{C,B}^{-1}(Ty)) = \ell$ and so, $Z_Y \cap \pi y_{C,B}^{-1}(y)$ and $Z_X \cap \pi x_{C,B}^{-1}(Ty)$ must have same cardinality. It is therefore possible to redefine T' on Z_Y so that T' extends T .

2. $\ell = \aleph_0$ or $\ell = \mathfrak{c}$, and there exists $E \in \mathcal{B}(C)$ and $F \in \mathcal{F}(C)$ such that $\mu_C(E) = \lambda_C(F) = 0$ and all fibers of the form $E \cap \pi x_{C,B}^{-1}(x_1)$ and $F \cap \pi y_{C,B}^{-1}(x_2)$ ($x_1 \in X(B)$, $x_2 \in Y(B)$) have cardinality $= \ell$.

Put $N_X = T'(F \cup Z_Y) \cup E$ and $N_Y = T'^{-1}(E) \cup (F \cup Z_Y)$. Then $\mu_C(N_X) = \lambda_C(N_Y) = 0$ and $T'(N_Y) = N_X$. Furthermore, for all $y \in Y(C)$, $N_Y \cap \pi y_{C,B}^{-1}(y)$ and $N_X \cap \pi x_{C,B}^{-1}(Ty)$ obviously have cardinality $= \ell$. Hence, it is possible to redefine T' on N_Y so that T' extends T .

Consequently, CCEC is satisfied and the proof is therefore complete. $\square$

The following result gives an interesting corollary of Theorems 5.7 and 5.8, to products of Souslin spaces.

Theorem 5.9. *Let i) $X = \Pi\{X_\alpha : \alpha \in A\}$ and $Y = \Pi\{Y_\alpha : \alpha \in A\}$ be uncountable products of strongly Hausdorff Souslin spaces, ii) μ and λ be respectively finite Baire measures on X and Y , and iii) φ be a (Baire) set isomorphism from X to Y .*

Then φ is induced by a σ -invariant completion (Baire) point isomorphism $T : Y \rightarrow X$.

Furthermore, if each X_α and Y_α is Polish, then T can be chosen to be a (Baire) point isomorphism.

Proof. By an argument first presented in [3, p. 115], we can assume that each X_α and Y_α is uncountable of cardinality c . Let $\mathcal{B}$ and $\mathcal{F}$ be respectively the Baire σ -algebras on X and on Y . Then, by [10, p. 43], $\mathcal{B}$ (and $\mathcal{F}$) coincides with the product of the Borel σ -algebras. We know that a product is a particular example of a projective limit (see, for example, remarks 4.3 b)), where the ordering on A is the relation of equality. Let P be the property: $B \subseteq A$ satisfies P if B is hereditary and P' be the property: (B, C) satisfies P' if B, C are countable subsets of A . Since, obviously, any subset of A is hereditary, conditions HC, TOUC, UC and EC are necessarily satisfied. (Observe that any $B \subseteq A$ is P -maximal.) To prove the first part of our result, it is therefore sufficient to show that FC is satisfied. Clearly, the proof of this fact reduces to the proof of the following:

Let 1) S_1, S_2 be two Souslin spaces such that S_2 is uncountable, and 2) m be a probability measure on $S = S_1 \times S_2$. Then, there exists a Borel set Z in S with $m(Z) = 0$ such that Z meets every vertical line segment $\pi s_{S_1}^{-1}(x)$ in a set of cardinality c , which is of the second category in itself when S_2 is Polish.

Proof. Take a sequence of compact sets $\{K_n\} \subseteq S_2$ such that $m_2(S_2 \setminus \bigcup K_n) = 0$, where m_2 denotes the measure induced by m on S_2 . If each K_n is countable, put $E = S_2 \setminus \bigcup K_n$. If not, there exists n such that K_n is uncountable. Each compact subset of S_2 being metrisable (see [19, p. 106]), K_n is therefore an uncountable Polish space and so, there exists a (necessarily uncountable) dense G_δ -set E in K_n which has m_2 -null measure. In both cases, put $Z = S_1 \times E$. Then, for all $x \in S_1$, the set $Z \cap \pi s_{S_1}^{-1}(x) = \{x\} \times E$ has cardinality c and it is obviously of the second category in itself whenever S_2 is Polish. $\square$

Let us now suppose that each X_α and Y_α is Polish (or even Lusin). Then, we prove that CEC is satisfied. Since any countable product of Polish spaces is Polish, the proof of this fact reduces to the proof of the following:

Let i) X_1, X_2, Y_1, Y_2 be uncountable Polish spaces with their Borel (=Baire)) σ -algebras, ii) m_X, m_Y be respectively finite Borel measures on $X_1 \times X_2$ and $Y_1 \times Y_2$, and iii) $T : Y_1 \rightarrow X_1$ be a point isomorphism such that the set isomorphism induced by T can be extended to a set isomorphism φ' from $Y_1 \times Y_2$ to $X_1 \times X_2$. Then there exists a point isomorphism $T' : Y_1 \times Y_2 \rightarrow X_1 \times X_2$ which extends T .

Proof. Using the above arguments as well as those used in the proof of Theorem 5.8, we get second category Borel sets $N_Y \subseteq Y_1 \times Y_2$, $N_X \subseteq X_1 \times X_2$ and a point isomorphism $T_1 : Y_1 \times Y_2 \rightarrow X_1 \times X_2$, such that $m_X(N_X) = m_Y(N_Y) = 0$, $T_1(N_Y) = N_X$, and $\pi x_1 \circ T_1 = T \circ \pi y_1$ on $(Y_1 \times Y_2) \setminus N_Y$. According to Mauldin [16, p. 230], there exist Borel parameterizations of N_X and N_Y i.e. there exist uncountable Borel sets $E_X \subseteq X_2$, $E_Y \subseteq Y_2$ and (Borel) isomorphisms $g_Y : Y_1 \times E_Y \rightarrow N_Y$, $g_X : X_1 \times E_X \rightarrow N_X$, such that for each $y \in Y_1$ and each $x \in X_1$, $g_Y(y, \cdot)$ maps $\{y\} \times E_Y$ onto $N_Y^y = \pi y_1^{-1}(y) \cap N_Y$ and $g_X(x, \cdot)$ maps $\{x\} \times E_X$ onto $N_X^x = \pi x_1^{-1}(x) \cap N_X$. Furthermore, by [11, p. 442], there exists a Borel isomorphism

$r : E_Y \rightarrow E_X$. Now, define $R : N_Y \rightarrow N_X$ by $Rx = g_X(Tx', r(y'))$, where $(x', y') = g_Y^{-1}x$. Then, obviously, R is a Borel isomorphism. Now, if $x \in Y_1$, then one easily proves that R maps N_Y^x onto N_X^{Tx} . Finally, put $T' = T_1$ on $(Y_1 \times Y_2) \setminus N_Y$ and for all $x \in N_Y$, put $T'x = Ry$ where $y = \pi_{Y_1}(x)$. Then $T' : Y_1 \times Y_2 \rightarrow X_1 \times X_2$ is a point isomorphism which extends T . $\square$

The proof of Theorem 5.9 is complete. $\square$

Remark 5.10. The result in Theorem 5.9 was first proved in [14, Theorem 5.11] using arguments relative to products, instead of the more general techniques used in the present paper. Note that this result generalizes the main theorem in [2] as well as [5, Theorem 1, p. 114].

A lattice $\mathcal{C}$ of $\mathbb{F}$ -valued functions is said to be a *strong lattice* if $|f| \in \mathcal{C}$ whenever $f \in \mathcal{C}$. Using this definition, we obtain from Theorems 5.1 and 5.9 the following very general result which is a slight generalization of Theorem 6.1 in [14].

Theorem 5.11. *Let i) $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{F}, \lambda)$ be σ -finite Baire measure spaces of arbitrary products (over the same set A) of strongly Hausdorff Souslin spaces, such that X, Y are uncountable, ii) $\mathcal{V}$ and $\mathcal{W}$ be respectively self-adjoint strong sublattices of $M(X, \mathcal{B}, \mathbb{F})$ and $M(Y, \mathcal{F}, \mathbb{F})$, which are closed under multiplication by characteristic functions of sets of finite measure, and which contain the class of all integrable simple functions, and iii) $U : \mathcal{V} \rightarrow \mathcal{W}$ be an e -linear support separating operator which satisfies conditions a) and b) of Theorem 4.5 and which is invertible in the sense given in the statement of Theorem 5.1.*

Then, there exists 1) a measurable function $h : Y \rightarrow \mathbb{F}$ having support equal to Y and 2) a completion (Baire) point isomorphism $T : Y \rightarrow X$, such that

$$Uf \cong h(f \circ T) \quad \text{for all } f \in \mathcal{V}$$

and

$$U^{-1}g \cong (1/h \circ T^{-1}) \cdot (g \circ T^{-1}) \quad \text{for all } g \in \mathcal{W}.$$

Furthermore, if each X_α and Y_α is Polish or, more generally, Lusin, then T can be chosen to be a (Baire) point isomorphism.

Proof. Identical to that of Theorem 6.1 in [14]. $\square$

Using the notations and definitions introduced just before Corollary 4.6, we obtain the following new result:

Corollary 5.12. *Let i) $(X, \mathcal{B}, \mu)$, $(Y, \mathcal{F}, \lambda)$ be as in Theorem 5.11, ii) ϕ_1, ϕ_2 be two strictly increasing continuous functions such that $\phi_1(0) = \phi_2(0) = 0$, and iii) $U : L_{\phi_1}(X, \mu, \mathbb{F}) \rightarrow L_{\phi_2}(Y, \lambda, \mathbb{F})$ be an invertible bounded (in the sense given in Corollary 4.6) linear support separating operator.*

Then there exists 1) a function $h \in M(Y, \mathcal{F}, \mathbb{F})$ and 2) a completion point isomorphism $T : Y \rightarrow X$ such that

$$Uf \cong h(f \circ T) \quad \text{for all } f \in L_{\phi_1}(X, \mu, \mathbb{F}).$$

Furthermore, if $\phi_1(\sqrt{t})$ and $\phi_2(\sqrt{t})$ are either strictly convex or strictly concave, and if we replace iii) by iii)' $U : L_{\phi_1}(X, \mu, \mathbb{F}) \rightarrow L_{\phi_2}(Y, \lambda, \mathbb{F})$ is an invertible linear operator satisfying: $I_{\phi_1}(Uf) = I_{\phi_2}(f)$, then there exists a completion point isomorphism $T : Y \rightarrow X$ such that

$$Uf \cong h(f \circ T) \text{ for all } f \in L_{\phi_1}(X, \mu, \mathbb{F}),$$

$$\text{where } \phi_2(|h|) = \frac{d(\mu \circ T)}{d\lambda} \text{ and } \phi_1(1/|h| \circ T^{-1}) = \frac{d(\lambda \circ T^{-1})}{d\mu}.$$

Proof. Obviously, the first part of our corollary follows directly from Theorem 5.12. Now, to obtain the second part, note that $U : L_{\phi_1}(X, \mu, \mathbb{F}) \rightarrow L_{\phi_2}(Y, \lambda, \mathbb{F})$ given by iii)' is such that U and U^{-1} are bounded. By Theorem 2.1 in [12], U and U^{-1} are also support separating. (The fact that $\phi_1(\sqrt{t})$ and $\phi_2(\sqrt{t})$ are either strictly convex or strictly concave is essential here.) Hence, we obtain $T : Y \rightarrow X$ by applying the first part of this corollary. Finally, the fact that $\phi_2(|h|) = d(\mu \circ T)/d\lambda$ follows easily from the relation $I_{\phi_1}(Uf) = I_{\phi_2}(f)$ applied to $f = \chi_E$ ($E \in \mathcal{B}$), and the identity $\phi_1(1/|h| \circ T^{-1}) = d(\lambda \circ T^{-1})/d\mu$ is proved similarly by using the relation $U^{-1}g \cong (1/h \circ T^{-1}) \cdot (g \circ T^{-1})$ ($g \in L_{\phi_2}(Y, \lambda, \mathbb{F})$) which is obtained from Theorem 5.11. The proof is complete. $\square$

6. Applications to homogeneous spaces. Let G be a locally compact σ -compact group. As in [4], we designate by $\mathcal{K}$ the collection of all compact normal subgroups of G , and by $\mathcal{H}$ the subcollection consisting of all $H \in \mathcal{K}$ such that G/H is metrisable with a countable basis. (Note that in this section, we suppose, for convenience, that all cosets are right cosets.) If H is any subgroup of G , we designate by $\mathcal{B}_H$ the Baire σ -algebra on G/H . For all subgroups $H, K \subseteq G$ such that $K \supseteq H$, we denote by $\pi_{H,K}$ the natural projection from G/H onto G/K .

Now, let L be an arbitrary closed subgroup of G which will be held fixed in the sequel. Then, if $\{K_i : i \in I\}$ is a directed subset of $\mathcal{H}$ (and even of $\mathcal{K}$) such that $K_i \supseteq K_j$ if $i \leq j$, then an easy topological argument shows that $L(\bigcap K_i) = \bigcap LK_i$ (see [4, p. 271]). Hence, by [6, p. 287], we deduce that $LK = \bigcap \{LH : H \in \mathcal{H}; H \supseteq K\}$ for all $K \in \mathcal{K}$. In particular, $L = \bigcap \{LH : H \in \mathcal{H}\}$.

For all $B \subseteq \mathcal{H}$, we put $G(B) = \text{projlim}_{H \in \mathcal{CB}} G/LH$ and $\mathcal{B}(B) = \text{projlim}_{H \in \mathcal{CB}} \mathcal{B}_{LH}$. (Recall that $\mathcal{CB} = \{H \in \mathcal{H} : H \supseteq K \text{ for some } K \in B\}$.) Clearly, G/LH is σ -compact and metrisable with a countable basis. $\mathcal{B}_{LH}$ therefore coincides with the (countably generated) Borel σ -algebra of the Lusin space G/LH . (In fact, G/LH is even Polish.) As in the preceding section, $\pi_{B,C}$ denotes the natural projection from $G(B)$ to $G(C)$.

We now adopt a convenient notation: If $B \subseteq \mathcal{H}$ and $H \in \mathcal{K}$ are such that $LH \subseteq \bigcap \{LK : K \in B\}$, then we write $G/LH = G(B)$ if

- 1) the natural projection $\pi_{LH,B}$ from G/LH to $G(B)$ is a homeomorphism and
- 2) $\mathcal{B}(B)$ coincides with the Baire σ -algebra on $G(B)$.

Hence, $G/LH = G(B)$ implies that $\pi_{LH,B}$ is a measurable isomorphism from $(G/LH, \mathcal{B}_{LH})$ to $(G(B), \mathcal{B}(B))$.

The first goal of this section is to give an alternative proof of the main result in [4, Theorem 1], using our Theorems 5.7 and 5.8. (Later on, we will use this proof

to obtain an important generalization of [4, Theorem 1].) Because of the use of these theorems, the proof given constitutes a simplification of the arguments used in [3] and [4]. Indeed, the techniques developed in the proof of Theorem 5.7 are based on the possibility of using projective limits over a not necessarily directed set. This simplifies the process of induction as well as the proof that the maximal element (B, T) is such that $B = A$. However, in order to use Theorems 5.7 and 5.8, strategic properties P (relative to the class of all subsets of $\mathcal{H}$) and P' (relative to the class of all couples of countable subsets of $\mathcal{H}$) are needed for which conditions HC, TOUC, EC, UC, and FC hold. The properties will be defined later but for now, some important theoretic results on groups are needed, proof of which are essentially given in [3] or [4].

Lemma 6.1. *If H, K are subgroups of G , then $Hx \cap Ky \neq \emptyset$ iff $\pi_{H, HK}(x) = \pi_{K, HK}(y)$.*

Lemma 6.2. *Let $K \in \mathcal{K}$ and $H_0 \in \mathcal{H}$. Then there exists $H \in \mathcal{H}$ such that $H \subseteq H_0$ and $L(H \cap K) = LH \cap LK$.*

Lemma 6.3. *Let $B \subseteq \mathcal{H}$ and put $H = \bigcap \{K : K \in B\}$. If $LH = \bigcap \{LK : K \in B\}$ and the natural projection (denoted $\pi_{G, B}$) from G into $G(B)$ is surjective, then $G/LH = G(B)$.*

Corollary 6.4. *If $B \subseteq \mathcal{H}$ is directed, then $G/LH = G(B)$, where $H = \bigcap \{K : K \in B\}$.*

Corollary 6.5. *For all $H \in \mathcal{K}$, $G/LH = G(\{LK : K \in \mathcal{H}; K \supseteq H\})$. In particular, $G/L = G(\mathcal{H})$.*

We now prove Theorem 1 in [4].

Theorem 6.6. *Let i) M be a locally compact homogeneous space acted upon by a locally compact σ -compact group G , ii) $\mathcal{B}$ be the Baire σ -algebra on M , and iii) μ be a finite (or even σ -finite) measure on $(M, \mathcal{B})$.*

Then any set automorphism φ on M , is induced by a completion (Baire) point automorphism $T : M \rightarrow M$.

Proof. First, by [17, p. 148], M is homeomorphic to G/L , where L is a closed subgroup of G . Furthermore, by Corollary 6.5, we have $G/L = G(\mathcal{H})$. Next, let P and P' be the properties defined by

$B \subseteq \mathcal{H}$ is a P -set iff $\pi_{\mathcal{H}, B}$ is surjective (or, equivalently, $\pi_{L, B}$ is surjective) and $LH = \bigcap \{LK : K \in B\}$, where $H = \bigcap \{K : K \in B\}$.
 $(B, C) \subseteq \mathcal{H} \times \mathcal{H}$ is a P' -couple iff B, C are countable subsets of $\mathcal{H}$ and C is directed (with respect to $\supseteq$).

Then, by Lemma 6.3 and Corollary 6.4, every P -set B is such that $G/LH = G(B)$ ($H = \bigcap \{K : K \in B\}$) and each directed set is a P -set. Furthermore, it is easy to show that the P -maximal sets are precisely the sets of the form $\{K : K \in \mathcal{H}; K \supseteq H\}$ where $H \in \mathcal{K}$ is such that whenever $N \in \mathcal{K}$ satisfies $LN = LH$ and $N \subseteq H$, then $N = H$. Now, we prove that P and P' satisfy the conditions HC, TOUC, EC, UC and FC. Our result will therefore follow from Theorems 5.7 and 5.8. Clearly, P' satisfies TOUC 2) and EC 2) and thus, the proof is reduced to the following steps.

Step 1. *P satisfies HC.*

Proof. Clearly, $\mathcal{H}$ is a P -set and by definition of P , $\pi_{\mathcal{H},B}$ is surjective which means that B is hereditary. $\square$

Step 2. *P satisfies TOUC 1).*

Proof. Let $\{B_i : i \in I\}$ be a totally ordered collection of P -sets. Put $B_i = \{K_{i,j} : j \in A_i\} \subseteq \mathcal{H}$, $B = \bigcup \{B_i : i \in I\}$, $K_i = \bigcap \{K_{i,j} : j \in A_i\} = \bigcap \{K : K \in B_i\}$, and $H = \bigcap \{K_i : i \in I\} = \bigcap \{K : K \in B\}$. Then

$$\begin{aligned} LH &= \bigcap \{LK_i : i \in I\} && (\text{since } \{B_i : i \in I\} \text{ is directed}) \\ &= \bigcap \bigcap \{LK_{i,j} : j \in A_i; i \in I\} && (\text{since each } B \text{ is a } P\text{-set}) \\ &= \bigcap \{LK : K \in B\}. \end{aligned}$$

We now prove that $\pi_{L,B}$ is surjective. First, we easily get that for all $K \in B$, the reciprocal image under $\pi_{L,LK}$ of any compact set is compact. Let $x \in G(B)$ and put $x_K = \pi_{B,LK}(x)$ ($K \in B$). If $\{K_i : i = 1, \dots, n\} \subseteq B$, then the fact that $\{B_i : i \in I\}$ is totally ordered implies that there is an i with $\{K_i : i = 1, \dots, n\} \subseteq B_i$. Since B_i is a P -set, π_{L,B_i} is surjective and hence $\bigcap \{\pi_{L,LK_i}^{-1}(x_{K_i}) : i = 1, \dots, n\} \neq \emptyset$. Consequently, $\{\pi_{L,LK}^{-1}(x_K) : K \in B\}$ is a collection of compact subsets of G/L possessing the finite intersection property and so, there exists $y \in \bigcap \{\pi_{L,LK}^{-1} : K \in B\}$. Since $\pi_{L,B}(y) = x$, the proof is complete. $\square$

Step 3. *P satisfies EC 1).*

Proof. Let B be P -maximal and C be a countable subset B . Then there exists $H \in \mathcal{K}$ such that $B = \{LK : K \in \mathcal{H}; K \supseteq H\}$. Put $D_1 = \{\bigcap K_i : \{K_i : i = 1, \dots, n\} \subseteq C\}$, $D_2 = \{\bigcap K_i : \{K_i : i = 1, \dots, n\} \subseteq D_1\}, \dots$ Finally, put $D = \bigcup D_n$. Then D is countable and directed (and so it is a P -set), and is obviously contained in B . $\square$

Step 4. *P satisfies UC.*

Proof. Let B be P -maximal and let $H \in \mathcal{K}$ be such that $B = \{LK : K \in \mathcal{H}; K \supseteq H\}$. Furthermore, let C be a countable P -set. By step 3, C is contained in a countable directed P -set D' . Put $N = \bigcap \{K : K \in D'\}$. Then applying Lemma 6.3 with $L = \{e\}$, we see that $G/N = \text{projlim}\{G/K : K \supseteq K' \text{ for some } K' \in D'\}$. Hence, G/N has a countable basis and consequently, $N \in \mathcal{H}$. By Lemma 6.2, there exists $M \in \mathcal{H}$ such that $N \supseteq M$ and $L(H \cap M) = LH \cap LM$. Put $D = D' \cup \{M\} \cup \{HM\} \supseteq C$. Then D is a countable directed P -set.

Now, let $x \in G(B \cup D)$ and put $x_B = \pi_{B \cup D, B}(x) \in G(B) = G/LH$, $x_D = \pi_{B \cup D, D}(x) \in G(D) = G/LM$. Observe that $HM \in B \cup D$ and $(LH)(LM) = LHM$. Hence, x_B and x_D have the same projection in G/LHM . By Lemma 6.1, there exists $y \in G/L$ such that $\pi_{L,B}(y) = x_B$ and $\pi_{L,D}(y) = x_D$. Hence, $\pi_{L, B \cup D}$ must be surjective. Since $L(H \cap M) = LH \cap LM$, it is obvious that $L(\bigcap \{K : K \in B \cap D\}) = \bigcap \{LK : K \in B \cup D\}$. Consequently, $B \cup D$ is a P -set. $\square$

Step 5. P and P' satisfy FC.

Proof. Let B, C be countable P -sets such that $B \subseteq C$ and C is directed. Put $K = \bigcap \{K' : K' \in C\}$ and $H = \bigcap \{K' : K' \in B\} \supseteq K$. Then $G/LK = G(C)$ and $G/LH = G(B)$. Since C is directed, then as in step 4, we can show that $K \in \mathcal{H}$. (Consequently, H is also a member of $\mathcal{H}$ since $G/H = (G/K)/(H/K)$.) Put $G' = G/K$, $L' = LK/K$ and $H' = H/K$. Then $G'/L' = G/LK$ and $G'/L'H' = G/LH$. The remainder of the proof of step 5 is identical to that of Lemma 6 in [4].

The proof of Theorem 6.6 is now complete. $\square$

Remark 6.7. Using an argument developed in the previous section, we see that the proof of Lemma 6 in [4] contains the fact that whenever $H \supseteq K$ and LH/LK is uncountable, then CEC holds with $X(B) = G/LH$ and $X(C) = G/LK$. However, when LH/LK is finite and Z is a Borel subset of $S \times (LH/LK)$ with ν -null measure (where S is a Borel section of G/LH in G/LK and ν is the measure induced by μ_K on $S \times (LH/LK)$), there is not necessarily a Borel parameterization of Z in $S \times (LH/LK)$. Hence, it is not guaranteed that CEC holds in this case. We must therefore be satisfied with a completion point isomorphism in the conclusion of Theorem 6.6.

The next objective of this section is to generalize Theorem 6.6 to an arbitrary product M of locally compact homogeneous spaces $\{M_\alpha : \alpha \in A\}$, where M_α is under the action of a locally compact σ -compact group G_α . We already know that M_α is homeomorphic to G_α/L_α , where L_α is a closed subgroup of G_α . Hence, M is homeomorphic to G/L , where $G = \prod \{G_\alpha : \alpha \in A\}$ and $L = \prod \{L_\alpha : \alpha \in A\}$. But G is not necessarily locally compact or σ -compact. The main theorem of this section (see Theorem 6.10 below) can therefore not be proved using the techniques of [3] or [4]. We will however prove this theorem using our flexible results obtained in section 5. The idea is to consider G/L as being a projective limit of Souslin (and even Polish) spaces and to define strategic properties P and P' which will suit our purpose. For a better understanding of the sequel, we refer the reader to the terminology introduced in section 2 about projective limits.

Theorem 6.10. *Let i) M be an arbitrary product of locally compact homogeneous spaces $\{M_\alpha\}_{\alpha \in A}$, where M_α is acted upon by a locally compact σ -compact group G_α , ii) $\mathcal{B}$ be the Baire σ -algebra on M , and iii) μ be an arbitrary σ -finite measure on M .*

Then any (Baire) set automorphism φ on M is induced by a completion (Baire) point automorphism $T : M \rightarrow M$.

Proof. First, we know that there exists a closed subgroup L_α of G_α such that M is homeomorphic to $\prod \{G_\alpha : \alpha \in A\} / \prod \{L_\alpha : \alpha \in A\}$ or, equivalently, to $\prod \{G_\alpha/L_\alpha : \alpha \in A\}$. We may thus suppose that $M = \prod \{G_\alpha/L_\alpha : \alpha \in A\}$. We may also suppose, without loss of generality, that each factor G_α/L_α has at least two points. For all $\alpha \in A$, denote by $\mathcal{K}_\alpha$ the collection of all compact subgroups of G_α and let $\mathcal{H}_\alpha$ be the subcollection consisting of all $H_\alpha \in \mathcal{K}_\alpha$ such that G_α/H_α has a countable basis. We distinguish two cases: 1) A is countable, and 2) A is uncountable.

1) A is countable

Consider the set $\mathcal{A} = \prod \{\mathcal{H}_\alpha : \alpha \in A\}$ and the relation $\leq$ on $\mathcal{A}$ defined by $(H_\alpha)_{\alpha \in A} \leq (K_\alpha)_{\alpha \in A}$ if $H_\alpha \supseteq K_\alpha$ for all $\alpha \in A$. Next, consider the projective

limit of Polish spaces $X = \text{projlim}_{\Lambda \in \mathcal{A}} X_\Lambda$, where $X_\Lambda = \Pi\{G_\alpha/L_\alpha K_\alpha : \alpha \in \Lambda\}$ ($\Lambda = (K_\alpha)_{\alpha \in A}$). Furthermore, for each $\Lambda \in \mathcal{A}$, denote by $\mathcal{B}_\Lambda$ the Borel σ -algebra on X_Λ (which coincides with the product of the Borel σ -algebras). For all $\alpha \in A$, let P be the property (relative to the subsets of $\mathcal{H}_\alpha$) defined the same way property P was defined in the proof of Theorem 6.6. Next, let P and P' be defined by

$B \subseteq \mathcal{A}$ is a P -set iff for all $\alpha \in A$, $B^\alpha = \{K_\alpha \in \mathcal{H}_\alpha : \text{there is a } (N_\lambda)_{\lambda \in A} \in B \text{ with } K_\alpha = N_\alpha\}$ is a P -set.

$(B, C) \subseteq \mathcal{A} \times \mathcal{A}$ is a P' -couple iff B, C are countable subsets of $\mathcal{A}$ such that C is directed (with respect to $\leq$ on $\mathcal{A}$).

If B is a P -set and if we put $H_\alpha = \bigcap\{K_\alpha : K_\alpha \in B^\alpha\}$, then using Lemma 6.3, we get $X(B) = \Pi\{G_\alpha/L_\alpha H_\alpha : \alpha \in A\}$ i.e. the natural projection map from $\Pi\{G_\alpha/L_\alpha H_\alpha : \alpha \in A\}$ to $X(B)$ is a homeomorphism as well as an invertible measurable transformation. (In particular, we get that X and M are isomorphic as measurable spaces.) One of the important consequences of this is that whenever B is P -maximal, then each B^α is P_α -maximal and further, $B = \Pi\{B^\alpha : \alpha \in A\}$. Thus, using the fact that P_α satisfies HC, TOUC 1), EC 1), UC for all $\alpha \in A$ (see the proof of theorem 6.6), we can easily verify that P also satisfies these conditions. For example, let us show that P satisfies EC 1). Let B be a P -maximal set and C be a countable subset of B . Let $\alpha \in A$ be such that $\text{cardinality}(C^\alpha) > 1$. Since B^α is P_α -maximal and $C^\alpha \subseteq B^\alpha$, then by step 3 of the proof of Theorem 6.6, there exists a countable directed P_α -set D such that $C^\alpha \subseteq D_\alpha \subseteq B^\alpha$. Put $H_\alpha = \bigcap\{K_\alpha : K_\alpha \in D_\alpha\}$ (which belongs to B^α) and $D = C \cup \{(H_\alpha)_{\alpha \in A}\}$. Then D is a countable P -set which is contained in B . Thus, P satisfies EC 1). Similarly, P satisfies UC and obviously, P also satisfies HC and TOUC 1). The fact that P' satisfies TOUC 2) and EC 2) is also very easy to establish. Hence, to complete the proof of case 1), it remains to show that P and P' satisfy FC.

Let B, C be countable P -sets such that $B \subseteq C$ and (B, C) is a P' -couple (i.e. C is directed). For all $\alpha \in A$, put $K'_\alpha = \bigcap\{K'_\alpha : K'_\alpha \in C^\alpha\}$ and $H_\alpha = \bigcap\{K'_\alpha : K'_\alpha \in B^\alpha\} \supseteq K_\alpha$. Since C^α is directed, then $K_\alpha \in \mathcal{H}_\alpha$ and since $H_\alpha \supseteq K_\alpha$, H_α also belongs to $\mathcal{H}_\alpha$. Further, we have already shown that $X(C) = \Pi\{G_\alpha/L_\alpha K_\alpha : \alpha \in A\} = G/LK$ and $X(B) = \Pi\{G_\alpha/L_\alpha H_\alpha : \alpha \in A\} = G/LH$, where $G = \Pi\{G_\alpha : \alpha \in A\}$, $L = \Pi\{L_\alpha : \alpha \in A\}$, $K = \Pi\{K_\alpha : \alpha \in A\}$, and $H = \Pi\{H_\alpha : \alpha \in A\}$. Put $G'_\alpha = G_\alpha/K_\alpha$, $L'_\alpha = L_\alpha K_\alpha/K_\alpha$ and $H'_\alpha = H_\alpha/K_\alpha$, and define G', L', H' in the obvious usual way. Then, $L'H'/L'$ is either finite or uncountable of cardinality c . Furthermore, $G'/L'H'$ has a Borel section R_α in G' . Hence, $R = \Pi\{R_\alpha : \alpha \in A\}$ is a Borel section of $G'/L'H'$ in G' . If we suppose that $L'H'/L'$ is uncountable, the proof that P, P' satisfy FC is identical to that of Lemma 6 in [4]. Thus, the result for case 1) follows from Theorems 5.7 and 5.8 combined with the fact that $M = X$.

2) A is uncountable

First, partition A into disjoint subsets β , such that each β has cardinality $\aleph_0$. If A_0 denotes the set of subsets of this partition, then $\text{cardinal}(A_0) = \text{cardinal}(A)$, and if $X^\beta = \Pi\{G_\alpha/L_\alpha : \alpha \in \beta\}$, then we know that the Borel σ -algebra on X^β coincides with the product of the Borel σ -algebras. Hence, M and $M_0 = \Pi\{X^\beta : \beta \in A_0\}$ are (Baire) isomorphic. Further, we saw in case 1) that each X^β is a projective limit of Polish spaces. In fact, $X^\beta = \text{projlim}_{\Lambda \in \mathcal{A}_\beta} X_\Lambda^\beta$, where $\mathcal{A}_\beta = \Pi\{\mathcal{H}_\alpha : \alpha \in \beta\}$ and

$X_\Lambda^\beta = \Pi\{G_\alpha/L_\alpha K_\alpha : \alpha \in \beta\}$ ($\Lambda = (K_\alpha)_{\alpha \in \beta}$). It is also known that the Baire σ -algebra on X^β coincides with $\mathcal{B}^\beta = \text{projlim}_{\Lambda \in \mathcal{A}_\beta} \mathcal{B}_\Lambda^\beta$, where $\mathcal{B}_\Lambda^\beta = \text{Borel } \sigma\text{-algebra on } X_\Lambda^\beta = \text{product of the Borel } \sigma\text{-algebras}$. Since each G_α/L_α has at least two points, then by removing from $\mathcal{H}_\alpha$ and $\mathcal{K}_\alpha$ all the subgroups K_α such that $L_\alpha K_\alpha = G_\alpha$, we easily see that the arguments developped in case 1) remain unchanged. In particular, the equations $X^\beta = \text{projlim}_{\Lambda \in \mathcal{A}_\beta} X_\Lambda^\beta$ and $\mathcal{B}^\beta = \text{projlim}_{\Lambda \in \mathcal{A}_\beta} \mathcal{B}_\Lambda^\beta$ are still valid and furthermore, all the Polish spaces X_Λ^β are uncountable of cardinality c .

Now, let $\mathcal{A} = \bigcup\{\mathcal{A}_\beta : \beta \in A_0\}$ and define $\leq$ as follows: $\Lambda \leq \Lambda'$ iff $\Lambda = \Lambda'$ or else, Λ, Λ' belong to the same $\mathcal{A}_\beta$ and $\Lambda \leq \Lambda'$ in $\mathcal{A}_\beta$ (see case 1)). For each $\Lambda \in \mathcal{A}$, put $X_\Lambda = X_\Lambda^\beta$ and $\mathcal{B}_\Lambda = \mathcal{B}_\Lambda^\beta$, where β is the only $\gamma \in A_0$ for which $\Lambda \in \mathcal{A}_\gamma$. Then, clearly, M_0 coincides with $X = \text{projlim}_{\Lambda \in \mathcal{A}} X_\Lambda$ (where each X_Λ is an uncountable Polish space) and $\mathcal{B}_0$, the Baire σ -algebra on M_0 , coincides with $\text{projlim}_{\Lambda \in \mathcal{A}} \mathcal{B}_\Lambda$. For all $\beta \in A_0$, let P_β be the property (relative to the subsets of $\mathcal{A}_\beta$) defined the same way property P was defined in case 1). Next, let P and P' be defined as follows:

$B \subseteq \mathcal{A}$ is a P -set iff $B \cap \mathcal{A}_\beta$ is a P_β -set for all β such that $B \cap \mathcal{A}_\beta \neq \emptyset$.
 $(B, C) \subseteq \mathcal{A} \times \mathcal{A}$ is a P' -couple iff B, C are countable subsets of $\mathcal{A}$ and $C \cap \mathcal{A}_\beta$ is directed whenever $C \cap \mathcal{A}_\beta \neq \emptyset$.

Obviously, if $B \subseteq \mathcal{A}$ is such that $B \cap \mathcal{A}_\beta \neq \emptyset$ (for some $\beta \in A_0$), then $B \cap \mathcal{A}_\beta$ is P_β -maximal whenever B is P -maximal. Thus, the fact that P satisfies HC, TOUC 1), EC 1) and UC for all $\beta \in A_0$ (see case 1)), easily implies that P satisfies the same conditions. Further, P' clearly satisfies TOUC 2) and EC 2) and so, it remains to show that P and P' satisfy FC.

Let B, C be countable P -sets such that $B \subseteq C$ and (B, C) is a P' -couple. Put $C' = \bigcup\{C \cap \mathcal{A}_\beta : B \cap \mathcal{A}_\beta \neq \emptyset\}$. Then, using the arguments developped in case 1), $X(B)$ and $X(C')$ are of the form $X_1 = \Pi\{G_\alpha/L_\alpha H_\alpha : \alpha \in \mathcal{C}\}$ and $Y_1 = \Pi\{G_\alpha/L_\alpha K_\alpha : \alpha \in \mathcal{C}\}$, where $\mathcal{C}$ is a countable subset of A and $H_\alpha \supseteq K_\alpha \in \mathcal{H}_\alpha$ (which implies $H_\alpha \in \mathcal{H}_\alpha$) for all $\alpha \in \mathcal{C}$. If $C = C'$, then FC follows easily as in case 1). If $C \setminus C' \neq \emptyset$, then $X(C) = Y_1 \times Y_2$, where Y_2 is an uncountable Polish space. Hence, there exists an uncountable Borel set $E_2 \subseteq Y_2$ such that $m(E_2) = 0$, where m is the contraction of μ to $Y_2 = X(\bigcup\{C \cap \mathcal{A}_\beta : B \cap \mathcal{A}_\beta \neq \emptyset\})$. Consequently, $\mu_C(Y_1 \times E_2) = 0$ and $(Y_1 \times E_2) \cap \pi_{C, B}^{-1}(x)$ has cardinality c for all $x \in X(B) = X_1$. FC is therefore satisfied and thus, the result for case 2) follows from Theorems 5.7 and 5.8 combined with the fact that $M = M_0 = X$.

The proof of Theorem 6.10 is complete. $\square$

Remark 6.11. Theorem 5.11 and Corollary 5.12 remain true if in the statements of these results, we put $(X, \mathcal{B}, \mu) = (Y, \mathcal{F}, \lambda) = (M, \mathcal{B}, \mu)$ where this last triplet is defined as in Theorem 6.10.

Résumé substantiel en français. Un problème intéressant en théorie de la mesure est le suivant: sous quelles conditions un automorphisme de σ -algèbres φ sur un espace mesuré σ -fini $(X, \mathcal{B}, \mu)$ peut-il être induit par une transformation $T : X \rightarrow X$ mesurable par rapport à la complétion de (X, μ) ? Un problème encore plus intéressant est de trouver les conditions sous lesquelles T peut être choisie inversible lorsque φ est inversible. Il est bien connu que le second problème ne peut être toujours résolu

(voir [18], [2, p. 97] et [3, Theorem 0]) et ce, même si $\mathcal{B}$ contient un sous-ensemble dénombrable, séparant et générateur (voir [7, Lemma 12]).

Von Neumann a été un des premiers mathématiciens à s'intéresser au second problème. Dans [20], il a démontré que tout automorphisme de σ -algèbres φ sur un espace mesuré fini $(X, \mathcal{B}, \mu)$, où X est un espace polonais et $\mathcal{B}$ est la σ -algèbre de Borel sur X , est induit par une transformation $T : X \rightarrow X$ mesurable et inversible. Ce résultat a été généralisé par Dorothy Maharam [15] au produit quelconque de tels espaces mesurés. Inspirés par les techniques utilisées par Maharam dans son article, Choksi [2] a amélioré le résultat de Maharam en démontrant que les mêmes conclusions s'appliquent lorsque la mesure sur le produit des espaces mesurables $(X_\alpha, \mathcal{B}_\alpha)$ (où X_α est un espace polonais et $\mathcal{B}_\alpha$ est la σ -algèbre de Borel sur X_α) est une mesure de Baire σ -finie quelconque et non seulement une mesure produit. Dans [3], il a été démontré que n'importe quel automorphisme de σ -algèbres de Baire sur un groupe compact est induit par une transformation ponctuelle inversible et mesurable de Baire. Afin de démontrer ce résultat, Choksi a adapté les techniques utilisées dans [2] aux limites projectives d'espaces topologiques. Dans [4], le même auteur a démontré que les mêmes conclusions s'appliquent à n'importe quel espace localement compact homogène sous l'action d'un groupe localement compact σ -compact.

L'un des objectifs de cet article est de présenter une synthèse de tous les résultats énoncés dans le paragraphe précédent. Cela est fait en démontrant un théorème très général sur les limites projectives d'espaces de Souslin. Ce théorème généralise les résultats principaux de [2, 15, 20] car il permet d'obtenir le corollaire suivant :

Soit 1) $X = \prod \{X_\alpha, \alpha \in A\}$ et $Y = \prod \{Y_\alpha, \alpha \in A\}$ deux produits d'espaces de Souslin complètement réguliers tels que X et Y soient non dénombrables, et 2) μ et λ deux mesures de Baire σ -finies sur X et Y respectivement. Alors, tout isomorphisme de σ -algèbres de X vers Y est induit par une transformation ponctuelle inversible qui est mesurable par rapport à la complétion de X . Lorsque X et Y sont des espaces polonais, la transformation ponctuelle peut être choisie de manière à être mesurable sur X .

Notre théorème contient également le résultat suivant :

Soit 1) M un produit arbitraire d'espaces localement compacts homogènes $\{M_\alpha\}_{\alpha \in A}$, où M_α est sous l'action d'un groupe G_α localement compact et σ -compact, 2) $\mathcal{B}$ la σ -algèbre de Baire sur M , et 3) μ une mesure arbitraire σ -finie sur M . Alors tout isomorphisme de σ -algèbres sur $(X, \mathcal{B}, \mu)$ est induit par une transformation ponctuelle inversible qui est mesurable par rapport à la complétion de M .

Ce résultat généralise les théorèmes principaux de [3] et [4] tout en simplifiant leurs preuves. En effet, lorsque ces théorèmes sont démontrés dans le cadre des limites projectives d'espaces de Souslin, les raisonnements spécifiques aux groupes sont minimisés et les autres détails sont automatiquement vérifiés dans notre théorème principal.

Dans cet article, nous démontrons également un résultat très général concernant les homomorphismes de σ -algèbres (non nécessairement inversibles) qui sont induits par une transformation ponctuelle (non nécessairement inversible). Ce résultat est énoncé dans le contexte des limites projectives d'espaces de Hausdorff complètement réguliers. Notons qu'il existe des exemples d'espaces mesurés σ -finis $(X, \mathcal{B}, \mu)$ pour lesquels il est possible de construire un homomorphisme de σ -algèbres sur X qui ne peut être

induit par une transformation ponctuelle $T : X \rightarrow X$ qui est mesurable par rapport à la complétion de X (voir [7, Lemma 12]).

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