

A TRANSPORT FUNCTIONAL EQUATION

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RÉSUMÉ. On étudie un problème à données initiales pour une équation aux dérivées partielles du premier ordre, avec un terme non local. On démontre l'existence et l'unicité de la solution $(\alpha(t), N(t, r))$. Le problème de prolongement en temps de cette solution est aussi analysé et résolu. On démontre quelques propriétés de la solution: non croissance de $\alpha(t)$, la compacité pour t fixé du support en r de $N(t, r)$, qui découle de l'hypothèse de support compact pour la donnée initiale $N_0(r) = N(0, r)$ et la non croissance en r de $N(t, r)$ qui découle, pour chaque t , d'une propriété similaire pour $N_0(r)$. Tous ces résultats mathématiques sont motivés par des études thermodynamiques d'un modèle de microfracture qui est brièvement décrit dans la dernière section de l'article.

ABSTRACT. An initial value problem for a first order PDE with a nonlocal term is studied. Existence and uniqueness of the solution $(\alpha(t), N(t, r))$ is proved. The t -continuation problem, for that solution, is also analyzed and solved. Several properties of the solution are also proven, namely the nonincreasing character of $\alpha(t)$, the compactness for each t of the r -support of $N(t, r)$, that is inherited from an assumed support compactness of the initial datum $N_0(r) = N(0, r)$, and the r -nonincreasing character of $N(t, r)$ that is inherited, for each t , from a similar assumed character of $N_0(r)$. All those mathematical results are motivated by thermodynamic studies of a microfracture model described elsewhere, which is briefly outlined in the final section of the paper.

1. Introduction and preliminaries. The following two equations

$$\frac{\partial N}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) = H(t, r, \alpha(t)) \quad ; \quad t \geq 0, r \geq 0 \quad (1)$$

$$\alpha(t) = \int_0^{+\infty} f(r) N(t, r) dr \quad (2)$$

with a constraint $\alpha(t) < \lambda$ and an initial condition $N(0, r) = N_0(r)$, couple together to give a functional condition in which λ is a given number, while $N_0(r)$, $f(r)$, $v(t, r, a)$, $H(t, r, a)$ are known functions, whose definition domains and regularity behavior will be stated later. From now on, $N_0(r)$ is a nonincreasing, nonnegative function, while

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$f(r)$ is a continuous increasing function, such that $f(0) = 0$. On the other hand, $v(t, r, a)$ is a nonnegative bounded function and $H(t, r, a)$ is a bounded, nonnegative and r -decreasing function.

Notation and hypotheses. The usual meaning of the symbol R_+ (positive real numbers) is here slightly modified to stand for the closed half line of all nonnegative real numbers. Given a positive real λ , the symbols D_λ , P_λ will respectively represent the slice identical to the triple cartesian product $R_+ \times R_+ \times [0, \lambda)$, and the semi-open strip $R_+ \times [0, \lambda)$. In short,

$$D_\lambda \equiv R_+ \times R_+ \times [0, \lambda) \quad ; \quad P_\lambda \equiv R_+ \times [0, \lambda) . \quad (3)$$

Now, the following hypothesis (H1), (H2), (H3) will be assumed:

(H1) – There exist three positive real numbers λ , c , c' , such that the two real functions $v(t, r, a)$ and $H(t, r, a)$ (see notations above) are nonnegative and continuously differentiable inside D_λ , with continuous extension to its closure, such that

$$\begin{aligned} \forall (t, r, a) \in D_\lambda, \quad v(t, 0, a) = 0, \quad 0 \leq v(t, r, a) \leq c \\ |\partial_t v(t, r, a)|, |\partial_r v(t, r, a)|, |\partial_a v(t, r, a)| \leq c' \end{aligned} \quad (4)$$

The initial datum $N_0(r)$ and the weight function $f(r)$ are assumed to be continuously differentiable in R_+ , $f(r)$ being also a positive and increasing function:

$$N_0 \in C^1[0, +\infty) \quad ; \quad f \in \{g \in C^1[0, +\infty) ; g > 0\}$$

$$\forall r \in R_+, \quad \forall s \leq r, \quad 0 \leq f(r) \leq f(s) \quad (5)$$

(H2) – Function H is zero whenever its second argument r is greater than a given number R_n . And the datum N_0 is a positive function with compact support.

(H3) – The two functions $H(t, r, a)$ and $N_0(r)$ are assumed to be r -nonincreasing in their respective domains of definition, that is

$$\forall (t, r, a) \in D_\lambda, \quad \forall s \geq r, \quad 0 \leq H(t, s, a) \leq H(t, r, a), \quad N_0(s) \leq N_0(r). \quad (6)$$

Remarks. a) The two equations in (1) represent a physical condition for a type of brittle material behavior models [2, 4]. In such a context, the known functions $N_0(r)$, $\alpha(t)$, $v(t, r, a)$, $H(t, r, a)$ are respectively called the initial crack distribution, the damage weight, the crack opening speed and the nucleation rate. Let us add that $N_0(r)$ represents the number of cracks of size larger than r per unit volume, that certain brittle material presents within a generic point of its spatial continuum, at time $t = 0$. $f(r)$ is designed to ascertain that the last integral expression in (1) produces a reasonable measure of the (scalar) local damage of the brittle material behavior to be mathematically simulated. The speed $v(t, r, a)$ refers to the opening of microcracks of size r , under the stress and strain conditions present at time t and with an actual material damage of value a . Finally, $H(t, r, a)$ stands for the time rate of cracks of size larger than r that are nucleated per

unit volume at time t , under the same physical conditions just stated in relation to $v(t, r, a)$.

b) The physical meaning of the upper bound c is that of a maximum speed for the opening of the microcracks, called Rayleigh limit, while the meaning of the upper bound R_n is that no cracks with radius above it are suddenly nucleated.

c) There do not seem to be many problems in the literature similar to the one described here. Only a few papers on Population Dynamics, e.g. [5, 8], show some distant parenthood with ours. If, for simplicity, we follow the founding paper [5], its equations on page 282 could be written in the form

$$\begin{aligned} \frac{\partial N}{\partial t}(t, a) + \frac{\partial N}{\partial a}(t, a) &= - \int_a^{+\infty} \lambda(c, P(t)) \rho(t, c) dc \quad ; \quad a > 0 \\ \frac{\partial N}{\partial t}(t, 0) + \frac{\partial N}{\partial a}(t, 0) &= \int_0^{+\infty} [\beta(c, P(t)) - \lambda(c, P(t))] \rho(t, c) dc \\ P(t) &= \int_0^{+\infty} \rho(t, c) dc \quad ; \quad N(t, a) = \int_a^{+\infty} \rho(t, c) dc \end{aligned} \quad (7)$$

also with an initial condition for $N(t, a)$. In these equations, our changing velocity coefficient $v(t, a, P(t))$ is missed (it is considered to be 1), while a death pattern not present in our model is considered (i.e. cracks are born in but never die out).

2. Existence of a t -local solution. The goal in this section is to show that a t -local solution $\alpha(t)$, $N(t, r)$ exists for the initial value problem

$$\begin{aligned} \frac{\partial N}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) &= H(t, r, \alpha(t)), \quad t \geq 0 \\ \alpha(t) &= \int_0^{+\infty} f(r) N(t, r) dr \quad ; \quad N(0, r) = N_0(r) \end{aligned} \quad (8)$$

when the functions $v(t, r, a)$, $H(t, r, a)$, $N_0(r)$, $f(r)$, verify hypothesis (H1) and (H2) stated in Section 1 and the inequality

$$\int_0^{+\infty} f(r) N_0(r) dr < \lambda \quad (9)$$

holds. A certain composition of mappings will play a key role in the proof of the existence of a t -local solution, one of the mappings, denoted by $\mathcal{F}$, will be defined after the following proposition.

Proposition 2.1. *Given a function $N_0(r)$ verifying (H1) and given also another continuously differentiable nonnegative function $\alpha(t)$, defined in some subinterval $[0, T)$ of R_+ , with graph contained in the strip P_λ , there is a unique continuously differentiable function $M(t, r)$ defined in $[0, T) \times R_+$ which verifies both the equation*

$$\forall (t, r) \in [0, T) \times R_+, \quad \frac{\partial M}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial M}{\partial r}(t, r) = H(t, r, \alpha(t)) \quad (10)$$

and the initial condition $M(0, r) = N_0(r)$.

Proof. It is just a piece of the elementary theory of first order linear partial differential equations (see [3, pp. 18–22] for instance). To recall this theory in a proper way, let us consider the flow $\Phi_\alpha(t; t_*, r_*)$ that solves the initial value problem

$$\forall t, t_*, r \geq 0, \quad \Phi_\alpha(t_*; t_*, r_*) = r_*; \quad \frac{d}{dt} \Phi_\alpha(t; t_*, r_*) = v(t, \Phi_\alpha(t; t_*, r_*), \alpha(t)) \quad (11)$$

When the last two arguments t_*, r_* are fixed, this flux $\Phi_\alpha(t; t_*, r_*)$ is the solution of the differential equation $dr/dt = v(t, r, \alpha(t))$, that satisfies the condition $r(t_*) = r_*$. The regularity hypothesis (H1) in Section 1, for function v , implies (see [6]) the existence, uniqueness and global t -continuation of the initial value problems for the just mentioned differential equation, which means that the associated flux Φ_α is *nonintersecting* and (its restriction for any fixed t, t_* in $[0, T)$) defines a one to one transformation, from R_+ into R_+ . Moreover, this one-to-one transformation is differentiable [6], and, given the assumption $v(t, 0, a) = 0$, is also a bijection from R_+ onto itself. $\square$

In fact, assume $M(t, r)$ is any solution to the PDE initial value problem defined in (10) and let us differentiate the composition $M(t, \Phi_\alpha(t, 0, r_0))$ with respect to t . The chain rule gives immediately

$$\begin{aligned} \frac{d}{dt} M(t, \Phi_\alpha) &= \frac{\partial M}{\partial t}(t, \Phi_\alpha) + \left(\frac{\partial M}{\partial r} \right) \frac{\partial \Phi_\alpha}{\partial t} \\ &= \frac{\partial M}{\partial t}(t, \Phi_\alpha) + v(t, \Phi_\alpha, \alpha(t)) \frac{\partial M}{\partial r} \\ &= H(t, \Phi_\alpha(t, 0, r_0), \alpha(t)). \end{aligned}$$

After integration, the mentioned PDE theory takes the form

$$M(t, \Phi_\alpha(t, 0, r_0)) = M(0, \Phi_\alpha(0, 0, r_0)) + \int_0^t H(\tau, \Phi_\alpha(\tau, 0, r_0), \alpha(\tau)) d\tau,$$

that is,

$$M(t, r) = N_0(\Phi_\alpha(0, t, r)) + \int_0^t H(\tau, \Phi_\alpha(\tau, t, r), \alpha(\tau)) d\tau \quad (12)$$

which is valid for any (t, r) in $[0, T) \times R_+$. This equation allows next to define what will be called the factor map.

Definition 2.1. With the notations and hypotheses given prior to Proposition 2.1 the symbol $\mathcal{F}$ will stand for the map assigning to any feasible function $\alpha(t)$ (i.e. with graph contained in the strip P_λ , recall definition (3)), the solution (12) of equation (10) that satisfies the initial condition $M(0, r) = N_0(r)$. When there is a possibility of confusion, the symbol $\mathcal{F}$ will be replaced by $\mathcal{F}_{N_0}$.

Remark. Notice that, from the hypotheses listed in Section 1, most of the hypotheses under (H1) have been used so far, but neither (H2) nor (H3) have been used yet.

Construction of a contracting map. In order to define a convenient contraction, we will now establish an intermediate result, which is also interesting in itself.

Proposition 2.2. *With the notations and assumptions above, compactness of the support of $N_0(r)$ entails, for any t in the interval $[0, T)$, the r -compactness of the support of $M(t, r)$. Further, if we denote Q the number R_n given in (H2) and L is the length of the support of N_0 , then the velocity of growth of the support of $M(t, r)$ has an upper bound expressed by*

$$\forall t \in [0, T), \forall r > Q + ct, \quad M(t, r) = 0 \quad (13)$$

where c is the upper bound for $v(t, r, a)$ (as defined in (H1)).

Proof. It will stem from formula (12) as soon as it is noticed that the diffeomorphisms $\Phi_\alpha(\tau, t, \cdot)$ preserve, as well as their inverses, the r -compactness of sets. Because then, the r -compactness of the support of functions N_0 and H is inherited by their compositions with the maps Φ_α that appear in formula (12). Further, if r is larger than $Q + ct$, then both $\Phi_\alpha(\tau, t, r)$ and $\Phi_\alpha(0, t, r)$ are larger than Q in view of the differential equation (11) and the bound (4) on $v(t, r, a)$. That is, the compositions $N_0(\Phi_\alpha(0, t, r))$ and $H(\tau, \Phi_\alpha(\tau, t, r), \alpha(\tau))$ that appear in (12) have both value zero, which yields relation (13) with respect to the support of $M(t, r)$. $\square$

Definition 2.2. Given any $T > 0$, let us denote by $\mathcal{J}_T$ the map assigning to any continuously differentiable function $M(t, r)$ in $[0, T) \times R_+$, with compact support in its second argument (r), the function $A(t)$ defined by the proper integral

$$A(t) = \int_0^{+\infty} f(r)M(t, r) dr$$

where t belongs to the interval $[0, T)$.

Lemma 2.1. *If the initial condition $N_0(r)$ has been chosen in problem (8) and inequality (9) holds, then there exist positive numbers $p > 0$, $T_0 > 0$ such that: for any positive T in $[0, T_0]$ and for any continuously differentiable function $\alpha(t)$ in $[0, T)$ that is admissible (i.e. graph within the strip P_λ defined in (3)), the inequality*

$$\forall t \in [0, T), \quad (\mathcal{J}_T(M_\alpha))(t) \leq \alpha(0) + pt \quad (14)$$

holds. Here $M_\alpha(t, r)$ stands for the solution of (10), with $\alpha(t)$ and $N_0(r)$ given.

Proof. We start from formula (12) which expresses $M_\alpha(t, r)$ as a sum of two terms. Given that $\Phi_\alpha(0, t, r)$ is the (reverse) flow of differential equation (11), the bound (4) of the equation right hand side $v(t, r, \alpha)$ gives

$$\forall T > 0, \forall t \in [0, T], \forall r \geq 0, \quad 0 \leq \Phi_\alpha(0, t, r) \leq r, \quad -c \leq \frac{\partial}{\partial t} \Phi_\alpha(0, t, r) \leq 0$$

so that the function $N_0(\Phi_\alpha(0, t, r))$ has r -compact support, of length at most $L + ct$ (where L is the length of the support of $N_0(r)$). It follows that if t is smaller than L/c ,

$$\begin{aligned} \left| \frac{d}{dt} \int_0^{+\infty} f(r)N_0(\Phi_\alpha(0, t, r)) dr \right| &= \left| \int_0^{L+ct} f(r)N'_0(\Phi_\alpha(0, t, r))\partial_t \Phi_\alpha(0, t, r) dr \right| \\ &\leq (2L)f(2L)m_0c, \end{aligned}$$

where m_0 is the supremum of N'_0 within its support. This gives a constant that is independent of the function $\alpha(t)$. Therefore

$$\left| \int_0^{+\infty} f(r) N_0(\Phi_\alpha(0, t, r)) dr \right| \leq \left| \int_0^{+\infty} f(r) N_0(\Phi_\alpha(0, 0, r)) dr \right| + 2Lf(2L)m_0ct,$$

where the last integral term is just $\alpha(0)$. On the other hand, the r -integral generated by the second term of (the right hand side of) formula (12), can be bounded as follows: firstly recall from hypothesis (H2) in Section 1 that the function $H(\tau, r, a)$ has r -compact support, of length R_n , and $H(\tau, \Phi_\alpha(\tau, t, r), a)$ has also (see Proposition 2.2) this property. Secondly, call h the maximum of $H(\tau, r, a)$ on the set defined by the bounds $0 \leq \tau \leq 1$, $0 \leq r \leq R_n$ and $0 \leq a \leq \lambda$, so that by choosing t smaller than 1 and R_n/c , one obtains

$$\begin{aligned} \left| \int_0^t d\tau \int_0^{+\infty} f(r) H(\tau, \Phi_\alpha(\tau, t, r), \alpha(\tau)) dr \right| &\leq \left| \int_0^t d\tau \int_0^{R_n+ct} f(r) h dr \right| \\ &\leq 2R_n f(2R_n) h t. \end{aligned}$$

The proof of (14) is done by taking T_0 as one half of the smallest of 1, L/c and R_n/c , and then, p larger than the sum of $2Lf(2L)m_0$ and $2R_n f(2R_n)h$. $\square$

Lemma 2.2. *Given problem (8), with an initial condition $N_0(r)$ that satisfies inequality (9), then there exists a positive T_I such that, for any smaller positive T_i , the set $\mathcal{S}$ defined by*

$$\mathcal{S} \equiv \left\{ \alpha \in C^1[0, T_i]; \begin{array}{l} \alpha(0) = \int_0^{+\infty} f(r) N_0(r) dr \\ \alpha(t) \leq \alpha(0) + (1+p)t \end{array} \right\}$$

is transformed into itself by the composition of maps $\mathcal{J}_{T_i} \circ \mathcal{F}$, with $\mathcal{F}$ as in Definition 2.1. Further, this set is made up of only admissible functions (with graph in the strip P_λ defined in (3)).

Proof. With p and T_0 as in the previous lemma, it is straightforward to see that

$$T_I \equiv \min \left\{ \frac{1}{2+2p} \left(\lambda - \int_0^{+\infty} f(r) N_0(r) dr \right), T_0 \right\}$$

is a number that meets all the conditions stated in this result, if we take into account the compactness ensured by Proposition 2.2. This compactness just means that $(\mathcal{F}(\alpha))(t, r)$ is a suitable input for the mapping $\mathcal{J}_T$. $\square$

To continue with the search of a convenient contraction, we state:

Lemma 2.3. *Given problem (8), with an initial condition $N_0(r)$ that satisfies inequality (9), then there exist two positive real numbers ℓ and T_r such that: for any two continuously differentiable functions $\alpha(t)$ and $\beta(t)$ defined in an interval of the form $[0, T)$ with $T > 0$, that are both admissible (i.e. with graphs inside P_λ), it holds*

$$\forall t \in [0, T_m], \forall r \geq 0, \quad |(\mathcal{F}(\beta))(t, r) - (\mathcal{F}(\alpha))(t, r)| \leq \ell t \|\beta - \alpha\|$$

where T_m is the smaller of both T_r and T , and $\|\cdot\|$ denotes the supremum norm over the interval $[0, T_m]$.

Proof. We first establish existence of some $T_p > 0$ such that for any positive T and for any two continuously differentiable functions α, β whose graphs along $[0, T]$ lie in the strip P_λ defined by (3), there holds: the two associated admissible flows Φ_α, Φ_β satisfy

$$\begin{aligned} \forall t \in [0, \min\{T_p, T\}), \forall r_0 \in R_+, \\ |\Phi_\beta(t, 0, r_0) - \Phi_\alpha(t, 0, r_0)| \leq 2c't \sup_{s \leq t} |\beta(s) - \alpha(s)| \end{aligned} \quad (15)$$

where c' is the bound for the modulus of the partial derivatives in hypothesis (H1).

To prove (15) we refer to standard studies on perturbation of solutions of initial value problems for ordinary differential equations, [6, pp. 86-87]. The three inequalities

$$|v(t, r, \beta(t))| \leq c \quad ; \quad |v(t, r, \alpha(t)) - v(t, s, \alpha(t))| \leq c'|r - s|$$

$$|v(t, r, \beta(t)) - v(t, r, \alpha(t))| \leq c'|\beta(t) - \alpha(t)|$$

are direct consequences from the ones in (4). From there, it follows that for any nonnegative r_0 ,

$$\begin{aligned} \forall t \in [0, \min\{T_p, T\}), \\ |\Phi_\beta(t, 0, r_0) - \Phi_\alpha(t, 0, r_0)| \leq \sup_{s \leq t} |\beta(s) - \alpha(s)| (-1 + \exp(c't)) \end{aligned}$$

which yields (15) for T_p small enough.

Now, we end the proof of the lemma. Abridged notations $M_\alpha(t, r) \equiv (\mathcal{F}_{N_0}(\alpha))(t, r)$, $M_\beta(t, r) \equiv (\mathcal{F}_{N_0}(\beta))(t, r)$ and $m_0 \equiv \sup\{|\partial_r N'_0(r)|; r \geq 0\}$ are set, where $N'_0(r)$ is assumed continuous with compact support. From formulae (12) and (15) the inequality

$$\begin{aligned} \forall t \in [0, \min\{T_p, T\}), \forall r_0 \in R_+, \\ |M_\beta(t, r) - M_\alpha(t, r)| \leq 2m_0c't\|\beta - \alpha\| \\ + \int_0^t |H(\tau, \Phi_\beta(\tau, t, r), \beta(\tau)) - H(\tau, \Phi_\alpha(\tau, t, r), \alpha(\tau))| d\tau \end{aligned}$$

holds true with T_p as before. Now continuous differentiability of $H(\tau, r, a)$, the fact that it has compact r -support, together with formula (12) imply

$$\begin{aligned} \forall t \in [0, \min\{T_p, T\}), \forall r_0 \in R_+, \\ |M_\beta(t, r) - M_\alpha(t, r)| \leq (2m_0c' + c'T_p H_0 + H_0)t\|\beta - \alpha\| \end{aligned}$$

where H_0 can be taken as follows: an upper bound in the product $[0, T_p] \times [0, R_n] \times [0, \lambda]$, for absolute value of the first partial derivatives of $H(\tau, r, a)$ (see (H2)). $\square$

Theorem 2.1. *There exists $T_i > 0$, so that the composition of maps $\mathcal{J}_{T_i} \circ \mathcal{F}_{N_0}$ is a contraction of $\mathcal{S}$ into itself, where $\mathcal{S}$ depends on T_i , as in definition above.*

Proof. $\mathcal{J}_{T_i} \circ \mathcal{F}_{N_0} : \mathcal{S} \longrightarrow \mathcal{S}$, for $0 < T_i < T_I$, T_I small enough (Lemma 2.2). Now, if $A(t)$ and $B(t)$ stand for the images of respective originals $\alpha(t)$ and $\beta(t)$, the

monotonicity hypothesis (4) on the weight f and compactness (13) ensure, in view of Lemma 2.3 that

$$\begin{aligned} |A(t) - B(t)| &\leq \int_0^{+\infty} f(r) |M_\alpha(t, r) - M_\beta(t, r)| dr \\ &= \int_0^{R+ct} f(r) |M_\alpha(t, r) - M_\beta(t, r)| dr \\ &\leq 2Rf(2R)\ell t \|\alpha - \beta\| \end{aligned}$$

where R is the larger of R_n (hypothesis (H2)) and the length L of the support of the initial datum $N_0(r)$. Now, there should be clear that a contraction is obtained by choosing

$$T_i = \min \left\{ T_I, T_r, \frac{1}{4\ell L f(2L)} \right\}$$

with T_I, T_r as in Lemmas 2.2 and 2.3. $\square$

We can conclude regarding problem (8). Existence and uniqueness of the function α in problem (8) follow as a fixed point of the contraction $\mathcal{J}_{T_i} \circ \mathcal{F}_{N_0}$ in some convenient $\mathcal{S}$. Then, N is deduced from α using formula (12). Thus, problem (8) is equivalent to proving existence of α .

3. Local uniqueness and t -continuation. Any solution α has to take at $t = 0$ a value equal to the integral of $f(r)N_0(r)$ in R_+ , with a derivative not larger than the number p appearing in inequality (14). Now, local uniqueness follows immediately as any solution outside the set $\mathcal{S}$ would have (by reducing $T_i > 0$ as needed in the definition of $\mathcal{S}$) a derivative at $t = 0$ not below $1 + p$, which contradicts the bound (14).

To begin with the continuation argument we will make use of hypothesis (H3). Those hypotheses have a clear physical meaning, namely the r -cumulative character of the distributions N_0 and $H(t, r, a)$. We first prove two important results, that are also of interest in themselves.

Proposition 3.1. *The part $N(t, r)$ of the solution of the previous local problem, is an r -nonincreasing function.*

Proof. It follows immediately from formula (12). In fact, the flow $\Phi_\alpha(\tau, t, r)$ is itself an r -nonincreasing function, and the same property was assumed both for the source term $H(\tau, s, a)$ and the initial damage distribution $N_0(s)$. $\square$

Proposition 3.2. *The part α of the solution of equation (8) is a nondecreasing function.*

Proof. Differentiation under the integral sign in the second equation in (8) together with the first equation there gives

$$\frac{d}{dt}\alpha(t) = \int_0^{+\infty} f(r) \left\{ H(t, r, \alpha(t)) - v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) \right\} dr, \quad (16)$$

where in view of Proposition 2.2, $N(t, r)$ has a compact support as a function of r , which justifies the integrand t -differentiation. Now, this integrand is nonnegative in view of Proposition 3.1 and the positivity hypothesis on v and H , which completes the proof. $\square$

Now, we state and prove the main theorem of this paper.

Theorem 3.1. *With the notations and hypotheses as above, the problem*

$$\begin{aligned} \frac{\partial N}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) &= H(t, r, \alpha(t)), \quad (t \geq 0) \\ \alpha(t) &= \int_0^{+\infty} f(r) N(t, r) dr; \quad N(t=0, r) = N_0(r); \quad \int_0^{+\infty} f(r) N_0(r) dr < \lambda \end{aligned}$$

has a unique solution, that can be continued in the t variable, as long as the function $\alpha(t)$ has a graph staying within the strip P_λ defined in Section 1.

Proof. Let us proceed in a recursive manner by writing $t_0 = 0$ and $\tau_0 = T_i/2$, where $T_i > 0$ exists (as it has been proved already) to give a unique solution for the problem (8)-(9) along the interval $[0, T_i)$. Next, assume that a positive integer, j , exists so that the following relations hold

$$\forall k \in \{0, \dots, j-1\}, \exists \tau_k, t_{k+1} > 0; \quad t_{k+1} = t_k + \tau_k$$

with the additional assumption that, in any of the closed intervals $[t_k, t_k + \tau_k]$ with k from 1 to $j-1$, there is only one solution for the problem

$$\begin{aligned} \frac{\partial N}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) &= H(t, r, \alpha(t)), \quad (t \geq 0) \\ \alpha(t) &= \int_0^{+\infty} f(r) N(t, r) dr; \quad N(t_k, r) = N(t_{k-1} + \tau_{k-1}, r) \equiv N_k(r) \end{aligned}$$

and further, that the inequality

$$\int_0^{+\infty} f(r) N(t_{k-1} + \tau_{k-1}, r) dr \equiv \int_0^{+\infty} f(r) N_k(r) dr < \lambda$$

holds for any of the k -indices above. Now, we try the next recursive step and write $t_j \equiv t_{j-1} + \tau_{j-1}$. In view of the α -increasing property above, either $\alpha(t_j) = \lambda$ (and threshold is then reached) or, else, $\alpha(t_j) < \lambda$, in which case a unique t -local solution can be again obtained, in a similar manner to prior theorem 2.1, but now starting at $(t_j, \alpha(t_j))$. Now, it becomes clear by induction, that α can be continued through contiguous intervals (t_{j-1}, t_j) , up to a point where damage threshold is reached. In fact, if the increasing sequence $\alpha(t_j)$ has a limit point $\alpha(t_*)$, either $\alpha(t_*) = \lambda$, or $\alpha(t_*) < \lambda$ and continuation can proceed.

Actually a little more has been proven above: solution α is continuously differentiable inside each (t_j, t_{j+1}) , as this function belongs to the corresponding S_j set. Also, the solution $N(t, r)$ is continuously differentiable in $(t_j, t_{j+1}) \times R_+$, in view of formula (12). To get a complete C^1 behavior, there only remains to show equality $\alpha'(t_j - 0) = \alpha'(t_j + 0)$. This follows from previous (16), as

$$\begin{aligned} \alpha'(t_j - 0) &= \int_0^{+\infty} f(r) \left\{ H(t_j, r, \alpha(t_j)) - v(t_j, r, \alpha(t_j)) \frac{\partial N}{\partial r}(t_j - 0, r) \right\} dr \\ &= \int_0^{+\infty} f(r) \left\{ H(t_j, r, \alpha(t_j)) - v(t_j, r, \alpha(t_j)) \frac{\partial}{\partial r} N_j(r) \right\} dr \\ &= \alpha'(t_j + 0) \end{aligned}$$

and t -continuation is completely established. $\square$

4. Physical motivation of the problem. The main purpose of this selfcontained, expository section is to show that a physical meaning can be given to the fact that, in problem (1), the time dependence of functions $v(t, r, a)$ and $H(t, r, a)$ is assumed to be known.

We follow [9, chaps. 2-3], to identify the core of every material behavior with a certain functional operator $\mathcal{MB}$. This operator takes any kinematic history that could have been possibly undertaken by the material, to another history of a dynamic character. This essentially means that any pair of configuration and temperature functions, $\bar{x}(\bar{X}, t)$ and $\theta(\bar{X}, t)$, is taken by the constitutive operator $\mathcal{MB}$ ($\bar{X}$ stands for a generic material point, while $\bar{x}$ stands for its present position), to a fourthlet of functions; namely the specific free energy $\psi(\bar{X}, t)$, the specific entropy $\eta(\bar{X}, t)$, the stress tensor $\tau(\bar{X}, t)$, and the heat vector flux $\bar{q}(\bar{X}, t)$.

Not any operator $\mathcal{MB}$ will do as a constitutive or material behavior operator. It must satisfy objectivity conditions that ensure independence from the observer, as formalized for instance in [7]. It must also satisfy the second principle of thermodynamics, expressed in a mathematical manner as in [1], via the Clausius Duhem inequality

$$\frac{\partial}{\partial t} \psi(\bar{X}, t) + \eta \frac{\partial}{\partial t} \theta(\bar{X}, t) + \tau \frac{\partial}{\partial t} F - \frac{1}{\rho \theta} \bar{q} \cdot \overline{\text{grad}} \theta \leq 0 \quad (17)$$

where F is the jacobian matrix of the configuration function and ρ is the density field (inversely proportional to the determinant of F). By dwelling into the technical details in [1], we also learn that the version of the stress tensor τ used in inequality (17) is the first Piola Kirchoff tensor.

What turns out to be a key theoretical idea is that previous inequality (17), far from being a burden to carry on, conveys implicit but very useful information on the constitutive operators, i.e. on the material itself. That is, those operators have to give, for *any* kinematic history $\theta, \bar{x}$, an image or dynamic response that always complies with inequality (17). Which leads, after some more or less difficult mathematical analysis (that works with the corresponding objectivity conditions as well) to what could be rightly called the theory of the material behavior under study. We may mention that some mild time derivative jumps have to be allowed in the set of all possible kinematic histories, for that theory to be useful enough.

A type of damage-elastic material behavior. Constitutive operators $\mathcal{MB}$ may be of many different kinds. A class of those, defined by means of the so-called hidden or internal variables, has got a lot of attention since the work in [1]. In a very sketchy summary, we could say that the constitutive operator is defined now, for this special type of materials, by giving four real functions $\hat{\psi}, \hat{\eta}, \hat{\tau}, \hat{q}$ of the t -present numerical values of $F, \theta, \bar{g} \equiv \overline{\text{grad}} \theta$ and some *hidden* vector $\bar{\alpha}$. The images of those functions give the present values of the corresponding dynamical quantities $\psi, \eta, \tau, \bar{q}$ and define the material operator, when completed with an evolution equation of the form

$$\left(\frac{\partial}{\partial t} \right) \bar{\alpha}(\bar{X}, t) = \vec{h}(F, \theta, \bar{g}, \bar{\alpha})$$

where $\vec{h}$ is another real function. In fact, we readily notice that making a choice for the kinematic history shapes up (for each material point $\bar{X}$) an ordinary differential

equation $(D/Dt)\vec{\alpha} = \vec{H}(t, \vec{\alpha})$, where D/Dt is the so-called material time derivative. That equation allows then (via the classical existence and uniqueness theorem for the initial value problem), to obtain the complete history of the four dynamical variables and the vector of internal ones.

It is worth mentioning, although not logically needed here, that when the thermodynamics analysis (i.e. compliance with inequality (17)) was carried forward in [1], there came out that the three functions $\hat{\psi}$, $\hat{\eta}$, $\hat{\tau}$ must be independent of the gradient $\bar{g}$, of the temperature field. Moreover, the relations of classical equilibrium thermodynamics

$$\hat{\eta} = -\frac{\partial}{\partial \theta} \hat{\psi}, \quad \hat{\tau} = \frac{\partial}{\partial \tau} \hat{\psi}$$

were recovered in a consistent, mathematically rigorous fashion.

We now use the above example of internal variables material behavior to motivate a new type of behavior, to be called *micro crack elastic* here, determined by a microphysical structure having a mathematical definition somewhat different from the one used in the internal variable approach. In fact both approaches are quite close from a physical point of view, but the one to be introduced now would require an infinite number of hidden variables and it would increase the difficulty of the objectivity analysis, [4], in a substantial manner.

To be more concrete, let $\hat{\psi}$, $\hat{\eta}$, $\hat{\tau}$, $\hat{q}$ be functions as above, depending on the present numerical values of F , θ , $\bar{g} \equiv \overline{\text{grad } \theta}$ and some scalar quantity α , termed damage, in such a way that the material behavior is determined by the set of equations

$$\begin{aligned} \psi(t, \bar{X}, t) &= \hat{\psi}(F, \theta, \overline{\text{grad } \theta}, \alpha) & ; & & \eta(t, \bar{X}, t) &= \hat{\eta}(F, \theta, \overline{\text{grad } \theta}, \alpha) \\ \tau(t, \bar{X}, t) &= \hat{\tau}(F, \theta, \overline{\text{grad } \theta}, \alpha) & ; & & \bar{q}(t, \bar{X}, t) &= \hat{q}(F, \theta, \overline{\text{grad } \theta}, \alpha) \end{aligned}$$

$$\frac{\partial N}{\partial t}(t, r) + v(F, \theta, \overline{\text{grad } \theta}, \alpha; r) \frac{\partial N}{\partial r}(t, r) = H(F, \theta, \overline{\text{grad } \theta}, \alpha; r)$$

$$(t \geq 0, r \geq 0) \quad ; \quad \alpha(t) = \int_0^{+\infty} f(r) N(t, r) dr$$

where, v , H , are known functions of their arguments. It should be clear now that having a well defined dynamic history depends on the unique solvability of all the initial value problems of type (1) that are given for all possible kinematic histories $\bar{x}(\bar{X}, t)$ and $\theta(\bar{X}, t)$. In other words, problem (1) is physically meaningful as it stands, i.e. *uncoupled* to the remaining of the continuum mechanics problem that considers the kinematical fields as unknowns. This latter problem is a most formidable one that needs, as is well known, to take into account the so-called conservation equations, sideways to the material behavior ones. Therefore, the reduced mathematical problem (yet complex enough) that was defined at the beginning of this paper has a clear thermo dynamical interpretation and meaning.

Editors note. Emilio Garbayo died a few months after he submitted this paper. We thank Eva Sanchez for preparing the final version of the paper.

Résumé substantiel en français. Dans ce travail, on présente des résultats mathématiques concernant une équation aux dérivées partielles du premier ordre qui est motivée par une modélisation de la distribution des fissures dans un domaine matériel continu. De façon plus précise, le modèle se formule par les équations suivantes :

$$\begin{aligned} \frac{\partial N}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) &= H(t, r, \alpha(t)); \quad t \geq 0, \quad r \geq 0 \\ \alpha(t) &= \int_0^{+\infty} f(r) N(t, r) dr \end{aligned}$$

avec les conditions $\alpha(t) < \lambda$, $N(0, r) = N_0(r)$, où les données $N_0(r)$, $f(r)$, $v(t, r, a)$, $H(t, r, a)$ sont des fonctions dont la régularité est précisée dans les hypothèses (H1), (H2) et (H3).

On montre l'existence locale en t d'une solution pour ce problème en faisant une transformation de celui-ci en un problème équivalent de point fixe pour l'inconnue $\alpha(t)$. Dans la section 2 on démontre que, pour T assez petit, la fonction $\alpha(t)$, $t \in [0, T]$ est le point fixe unique d'une certaine application contractante.

Le résultat principal de l'article est le théorème 3.1 de la section 3:

Théorème 3.1. *Avec les notations et les hypothèses déjà introduites, le problème :*

$$\begin{aligned} \frac{\partial N}{\partial t}(t, r) + v(t, r, \alpha(t)) \frac{\partial N}{\partial r}(t, r) &= H(t, r, \alpha(t)), \quad (t \geq 0) \\ \alpha(t) &= \int_0^{+\infty} f(r) N(t, r) dr; \quad N(0, r) = N_0(r); \quad \int_0^{+\infty} f(r) N_0(r) dr < \lambda \end{aligned}$$

a une solution unique qui peut être prolongée selon la variable t à tout t tel que la fonction $\alpha(t)$ ait son graphe dans l'ensemble P_λ défini par la formule (3) de la section 1.

La section 4 contient une description résumée des problèmes physiques qui ont donné lieu à l'équation proposée.

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REFERENCES

1. B. D. Coleman and M. E. Gurtin, *Thermodynamics with internal state variables*, J. Chem. Phys. **47** (1967), 597–613.
2. D. R. Curran, L. Seaman and D. A. Shockey, *Dynamic failure of solids*, Phys. Rep. **147** (1987), 253–388.
3. P. R. Garabedian, *Partial Differential Equations*, Chelsea, New York, 1986.
4. E. Garbayo and L. Seaman, *Objective rates in a crack model* (to appear).

5. M. E. Gurtin and R. C. MacCamy, *Non-linear age-dependent population dynamics*, Arch. Rational Mech. Anal. **54** (1974), 281–300.
6. E. Hille, *Lectures on ordinary differential equations*. Addison-Wesley, Reading, Mass., 1969.
7. W. Noll, *On the continuity of the solid and fluid states*, J. Rational Mech. Anal. **4** (1955), 3–81.
8. H. R. Thieme, *Analysis of age-structured population models with an additional structure*, Mathematical Population Dynamics (New Brunswick, NJ, 1989) (O. Arino *et al.*, eds.), Lecture Notes in Pure and Appl. Math. vol. 131, Marcel Dekker, New York, 1991, pp. 115–126.
9. C. Truesdell, *Rational thermodynamics: a course of lectures on selected topics*, McGraw-Hill, New York, 1969.

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