

REMARKS ON FINITELY SUBADDITIVE OUTER MEASURES

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RÉSUMÉ. Soit $\mathcal{L}$ un treillis de parties d'un ensemble X . Ce travail traite des relations entre les mesures extérieures sous-additives définies sur $\mathcal{L}$ dans le cas où $\mathcal{L}$ est un treillis normal. Nous présentons aussi des conditions sous lesquelles un ensemble est mesurable par rapport à une mesure extérieure sous-additive.

ABSTRACT. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X . This paper is concerned with finitely subadditive outer measure and relations between them, assuming $\mathcal{L}$ is a normal lattice. The last part of the paper gives conditions for a set to be measurable with respect to a finitely subadditive outer measure.

1. Introduction. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X . If $\mathcal{L}$ is a normal lattice, then in previous papers consequences pertaining to $I(\mathcal{L})$ - the set of non-trivial finitely additive zero-one valued measures on $A(\mathcal{L})$, the algebra generated by $\mathcal{L}$ have been investigated (see e.g. [3, 5, 6]).

In the first part of this paper we define a measure $\mu \in M(\mathcal{L})$ and then define an auxiliary finitely subadditive outer measure μ' associated with it which prove useful in extending some of the above results from $I(\mathcal{L})$ to $M(\mathcal{L})$. This along with another outer measure μ'' is considered in detail.

In the second part of the paper we study in detail the finitely subadditive outer measure $\hat{\mu}$, assuming $\mathcal{L}$ is a normal lattice.

The last part of the paper gives conditions for a set to be measurable with respect to a finitely subadditive outer measure.

2. Background and notations. We begin with some standard background material for the reader's convenience. We note that our notation is consistent with [1, 2, 4] and [7].

Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X . It is assumed that $\emptyset, X \in \mathcal{L}$. The lattice $\mathcal{L}$ is called: *normal* if for any $L_1, L_2 \in \mathcal{L}$ with $L_1 \cap L_2 = \emptyset$ there exist $L_3, L_4 \in \mathcal{L}$ with $L_1 \subset L_3'$, $L_2 \subset L_4'$ and $L_3' \cap L_4' = \emptyset$ (where prime denotes complement). $\mathcal{L}$ is *almost countably compact* if $\mu \in I_R(\mathcal{L}')$ implies $\mu \in I_\sigma(\mathcal{L})$. (See below for the definitions of $I_R(\mathcal{L}')$ and $I_\sigma(\mathcal{L})$.)

We give now some measure terminology. $M(\mathcal{L})$ denotes the set of finite valued bounded finitely additive measures on $A(\mathcal{L})$. Without loss of generality we may assume throughout that all measures are non-negative. A measure $\mu \in M(\mathcal{L})$ is called:

σ -smooth on $\mathcal{L}$ if for all sequences $\{L_n\}$ of sets of $\mathcal{L}$ with $L_n \downarrow \emptyset$, $\mu(L_n) \rightarrow 0$.

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σ -smooth on $A(\mathfrak{L})$ if for all sequences $\{A_n\}$ of sets of $A(\mathfrak{L})$ with $A_n \downarrow \emptyset$, $\mu(A_n) \rightarrow 0$.

$\mathfrak{L}$ -regular if for any $A \in A(\mathfrak{L})$, $\mu(A) = \sup\{\mu(L) \mid L \subset A, L \in \mathfrak{L}\}$.

We denote by $M_R(\mathfrak{L})$ the set of $\mathfrak{L}$ -regular measures of $M(\mathfrak{L})$; $M_\sigma(\mathfrak{L})$ the set of σ -smooth measures on $\mathfrak{L}$ of $M(\mathfrak{L})$; $M^\sigma(\mathfrak{L})$ the set of σ -smooth measures on $A(\mathfrak{L})$ of $M(\mathfrak{L})$; $M_R^\sigma(\mathfrak{L})$ the set of $\mathfrak{L}$ -regular measures of $M^\sigma(\mathfrak{L})$.

In addition, $I(\mathfrak{L})$, $I_R(\mathfrak{L})$, $I_\sigma(\mathfrak{L})$, $I^\sigma(\mathfrak{L})$ and $I_R^\sigma(\mathfrak{L})$ are the subsets of the corresponding M 's which consist of the non-trivial zero-one valued measures.

Related matters are considered in [3, 5, 6] and [7].

3. Finitely subadditive and cover regular outer measures. In this section we define finitely subadditive outer measures (**f. s. a.**) and cover regular outer measures in contrast to an ordinary outer measure which is countably subadditive. Some examples of **f. s. a.** outer measures are given and studied in detail.

Definition 1. μ defined on $\mathcal{P}(X)$ is a **f. s. a.** outer measure if:

- a) μ nondecreasing.
- b) $\mu\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mu(E_i)$, for any $E_1, E_2, \dots, E_n \subset X$.
- c) $\mu(\emptyset) = 0$.

Definition 2. Let $\mathbf{S}_\nu$ be the set of ν -measurable sets, with ν a **f. s. a.** outer measure. ν is called cover regular if for any $S \subset X$, there exists $E \in \mathbf{S}_\nu$ such that $S \subset E$ and $\nu(S) = \nu(E)$.

Let ν be a **f. s. a.** outer measure. We say that a set E is measurable with respect to ν if for any $A \subset X$:

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E')$$

We state the next two easy to prove theorems:

Theorem 1. Let ν be a **f. s. a.** outer cover regular measure and suppose $\nu(X)$ finite. Then $E \in \mathbf{S}_\nu$ iff $\nu(X) = \nu(E) + \nu(E')$.

Theorem 2. Let ν be a cover regular outer measure. Then for any sequence of sets $E_n \subset X$, $E_n \uparrow$ we have $\nu(\lim E_n) = \lim \nu(E_n)$.

Definition 3. Let $\mu \in M(\mathfrak{L})$ and define $\mu'(E) = \inf \sum_{i=1}^n \mu(L'_i)$, $E \subset \bigcup L'_i$, $L_i \in \mathfrak{L}$, $E \subset X$. Clearly, μ' is a **f. s. a.** outer measure. Let $\mathbf{S}_{\mu'}$ be the set of all μ' -measurable sets, where E is measurable with respect to μ' if for any $A \subset X$: $\mu'(A) = \mu'(A \cap E) + \mu'(A \cap E')$.

Theorem 3. $E \in \mathbf{S}_{\mu'}$ iff $\mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E')$ for all $A \in \mathfrak{L}$.

Proof.

- a) If $E \in \mathbf{S}_{\mu'}$, then the relation clearly holds.
- b) Let B be arbitrary in X , $B \subset A'$, $A \in \mathfrak{L}$. We have: $\mu(A') = \mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E') \geq \mu'(B \cap E) + \mu'(B \cap E')$. Since this is true for any $B \subset A'$, $A \in \mathfrak{L}$, $\mu'(B) \geq \mu'(B \cap E) + \mu'(B \cap E')$.

By the definition of μ' as an outer measure we have for $B = (B \cap E) \cup (B \cap E')$: $\mu'(B) \leq \mu'(B \cap E) + \mu'(B \cap E')$. Therefore $\mu'(B) = \mu'(B \cap E) + \mu'(B \cap E')$, B arbitrary in X . $\square$

Definition 4. Let $\mu \in M(\mathfrak{L})$ and define the inner measure $\mu_i(E) = \sup \mu(L)$, $L \subset E$, $L \in \mathfrak{L}$, $E \subset X$.

Theorem 4. The following statements are true:

- a) $\mu(X) = \mu_i(L) + \mu'(L')$, $L \in \mathfrak{L}$.
- b) $\mu(X) = \mu_i(L') + \mu'(L)$, $L \in \mathfrak{L}$.

Proof. Clear. $\square$

Definition 5. Let $\mu \in M_\sigma(\mathfrak{L})$, $E \subset X$ and define $\mu''(E) = \inf \sum_{i=1}^{\infty} \mu(L'_i)$, $E \subset \bigcup L'_i$, $L_i \in \mathfrak{L}$.

Clearly μ'' is an outer measure. Let $\mathbf{S}_{\mu''}$ be the set of μ'' -measurable sets. Similarly to Theorem 3 and its proof we have:

Theorem 5. $E \in \mathbf{S}_{\mu''}$ iff $\mu''(A') \geq \mu''(A' \cap E) + \mu''(A' \cap E')$ for all $A \in \mathfrak{L}$.

Theorem 6. Let $\mu \in M(\mathfrak{L})$, $E \subset X$. Then $E \in \mathbf{S}_{\mu'}$ iff $\mu_i(E) = \mu'(E)$.

Proof. Suppose $E \in \mathbf{S}_{\mu'}$. Then $\mu(X) = \mu'(E') + \mu'(E)$ and by Theorem 4 $\mu'(E') = \mu_i(E')$. Hence $\mu(X) - \mu'(E') = \mu(X) - \mu_i(E')$, and then $\mu_i(E) = \mu'(E)$.

Conversely, suppose $\mu_i(E) = \mu'(E)$. Then, given $\varepsilon > 0$, $\exists \hat{L} \in \mathfrak{L}$, $\hat{L} \subset E$ and $\mu(\hat{L}) + 1/2\varepsilon > \mu'(E)$. Also, by the definition of μ' , $\exists M \in \mathfrak{L}$ such that $M' \supset E \supset \hat{L}$ and $\mu(M') < \mu'(E) + 1/2\varepsilon$. Now let $A' \in \mathfrak{L}'$. Then $\mu'(A' \cap E) \leq \mu'(A' \cap M') = \mu(A' \cap M') = \mu(A') + \mu(M') - \mu(A' \cup M') \leq \mu(A') + \mu'(E) + 1/2\varepsilon - \mu(A' \cup \hat{L}) = \mu(A' \cap \hat{L}) + \varepsilon$. $E' \subset L'$, hence $\mu'(A' \cap E) + \mu'(A' \cap E') \leq \mu(A' \cap \hat{L}) + \varepsilon + \mu'(A' \cap \hat{L}') = \mu(A' \cap \hat{L}) + \varepsilon + \mu(A' \cap \hat{L}') = \mu(A') + \varepsilon$, ε arbitrary. $\square$

Definition 6. Let $\mu \in M(\mathfrak{L})$, $E \subset X$ and define $\hat{\mu}(E) = \inf \mu_i(L')$, $E \subset L'$, $L \in \mathfrak{L}$.

Theorem 7. The following statements are true:

- a) $\mu = \mu_i \leq \hat{\mu} \leq \mu'$ on $\mathfrak{L}$.
- b) $\hat{\mu} = \mu_i \leq \mu = \mu'$ on $\mathfrak{L}'$.

Proof. Let $A \in \mathfrak{L}$. $\mu_i(A) = \sup \mu(A) = \mu(A)$, hence $\mu = \mu_i$ on $\mathfrak{L}$. Take now $\hat{\mu}(A) = \inf \mu_i(L')$, $A \subset L'$, $L \in \mathfrak{L}$. Hence $\mu_i(A) \leq \mu_i(L')$ and since $\mu_i(A) \leq \hat{\mu}(A)$ we get $\mu_i \leq \hat{\mu}$ on $\mathfrak{L}$.

Now $\mu'(A) = \inf \mu(L')$ implies $\mu(A) \leq \mu(L')$, all $A \subset L'$, $L \in \mathfrak{L}$, hence $\mu \leq \mu'$ on $\mathfrak{L}$.

For $B \in \mathfrak{L}'$, $\mu'(B) = \mu(B)$, i.e. $\mu = \mu'$ on $\mathfrak{L}'$. $\mu'(A) = \inf \mu(L') = \inf \mu'(L')$, $A \subset L'$, which implies $\hat{\mu} \leq \mu'$ on $\mathfrak{L}$. By all of the above we get $\mu = \mu_i \leq \hat{\mu} \leq \mu'$ on $\mathfrak{L}$.

Now, $\hat{\mu}(B) = \inf \mu_i(B) = \mu_i(B)$, i.e. $\hat{\mu} = \mu_i$ on $\mathfrak{L}'$. Also $\mu_i(B) = \sup \mu(L)$, $L \subset B$ implies $\mu(B) \geq \sup \mu(L) = \mu_i(B)$, hence $\mu \geq \mu_i$. Therefore $\hat{\mu} = \mu_i \leq \mu = \mu'$ on $\mathfrak{L}'$. $\square$

4. Finitely subadditive outer measures and normal lattices. Properties of different finitely subadditive outer measures and relations between them are now investigated, assuming $\mathfrak{L}$ is a normal lattice.

Theorem 8. *Let $\mathfrak{L}$ be normal and let $\mu \in M(\mathfrak{L})$. Then the following statements are true:*

- a) μ_i is finitely additive on $\mathfrak{L}'$.
- b) $\hat{\mu}$ is a **f. s. a.** outer measure.
- c) If $\mathbf{S}_{\hat{\mu}}$ denotes the set of $\hat{\mu}$ -measurable sets, then $E \in \mathbf{S}_{\hat{\mu}}$ iff $\hat{\mu}(A') \geq \hat{\mu}(A' \cap E) + \hat{\mu}(A' \cap E')$ for all $A \in \mathfrak{L}$.
- d) $\mathfrak{L}' \subset \mathbf{S}_{\hat{\mu}}$.

Proof. a) $\mu_i(E) \geq 0$ for any $E \subset X$. For $E_1 \subset E_2$, arbitrary, we have $\mu_i(E_1) \leq \mu_i(E_2)$. Let $A_1 \subset A'_2 \cup B'_2$, $A_1, A_2, B_2 \in \mathfrak{L}$. Then $A_1 = B_1 \cup C_1$, where $B_1, C_1 \in \mathfrak{L}$, $B_1 \subset A'_2$, $C_1 \subset B'_2$. Therefore $\mu(A_1) \leq \mu(B_1) + \mu(C_1) \leq \mu_i(A'_2) + \mu_i(B'_2)$ and $\mu_i(A'_2 \cup B'_2) = \sup\{\mu(A), A \subset A'_2 \cup B'_2, A \in \mathfrak{L}\} \leq \mu_i(A'_2) + \mu_i(B'_2)$. Now, let $E_1, E_2 \subset X$ with $E_1 \cap E_2 = \emptyset$ and define $\mu_i(E_j) = \sup\{\mu(L_j), L_j \subset E_j, L_j \in \mathfrak{L}, j = 1, 2\}$. Given $\varepsilon > 0$ arbitrary, there exist $M \subset E_1$ and $N \subset E_2$ such that $\mu(M) + 1/2\varepsilon > \mu_i(E_1)$ and $\mu(N) + 1/2\varepsilon > \mu_i(E_2)$, hence $\mu_i(E_1) + \mu_i(E_2) < \mu(M) + \mu(N) + \varepsilon = \mu(M \cup N) + \varepsilon$, since $M \cap N = \emptyset$. ε arbitrary small implies $\mu_i(E_1) + \mu_i(E_2) \leq \mu_i(E_1 \cup E_2)$. Finally $\mu_i(\emptyset) = \mu(\emptyset) = 0$.

b) Clearly $\hat{\mu}(E) \geq 0$ for all $E \subset X$, $\hat{\mu}(\emptyset) = 0$, and $\hat{\mu}(E_1) \leq \hat{\mu}(E_2)$ for any $E_1 \subset E_2 \subset X$. Now let $E_i \subset X, i = 1, 2, \dots, N$ define $\hat{\mu}(E_i) = \inf\{\mu_i(L'_i), L_i \in \mathfrak{L}, E_i \subset L'_i\}$. Given $\varepsilon > 0$ we have $\hat{\mu}(E_i) + \frac{1}{N}\varepsilon \geq \mu_i(L'_i)$; then $\hat{\mu}(\bigcup E_i) \leq \mu_i(\bigcup L'_i) \leq \sum \mu_i(L'_i) \leq \sum \hat{\mu}(E_i) + \varepsilon$.

c) If $E \in \mathbf{S}_{\hat{\mu}}$, clearly the relation holds. Conversely let B be arbitrary in X , $B \subset A'$, $A \in \mathfrak{L}$. We have: $\hat{\mu}(A') \geq \hat{\mu}(A' \cap E) + \hat{\mu}(A' \cap E') \geq \hat{\mu}(B \cap E) + \hat{\mu}(B \cap E')$ for any $B \subset A'$, $A \in \mathfrak{L}$, so $\hat{\mu}(B) \geq \hat{\mu}(B \cap E) + \hat{\mu}(B \cap E')$.

By the definition of $\hat{\mu}$ as a **f. s. a.** outer measure we have for $B = (B \cap E) \cup (B \cap E')$: $\hat{\mu}(B) \leq \hat{\mu}(B \cap E) + \hat{\mu}(B \cap E')$. Therefore $\hat{\mu}(B) = \hat{\mu}(B \cap E) + \hat{\mu}(B \cap E')$.

d) Let $A, D \in \mathfrak{L}$ and $L' \in \mathfrak{L}'$ with $D \subset A' \cap L'$ and let $F \in \mathfrak{L}$, $F \subset A' \cap D'$. We have $A' \cap L', A' \cap D' \in \mathfrak{L}'$ and $D \cap F = \emptyset$. $\hat{\mu}(A') = \mu_i(A') \geq \mu(D \cup F)$, and $\hat{\mu}(A') \geq \mu(D) + \sup\{\mu(F), F \subset A' \cap D', F \in \mathfrak{L}\} = \mu(D) + \mu_i(A' \cap D') = \mu(D) + \hat{\mu}(A' \cap D') \geq \mu(D) + \hat{\mu}(A' \cap L)$. Therefore $\hat{\mu}(A') \geq \sup\{\mu(D), D \subset A' \cap L', D \in \mathfrak{L}\} + \hat{\mu}(A' \cap L) = \mu_i(A' \cap L') + \hat{\mu}(A' \cap L) = \hat{\mu}(A' \cap L') + \hat{\mu}(A' \cap L)$ i.e. $\mathfrak{L}' \subset \mathbf{S}_{\hat{\mu}}$. $\square$

Theorem 9. *Suppose $\mathfrak{L}$ is normal and let $\mu \in M(\mathfrak{L})$. Then:*

- a) $\nu = \hat{\mu}|_{A(\mathfrak{L})} \in M_R(\mathfrak{L})$.
- b) $\nu(X) = \mu(X)$.
- c) $\mu \leq \nu$ on $\mathfrak{L}$.
- d) $\hat{\mu} = \mu'$ on $\mathfrak{L}$.

Proof. a) $\nu(L') = \hat{\mu}(L') = \mu_i(L') = \sup\{\mu(M), M \subset L', M \in \mathfrak{L}\}$. Therefore $\nu(L') = \sup\{\hat{\mu}(M), M \subset L', M \in \mathfrak{L}\} = \sup\{\nu(M), M \subset L', M \in \mathfrak{L}\}$, i.e. $\nu \in M_R(\mathfrak{L})$.

b) $\nu(X) = \hat{\mu}(X) = \mu(X)$, since $\mu(X) \leq \hat{\mu}(X)$ and $\hat{\mu}(X) \leq \mu(X)$.

c) $\mu \leq \hat{\mu}$ on $\mathfrak{L}$ implies $\mu \leq \nu$ on $\mathfrak{L}$.

d) We have $\nu = \hat{\mu}|_{A(\mathfrak{L})} \in M_R(\mathfrak{L})$, hence $\mu \leq \nu = \nu' = \mu'$ on $\mathfrak{L}$, therefore $\hat{\mu} = \mu'$ on $\mathfrak{L}$. $\square$

Theorem 10. Let $\mu \in M(\mathfrak{L})$ and define $\mathbf{L}_\mu = \{L \in \mathfrak{L}, \mu(L) = \mu'(L)\}$. Then:

- a) $\mathbf{S}_\mu \cap \mathfrak{L} = \mathbf{L}_\mu$.
- b) $\mathfrak{L} \subset \mathbf{S}_{\mu'}$ iff $\mu \in M_R(\mathfrak{L})$.

Proof. a) $L \in \mathbf{S}_{\mu'} \cap \mathfrak{L}$, hence $\mu'(L) = \mu_i(L) = \mu(L)$. Thus $L \in \mathbf{L}_\mu$ implies $\mathbf{S}_{\mu'} \cap \mathfrak{L} \subset \mathbf{L}_\mu$. Conversely let $L \in \mathbf{L}_\mu$ and since $\mu = \mu_i$ on $\mathfrak{L}$ we get $\mu'(L) = \mu_i(L)$ hence $L \in \mathbf{S}_{\mu'}$.

b) Suppose $\mathfrak{L} \subset \mathbf{S}_{\mu'}$; then for any $L \in \mathfrak{L}$, $\mu'(L) = \mu_i(L) = \mu(L)$, therefore $\mu \in M_R(\mathfrak{L})$. Conversely if $\mu \in M_R(\mathfrak{L})$, $\mu(A) = \inf\{\mu(L'), A \subset L', L \in \mathfrak{L}\} = \mu'(A)$. $\square$

Definition 7. Let $\mu \in M(\mathfrak{L})$, $E \subset X$ and define $\mu_j(E) = \sup\{\mu'(L), L \subset E, L \in \mathfrak{L}\}$.

Definition 8. Let $\mu \in M(\mathfrak{L})$ and define $M_\omega(\mathfrak{L})$ the set of all $\mu \in M(\mathfrak{L})$ such that $\mu(L') = \sup\{\mu'(A), A \subset L', A, L \in \mathfrak{L}\}$.

Theorem 11. Let $\mu \in M(\mathfrak{L})$. Then:

- a) $\mu = \mu_j$ on $\mathfrak{L}'$ iff $\mu \in M_\omega(\mathfrak{L})$.
- b) $\mu = \mu'$ on $\mathfrak{L}$ iff $\mu \in M_R(\mathfrak{L})$.

Proof. Clearly. $\square$

Theorem 12. Let $\mu, \nu \in M(\mathfrak{L})$ with $\mu \leq \nu$ on $\mathfrak{L}$ and $\mu(X) = \nu(X)$. Then:

- a) $\mathbf{S}_{\mu'} \subset \mathbf{S}_{\nu'}$.
- b) $\mathbf{L}_\mu \subset \mathbf{L}_\nu$.

Proof. a) $\nu' \leq \mu'$ on all sets and then we have $\mu \leq \nu \leq \nu' \leq \mu'$ on $\mathfrak{L}$. Now suppose $E \in \mathbf{S}_{\mu'}$; $\mu'(E) = \mu_i(E)$, hence $\nu'(E) = \nu_i(E)$ so $E \in \mathbf{S}_{\nu'}$.

b) If $L \in \mathbf{L}_\mu$ then $\mu(L) = \mu'(L)$ and then it follows $\nu(L) = \nu'(L)$, therefore $L \in \mathbf{L}_\nu$. $\square$

Theorem 13. If $\mathfrak{L}$ is normal then $\mu_j \leq \hat{\mu}$ on $\mathfrak{L}$.

Proof. Let $A \in \mathfrak{L}$. $\mu_j(A) = \sup \mu'(L), L \subset A, L \in \mathfrak{L}$. Hence $\sup \mu'(L) \leq \mu'(A)$, which implies $\mu_j(A) \leq \mu'(A)$, and since A arbitrary it follows $\mu_j \leq \mu'$ on $\mathfrak{L}$. $\mathfrak{L}$ is normal, hence $\hat{\mu} = \mu'$ on $\mathfrak{L}$, hence $\mu_j \leq \hat{\mu}$ on $\mathfrak{L}$. Let $B \in \mathfrak{L}'$; $\mu_i(B) = \sup \mu(L), L \subset B, L \in \mathfrak{L}$, and $\mu_j(B) = \sup \mu'(B)$, hence $\mu_i \leq \mu_j$ on $\mathfrak{L}'$. But $\hat{\mu} = \mu_i$ on $\mathfrak{L}'$ implies $\hat{\mu} \leq \mu_j$ on $\mathfrak{L}'$. Therefore we have: $\mu = \mu_i \leq \mu_j \leq \hat{\mu} = \mu'$ on $\mathfrak{L}$ and $\hat{\mu} = \mu_i \leq \mu_j \leq \mu = \mu'$ on $\mathfrak{L}'$. $\square$

Theorem 14. If $\mathfrak{L}$ is normal and $\mu \in M_\omega(\mathfrak{L})$ then $\mu \in M_R(\mathfrak{L})$.

Proof. Clearly, by previous work. $\square$

5. Sets measurable with respect to a finitely subadditive outer measure. In this section we consider conditions for a set to be measurable with respect to a **f. s. a.** outer measure. Although μ' might not be a regular **f. s. a.** outer measure, it is still true that an arbitrary set $E \subset X$ is μ' -measurable if and only if it splits just X additively.

Let ν be a **f. s. a.** outer measure, $\nu(X)$ finite. Let $\mathbf{S}_\nu$ be the set of all ν -measurable sets. Then clearly $\mathbf{S}_\nu$ is an algebra and the restriction $\nu|_{\mathbf{S}_\nu} = \mu$ is a finitely additive measure.

Definition 9. Let ν be a **f. s. a.** outer measure and let $E \subset X$. Define $\nu^0(E) = \inf \nu(M)$, $E \subset M$, $M \in \mathbf{S}_\nu$; $\nu_0(E) = \sup \nu(M)$, $M \subset E$, $M \in \mathbf{S}_\nu$.

Theorem 15. Let ν be a **f. s. a.** outer measure. Then:

- a) $\nu_0(G) = \nu(E) - \nu^0(E - G)$, $G \subset X$ and $G \subset E \in \mathbf{S}_\nu$.
(In particular $\nu_0(G) = \nu(X) - \nu^0(G')$.)
- b) $\nu_0 \leq \nu \leq \nu^0$.
- c) ν^0 is a **f. s. a.** outer measure; ν_0 is an inner measure.
- d) ν_0 supermodular i.e. $\nu_0(E \cup F) + \nu_0(E \cap F) \geq \nu_0(E) + \nu_0(F)$; ν^0 submodular i.e. $\nu^0(E \cup F) + \nu^0(E \cap F) \leq \nu^0(E) + \nu^0(F)$, for all $E, F \in \mathbf{S}_\nu$.

Proof. Omitted. $\square$

Theorem 16. Let $\mu \in M(\mathfrak{L})$ and $E \subset X$. Then

- a) $\mu(X) = \mu_i(E) + \mu'(E')$
- b) $E \in \mathbf{S}_{\mu'}$ iff $\mu(X) = \mu'(E) + \mu'(E')$.

Proof. Clearly. $\square$

Theorem 17. Let ν be a **f. s. a.** outer measure. Then:

- a) If $E \in \mathbf{S}_\nu$ then $\nu_0(E) = \nu^0(E)$
- b) Suppose ν satisfies condition

$$(*) \quad E \in \mathbf{S}_\nu \text{ iff } \nu(X) = \nu(E) + \nu(E').$$

Then $E \in \mathbf{S}_\nu$ iff $\nu_0(E) = \nu^0(E)$.

- c) If ν satisfies condition $(*)$ then ν^0 also satisfies condition $(*)$.

Proof. a) If $E \in \mathbf{S}_\nu$, $\nu^0(E) = \inf \nu(M) = \nu(E) = \sup \nu(M) = \nu_0(E)$, where $E \subset M$, $M \in \mathbf{S}_\nu$.

b) In general we have $\nu_0 \leq \nu \leq \nu^0$ and since $\nu_0(E) = \nu^0(E)$ it follows $\nu(X) = \nu_0(E) + \nu^0(E') = \nu^0(E) + \nu^0(E') = \nu(E) + \nu(E')$.

c) Let $E \in \mathbf{S}_\nu$ and $\varepsilon > 0$. For any set $A \subset X$, there exists $B \in \mathbf{S}_\nu$ such that $A \subset B$ and $\nu(B) \leq \nu^0(A) + \varepsilon$, hence $\nu^0(A \cap E) + \nu^0(A \cap E') \leq \nu^0(B \cap E) + \nu^0(B \cap E') = \nu(B \cap E) + \nu(B \cap E') = \nu(B) \leq \nu^0(A) + \varepsilon$, therefore $E \in \mathbf{S}_{\nu^0}$. Now suppose $\nu^0(X) = \nu^0(E') + \nu^0(E)$ and then $\nu(X) \geq \nu(E) + \nu(E')$. Since ν satisfies condition $(*)$ it follows that $E \in \mathbf{S}_\nu$ and then by the above $E \in \mathbf{S}_{\nu^0}$. $\square$

Theorem 18. Let ν be a **f. s. a.** outer measure and let $\nu(X)$ be finite. Then $\mathbf{S}_{\nu_0} = \{A \subset X, \nu_0(A) = \nu^0(A)\}$.

Proof. Let $A \in \mathbf{S}_{\nu_0}$; then $\nu_0(B) \geq \nu_0(A \cap B) + \nu_0(A' \cap B)$, $B \subset X$. It follows $\nu(X) = \nu_0(A) + \nu(X) - \nu^0(A)$, i.e. $\nu_0(A) = \nu^0(A)$. Conversely suppose $\nu_0(A) = \nu^0(A)$ and take arbitrary $E \subset X$: $\nu_0(A \cap E) + \nu_0(A' \cap E) \geq \nu_0(A) + \nu_0(E) - \nu_0(A \cup E) + \nu_0(A') + \nu_0(E) - \nu_0(A' \cup E) \geq \nu(X) + 2\nu_0(E) - \nu_0[(A \cup E) \cup (A' \cup E)] - \nu_0[(A \cup E) \cap (A' \cup E)] = \nu_0(E)$. Clearly $\nu_0(E) \geq \nu_0(A \cap E) + \nu_0(A' \cap E)$, therefore $A \in \mathbf{S}_{\nu_0}$. $\square$

Theorem 19. If ν satisfies condition $(*)$ then: $\mathbf{S}_{\nu_0} = \mathbf{S}_\nu = \mathbf{S}_{\nu^0}$.

Proof. Let $E \in \mathbf{S}_\nu$; since ν satisfies condition $(*)$, $\nu_0(E) = \nu^0(E)$ hence $\mathbf{S}_\nu = \mathbf{S}_{\nu_0}$ by previous work. Now let $E \in \mathbf{S}_{\nu^0}$; $\nu(X) = \nu^0(X) = \nu^0(E) + \nu^0(E')$, hence $\nu(X) = \nu(E) + \nu(E')$ and since ν satisfies condition $(*)$ it follows $E \in \mathbf{S}_\nu$. $\square$

Theorem 20. μ' satisfies condition $(*)$.

Proof. Clearly by Theorem 16. $\square$

Definition 10. Let ν be a **f. s. a.** outer measure. ν is called regular if $\nu = \nu^0$.

Theorem 21. Let ν be a **f. s. a.** outer measure.

- a) If ν is cover regular then ν is regular.
- b) If ν is regular then ν satisfies condition $(*)$.

Proof. a) $\nu(E) \leq \nu^0(E) = \inf \nu(M)$, $E \subset M$, $M \in \mathbf{S}_\nu$. ν cover regular hence for $E \subset X$ there exists $M \in \mathbf{S}_\nu$, $E \subset M$ and $\nu(M) = \nu(E)$.

b) Let A be any set and let $\varepsilon > 0$; there exists $M \in \mathbf{S}_\nu$ such that $A \subset M$ and $\nu(A) = \nu^0(A) > \nu(M) - 1/2\varepsilon$. Now suppose E such that $\nu(X) = \nu(E) + \nu(E')$. Hence $\nu(X) = \nu(E) + \nu(E') = \nu(E \cap M) + \nu(E \cap M') + \nu(E' \cap M) + \nu(E' \cap M') = \nu(M) + \nu(M')$. Therefore $\nu(A \cap E) + \nu(A \cap E') \leq \nu(E \cap M) + \nu(E' \cap M) \leq \nu(M) < \nu(A) + \varepsilon$, ε arbitrary small. $\square$

Theorem 22. Suppose that ν satisfies condition $(*)$. Then:

- a) $\nu^{00} = \nu^0$
- b) ν^0 satisfies condition $(*)$.

Proof. a) $\nu^{00}(E) = (\nu^0)^0(E) = \inf \nu^0(M)$, $E \subset M$, $M \in \mathbf{S}_{\nu^0}$, hence since ν satisfies condition $(*)$ we get $M \in \mathbf{S}_\nu$ and then $\nu^0(M) = \nu(M)$ i.e. $(\nu^0)^0 = \nu^0$.

b) If ν^0 satisfies condition $(*)$, by definition ν^0 is regular and by Theorem 21 ν^0 satisfies condition $(*)$. $\square$

Note that the proof of Theorem 22 (b) gives an alternate proof for Theorem 17 (c).

Résumé substantiel en français.

Introduction. Soit $\mathfrak{L}$ un treillis de parties d'un ensemble X , tel que $\emptyset \in \mathfrak{L}$, $X \in \mathfrak{L}$. Soit $I(\mathfrak{L})$ l'ensemble des mesures finiment additives, non-triviales et à valeurs dans $\{0, 1\}$, définies sur l'algèbre $A(\mathfrak{L})$ engendrée par $\mathfrak{L}$. L'ensemble $I(\mathfrak{L})$ a beaucoup été étudié (voir, par exemple, [3, 5, 6]) dans le cas où $\mathfrak{L}$ est un treillis normal. Désignons par $M(\mathfrak{L})$ l'ensemble des mesures finies, bornées, finiment additives et non négatives sur $A(\mathfrak{L})$.

Dans la première partie de ce travail, nous définissons une mesure $\mu \in M(\mathfrak{L})$ et lui associons une mesure extérieure auxiliaire finiment additive μ' qui permet d'étendre à $M(\mathfrak{L})$ certains résultats concernant $I(\mathfrak{L})$. D'autres mesures extérieures utiles μ'' et $\hat{\mu}$ sont aussi analysées. Nous terminons par l'énoncé de certaines conditions assurant la mesurabilité d'un ensemble par rapport à une mesure extérieure donnée. La terminologie, les notions et les notations utilisées dans le présent travail sont classiques et en accord avec celles de [1, 2, 4, 7]. Pour faciliter la lecture, la section 2 plus haut (Background and notations) contient quelques rappels : treillis normal; treillis presque dénombrablement compact; l'ensemble $M_R(\mathfrak{L}) \subseteq M(\mathfrak{L})$ des mesures $\mathfrak{L}$ -régulières; l'ensemble $M_\sigma(\mathfrak{L}) \subseteq M(\mathfrak{L})$ des mesures σ -continues; l'ensemble $M^\sigma(\mathfrak{L}) \subseteq M(\mathfrak{L})$ des mesures σ -continues définies sur $A(\mathfrak{L})$; l'ensemble $M_R^\sigma(\mathfrak{L})$ des mesures $\mathfrak{L}$ -régulières de $M^\sigma(\mathfrak{L})$. Les ensembles $I(\mathfrak{L})$, $I_R(\mathfrak{L})$, $I_\sigma(\mathfrak{L})$, $I^\sigma(\mathfrak{L})$ et $I_R^\sigma(\mathfrak{L})$ désignent les sous-ensembles des M correspondants formés seulement des mesures non-triviales à valeurs dans $\{0, 1\}$.

Mesures extérieures finiment sous-additives et recouvrantes régulières.

Définition 1. Une fonction μ définie sur $\mathcal{P}(X)$ est une mesure extérieure finiment sous-additive (**f. s. a.**) si

- a) μ est non décroissante,
- b) $\mu\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mu(E_i)$ pour tous $E_1, \dots, E_n \subset X$, et
- c) $\mu(\emptyset) = 0$.

Définition 2. Soit $\mathbf{S}_\nu$ l'ensemble des ensembles ν -mesurables où ν est une mesure **f. s. a.** donnée. La mesure extérieure ν est recouvrante régulière si pour tout $S \subset X$, il existe $E \in \mathbf{S}_\nu$ tel que $S \subset E$ et $\nu(S) = \nu(E)$. Un ensemble E est mesurable par rapport à ν si pour tout $A \subset X$ on a $\nu(A) = \nu(A \cap E) + \nu(A \cap E')$, où E' désigne le complément de E .

Théorème 1. Soit ν une mesure extérieure **f. s. a.** recouvrante régulière et supposons que $\nu(X) < \infty$. Alors $E \in \mathbf{S}_\nu$ si et seulement si $\nu(X) = \nu(E) + \nu(E')$.

Théorème 2. Soit ν une mesure extérieure recouvrante régulière. Alors pour toute suite d'ensembles $E_n \subset X$, $E_n \uparrow$, on a $\nu(\lim E_n) = \lim \nu(E_n)$.

Définition 3. Soit $\mu \in M(\mathfrak{L})$ et définissons μ' par $\mu'(E) = \inf \sum_{i=1}^n \mu(L'_i)$, $E \subset \bigcup L'_i$, $L_i \in \mathfrak{L}$, $E \subset X$. Clairement, μ' est une mesure extérieure **f. s. a.**

Théorème 3. Soit $\mathbf{S}_{\mu'}$ l'ensemble des ensembles μ' -mesurables. Alors $E \in \mathbf{S}_{\mu'}$ si et seulement si $\mu'(A') \geq \mu'(A' \cap E) + \mu'(A' \cap E')$ pour tout $A \in \mathfrak{L}$.

Définition 4. Soit $\mu \in M(\mathfrak{L})$ et définissons une mesure intérieure μ_i par $\mu_i(E) = \sup \mu(L)$, $L \subset E$, $L \in \mathfrak{L}$, $E \subset X$.

Théorème 4. Les énoncés suivants sont vrais :

- a) $\mu(X) = \mu_i(L) + \mu'(L')$, $L \in \mathfrak{L}$;
- b) $\mu(X) = \mu_i(L') + \mu'(L)$, $L \in \mathfrak{L}$.

Définition 5. Soit $\mu \in M_\sigma(\mathcal{L})$, $E \subset X$ et définissons μ'' par $\mu''(E) = \inf \sum_{i=1}^\infty \mu(L'_i)$, $E \subset \bigcup L'_i$, $L_i \in \mathcal{L}$. Soit $\mathbf{S}_{\mu''}$ l'ensemble des ensembles μ'' -mesurables.

Théorème 5. $E \in \mathbf{S}_{\mu''}$ si et seulement si $\mu''(A') \geq \mu''(A' \cap E) + \mu''(A' \cap E')$ pour tout $A \in \mathcal{L}$.

Théorème 6. Soit $\mu \in M(\mathcal{L})$, $E \subset X$. Alors $E \in \mathbf{S}_{\mu'}$ si et seulement si $\mu_i(E) = \mu'(E)$.

Définition 6. Soit $\mu \in M(\mathcal{L})$, $E \subset X$ et définissons $\hat{\mu}$ par $\hat{\mu}(E) = \inf \mu_i(L')$, $E \subset L'$, $L \in \mathcal{L}$.

Théorème 7. Les énoncés suivants sont vrais :

- a) $\mu = \mu_i \leq \hat{\mu} \leq \mu'$ sur $\mathcal{L}$;
- b) $\hat{\mu} = \mu_i \leq \mu = \mu'$ sur $\mathcal{L}'$.

Mesures extérieures finiment sous-additives et treillis normaux.

Théorème 8. Soit $\mathcal{L}$ un treillis normal et $\mu \in M(\mathcal{L})$. Alors :

- a) μ_i est finiment sous-additive sur $\mathcal{L}'$;
- b) $\hat{\mu}$ est une mesure extérieure **f. s. a.**;
- c) si $\mathbf{S}_{\hat{\mu}}$ désigne l'ensemble des ensembles $\hat{\mu}$ -mesurables, alors $E \in \mathbf{S}_{\hat{\mu}}$ si et seulement si $\hat{\mu}(A') \geq \hat{\mu}(A' \cap E) + \hat{\mu}(A' \cap E')$ pour tout $A \in \mathcal{L}$;
- d) $\mathcal{L}' \subset \mathbf{S}_{\hat{\mu}}$.

Théorème 9. Supposons que $\mathcal{L}$ est un treillis normal et $\mu \in M(\mathcal{L})$. Alors :

- a) $\nu = \hat{\nu}|A(\mathcal{L}) \in M_R(\mathcal{L})$;
- b) $\nu(X) = \mu(X)$;
- c) $\mu \leq \nu$ sur E ;
- d) $\hat{\mu} = \mu'$ sur $\mathcal{L}$.

Théorème 10. Soit $\mu \in M(\mathcal{L})$ et posons $\mathbf{L}_\mu = \{L \in \mathcal{L} : \mu(L) = \mu'(L)\}$. Alors :

- a) $\mathbf{S}_{\mu'} \cap \mathcal{L} = \mathbf{L}_\mu$;
- b) $\mathcal{L} \subset \mathbf{S}_{\mu'}$ si et seulement si $\mu \in M_R(\mathcal{L})$.

Définition 7. Soit $\mu \in M(\mathcal{L})$, $E \subset X$ et définissons μ_j par $\mu_j(E) = \sup\{\mu'(L) : L \subset E, L \in \mathcal{L}\}$.

Définition 8. Soit $\mu \in M(\mathcal{L})$ et définissons $M_\omega(\mathcal{L})$ par l'ensemble des $\mu \in M(\mathcal{L})$ telles que $\mu(L') = \sup\{\mu'(A) : A \subset L', A \in \mathcal{L}\}$, $L \in \mathcal{L}$.

Théorème 11. Soit $\mu \in M(\mathcal{L})$. Alors :

- a) $\mu = \mu_j$ sur $\mathcal{L}'$ si et seulement si $\mu \in M_\omega(\mathcal{L})$;
- b) $\mu = \mu'$ sur $\mathcal{L}$ si et seulement si $\mu \in M_R(\mathcal{L})$.

Théorème 12. Soit $\mu, \nu \in M(\mathcal{L})$ telles que $\mu \leq \nu$ sur $\mathcal{L}$ et $\mu(X) = \nu(X)$. Alors :

- a) $\mathbf{S}_{\mu'} \subset \mathbf{S}_{\nu'}$;
- b) $\mathbf{L}_\mu \subset \mathbf{L}_\nu$.

Théorème 13. *Si $\mathcal{L}$ est normal, alors $\mu_j \leq \hat{\mu}$ sur $\mathcal{L}$.*

Théorème 14. *Si $\mathcal{L}$ est normal et $\mu \in M_\omega(\mathcal{L})$, alors $\mu \in M_R(\mathcal{L})$.*

Ensembles mesurables par rapport à une mesure extérieure finiment sous-additive.

Définition 9. Soit ν une mesure extérieure **f. s. a.** et $E \subset X$. Définissons ν^0, ν_0 par $\nu^0(E) = \inf \nu(M), E \subset M, M \in \mathbf{S}_\nu, \nu_0(E) = \sup \nu(M), M \in \mathbf{S}_\nu$.

Théorème 15. *Soit ν une mesure extérieure **f. s. a.** Alors :*

- a) $\nu_0(G) = \nu(E) - \nu^0(E - G), G \subset X$ et $G \subset E \in \mathbf{S}_\nu$;
- b) $\nu_0 \leq \nu \leq \nu^0$;
- c) ν^0 est une mesure extérieure **f. s. a.** et ν_0 est une mesure intérieure;
- d) ν_0 est surmodulaire, c'est-à-dire $\nu_0(E \cup F) + \nu_0(E \cap F) \geq \nu_0(E) + \nu_0(F)$,
 ν^0 est sous-modulaire, c'est-à-dire $\nu^0(E \cup F) + \nu^0(E \cap F) \leq \nu^0(E) + \nu^0(F)$,
pour tous $E, F \in \mathbf{S}_\nu$.

Théorème 16. *Soit $\mu \in M(\mathcal{L})$ et $E \subset X$. Alors*

- a) $\mu(X) = \mu_i(E) + \mu'(E')$,
- b) $E \in \mathbf{S}_{\mu'}$ si et seulement si $\mu(X) = \mu'(E) + \mu'(E')$.

Théorème 17. *Soit ν une mesure extérieure **f. s. a.** Alors :*

- a) si $E \in \mathbf{S}_\nu$, alors $\nu_0(E) = \nu^0(E)$;
- b) si ν satisfait la condition

$$(*) \quad E \in \mathbf{S}_\nu \text{ si et seulement si } \nu(X) = \nu(E) + \nu(E'),$$

alors $E \in \mathbf{S}_\nu$ si et seulement si $\nu_0(E) = \nu^0(E)$;

- c) si ν satisfait la condition $(*)$ alors ν^0 satisfait aussi la condition $(*)$.

Théorème 18. *Soit ν une mesure extérieure **f. s. a.** et supposons que $\nu(X) < \infty$. Alors $\mathbf{S}_{\nu_0} = \{A \subset X : \nu_0(A) = \nu^0(A)\}$.*

Théorème 19. *Si ν satisfait la condition $(*)$, alors $\mathbf{S}_{\nu_0} = \mathbf{S}_\nu = \mathbf{S}_{\nu^0}$.*

Théorème 20. μ' satisfait la condition $(*)$.

Définition 10. Une mesure extérieure **f. s. a.** ν est dite régulière si $\nu = \nu^0$.

Théorème 21. *Soit ν une mesure extérieure **f. s. a.**, alors :*

- a) si ν est recouvrante régulière alors ν est régulière;
- b) si ν est régulière, alors ν satisfait la condition $(*)$.

Théorème 22. *Supposons que ν satisfait la condition $(*)$. Alors :*

- a) $\nu^{00} = \nu^0$;
- b) ν^0 satisfait la condition $(*)$.

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