

EXAMPLES OF INDECOMPOSABLE NON-IRREDUCIBLE LORENTZIAN MANIFOLDS

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RÉSUMÉ. Localement, la métrique d'une variété lorentzienne indécomposable non irréductible est construite. Des cas particuliers sont étudiés. D'autres exemples d'une telle variété sont donnés, et on démontre qu'une variété lorentzienne d'Einstein ne peut être ni de type 3 ni de type 4.

ABSTRACT. Locally, the metric tensor is constructed for an indecomposable non-irreducible Lorentzian manifold. Particular cases are examined. New examples of such manifolds are given, and we show that an Einstein Lorentzian manifold cannot be neither of type 3 nor type 4.

1. Introduction. H. Wu showed in [17] that the (restricted) holonomy representation of a pseudo-Riemannian manifold is the exterior direct sum of a trivial representation and a finite number of indecomposable representations (see §2 for definitions). On the other hand, all the irreducible (restricted) holonomy representations of a pseudo-Riemannian manifold are classified (see [1, 3, 5, 6, 8, 9, 10, 13]). So our investigation is reduced to that of indecomposable, non-irreducible holonomy groups. Then the following questions remain open problems.

- 1) Which are the possible candidates of indecomposable, non-irreducible holonomy groups for a pseudo-Riemannian manifold ?
- 2) Among those candidates, which are restricted holonomy groups of at least one pseudo-Riemannian manifold ?

Up to now, the answer to these problems is not known, except in the symmetric Lorentzian case [12], or in the symmetric pseudo-Riemannian case in dimensions ≤ 7 [11]. In [4], the author and L. Berard Bergery have given the solution to the first question, in the Lorentzian case. More precisely, they have given a complete classification of connected subgroups of the orthogonal group $O(1, n + 1)$, which are not irreducible, but do not leave invariant any non-degenerate proper subspace of $\mathbb{R}^{n+2}$ (Theorems 2 and 3). In this paper we do not give the solution to the second question, but give locally some information about the structure of a Lorentzian manifold read from the structure of the holonomy group which verifies both the theorems 2 and 3. In other words, we construct a coordinate system in which we have a reduced form of a metric whose

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holonomy group verifies the conclusions of both the theorems 2 and 3 (Theorem 5). Furthermore, when the dimension of the holonomy group is small compared to that of $O(1, n+1)$, like in the symmetric case, then the Lorentzian manifold is locally almost determined. We examine some particular cases (Corollary 9). In [4], some examples of metrics are constructed, in dimensions ≤ 7 , when the holonomy verifies the conclusions of both the theorems 2 and 3. In §4, we give some new examples, and we show in §5, that an Einstein Lorentzian manifold cannot be neither of type 3 nor type 4 (Theorem 11).

2. Preliminaries. We recall briefly the main definitions, and refer to classical textbooks like [3, chapter 10] or [15, chapter IV] for more details. Let $(M, \langle \rangle)$ be a connected pseudo-Riemannian manifold of dimension m . There is a unique torsion-free metric connection D , called the Levi-Civita connection, and it gives rise to a parallel transport τ_c along each piecewise smooth paths $c : [0, 1] \rightarrow M$. For $x \in M$, the restricted holonomy group H_x is the subgroup of $SO(T_x M, \langle \rangle_x)$ generated by parallel transports along all the x -based loops which are homotopic to the identity. Also, if c is as above, with $c(0) = x$ and $c(1) = y$, the map τ_c induces an isomorphism of $T_x M$ with $T_y M$ which identifies H_x with H_y . Choose an $\theta \in M$ and an isomorphism $\Phi : T_\theta M \rightarrow \mathbb{R}^m$. Since M is connected, the conjugacy class of the subgroup $\mathbf{H} \subset SO(m)$ which corresponds under Φ to $H_\theta \subset SO(T_\theta M, \langle \rangle_\theta)$ is independent of the choice of θ or Φ . By abuse of language, we speak of $\mathbf{H}$ as the restricted holonomy group (or the restricted holonomy representation) of $(M, \langle \rangle)$. The Lie algebra of H_x is called the holonomy algebra at x . A theorem of W. Ambrose and I. M. Singer [2] asserts that the holonomy algebra $\mathcal{H}$ at x is generated by all $\tau_c^{-1} \circ R_y(X, Y) \circ \tau_c$, where c runs through all C^1 -piecewise paths starting from x , and ending at y ; X, Y run through $T_y M$, and R is a curvature tensor of $(M, \langle \rangle)$. If $(M, \langle \rangle)$ is a real analytic pseudo-Riemannian manifold, then $\mathcal{H}$ is generated by the set of all covariant derivatives $D_{X_1} \cdots D_{X_k} R(X, Y)$ of the curvature tensor evaluated at x (see [15, section II-10 and III-9]).

Definition 1. A connected pseudo-Riemannian manifold $(M, \langle \rangle)$ is called *irreducible* if its restricted holonomy representation $\mathbf{H}$ at a point θ is irreducible; *indecomposable* if $\mathbf{H}$ does not leave invariant any *nondegenerate* proper subspace of the tangent space $T_\theta M$.

Now we suppose $(M, \langle \rangle)$ to be indecomposable, but not irreducible. Then $\mathbf{H}$ leaves invariant at least one degenerate subspace E of $T_\theta M$. Hence $\mathbf{H}$ leaves also invariant the orthogonal complement $E^\perp$ and the intersection $E \cap E^\perp = \mathbb{R}e_0$, which is a one-dimensional isotropic subspace. It is then clear that, up to conjugacy, the holonomy algebra $\mathcal{H}$ of $\mathbf{H}$ is contained in

$$\mathcal{G} = \left\{ \begin{pmatrix} a & {}^t X & 0 \\ 0 & A & X \\ 0 & 0 & -a \end{pmatrix}, a \in \mathbb{R}, X \in \mathbb{R}^n, A \in so(n) \right\}$$

The connected Lie group associated to $\mathcal{G}$ is denoted by $\mathbf{G}$. In order to simplify the notations, we identify $\mathcal{G}$ with $\mathbb{R} \oplus so(n) \oplus \mathbb{R}^n$, and we denote the above element of $\mathbf{G}$ by (a, A, X) . Notice that $\mathcal{A} = \{(a, 0, 0), a \in \mathbb{R}\}$ is a subalgebra of $\mathcal{G}$; $\mathcal{N} = \{(0, 0, X), X \in \mathbb{R}^n\}$ is an abelian ideal of $\mathcal{G}$ and $\mathcal{K} = \{(0, A, 0), A \in so(n)\}$ is a subalgebra of $\mathcal{G}$ which is isomorphic to $so(n)$.

We shall denote by $\mathbf{A}$ (resp. $\mathbf{N}$, $\mathbf{K}$) the connected subgroup of $\mathbf{G}$ generated by $\mathcal{A}$ (resp. $\mathcal{N}$, $\mathcal{K}$). The complete classification of connected subgroups of the orthogonal group $SO_0(< >)$, which leave invariant a null-line $\mathbb{R}e_0$, but do not leave invariant any non-degenerate proper subspace has been established. More precisely, we have :

Theorem 2 [4]. *Let $\mathcal{H}$ be the Lie algebra of a connected subgroup $\mathbf{H}$ of $SO_0(< >)$ which leaves invariant $\mathbb{R}e_0$ and does not leave invariant any non-degenerate subspace of $\mathbb{R}^m$. Then $\mathcal{H}$ is in $\mathcal{G}$ and either $\mathcal{H}$ contains $\mathcal{N}$, or there exists a non-trivial decomposition $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, where $n = p + q$, a non-trivial abelian subalgebra Λ of $so(p)$, a semi-simple subalgebra $\mathcal{C}$ of $so(p)$ (eventually 0), commuting with Λ and a surjective linear map $\varphi : \Lambda \rightarrow \mathbb{R}^q$ such that, up to conjugacy in $\mathcal{G}$, $\mathcal{H}$ is the subalgebra of $\mathcal{G}$ of the following “block” form*

$$\left\{ \begin{pmatrix} 0 & {}^tX & {}^t\varphi(A) & 0 \\ 0 & A+B & 0 & X \\ 0 & 0 & 0 & \varphi(A) \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ for } A \in \Lambda, B \in \mathcal{C} \text{ and } X \in \mathbb{R}^p \right\}. \quad (1)$$

Theorem 3 [4]. *Let $\mathbf{H}$ be the restricted holonomy group at a point θ of an indecomposable Lorentzian manifold $(M, < >)$, and assume that $\mathbf{H}$ leaves invariant a null-line $\mathbb{R}e_0$. Then the projection $\mathcal{B}$ of $\mathcal{H}$ into $\mathcal{K}$ with respect to $\mathcal{A} \oplus \mathcal{N}$ is completely reducible.*

Now, let $(e_0, e_1, \dots, e_{n+1})$ be a basis of $T_\theta M$ in which the metric $< >$ at a point θ reads

$$2dx_0dx_{n+1} - \sum_{i=1}^n dx_i^2$$

$\mathbf{H}$ leaves invariant the null-line $T_\theta^0 = \mathbb{R}e_0$ and its orthogonal T_θ generated by $e_0, \dots, e_n$. So there exist two parallel distributions T^0 and T respectively of dimension 1 and $(n+1)$, which are invariant under parallel transport. We have then:

Proposition 4 [3, p. 10–21]. *If X, Y are two vector fields, and Y belongs to T (respectively T^0), then $D_X Y$ belongs to T (respectively T^0).*

T is involutive. Then, according to Frobenius’s theorem, there exists a maximal integral submanifold V of T , through any point θ . V is $(n+1)$ -dimensional and totally geodesic. We shall denote by C the quotient space M/T (or M/V) of M under the equivalence which identifies two points of M if they are on the same leaf V . If we replace M by a neighborhood of θ , C is 1-dimensional Hausdorff manifold. We shall denote by π the projection: $M \rightarrow C$.

3. Construction of a coordinate system.

Theorem 5. *Let $(M, < >)$ be an $(n+2)$ -dimensional Lorentzian manifold whose (restricted) holonomy group $\mathbf{H}$ is in $\mathbf{G}$. Then there exists some coordinate system $(x_0, x_1, \dots, x_{n+1})$ in which the metric $< >$ has the reduced form*

$$ds^2 = 2 dx_0 dx_{n+1} - g_{x_{n+1}} + 2 \sum_{i=1}^n f_i dx_i dx_{n+1} + h dx_{n+1}^2 \quad (2)$$

where $g_{x_{n+1}}$ is a Riemannian metric on $\mathbb{R}^n = \{x_1, \dots, x_n\}$ depending on the coordinate x_{n+1} as a parameter. The $\langle \partial_i, \partial_j \rangle$'s and f_i 's are independent of x_0 , for $i, j \in \{1, \dots, n\}$.

Proof. With the same notations as in §2, let γ be the geodesic in the direction of e_{n+1} . Let E_0, E_i and E_{n+1} be the vector fields that are parallel along γ such that $E_0(\theta) = e_0$, $E_i(\theta) = e_i$ and $E_{n+1}(\theta) = e_{n+1}$. Therefore E_0 belongs to T^0 , E_i belongs to T and the metric is flat along γ . Now let $\mathbf{P}$ be the hyperplane generated by the geodesics parallel to the E_i 's. We parameterize a part c of the curve C by $\gamma : c(t) = \pi(\gamma(t))$. We place some coordinates on $\mathbf{P}$ as follows. Let y be a point on $\mathbf{P}$, such that $\pi(y) = c(t)$, we put $z = \gamma(t)$. y and z are in the same integral submanifold associated to T , and we choose the coordinates $(x_1, \dots, x_n)$ corresponding to the normal neighborhood $W_{\gamma(t)}$ of the E_i 's. We take the parameter t as the coordinate x_{n+1} . This choice being made, we set $E_{n+1} = \partial/\partial t$ the vector field for those coordinates on $\mathbf{P}$. Along $\mathbf{P}$, we choose E_0 such that $\langle E_0, E_{n+1} \rangle = 1$ and $E_0 \in T^0$. Let us then look at the geodesics generated by E_0 . We define the coordinates of a point x neighboring θ as follows. x_0 is the parameter of the geodesic starting from $y \in \mathbf{P}$ in the direction of E_0 passing through x .

For the other coordinates, we take those of y . Now look at the properties of the vector fields ∂_0, ∂_i and ∂_{n+1} . First, by construction, ∂_0 satisfies everywhere:

$$D_{\partial_0} \partial_0 = 0, \quad \partial_0 \text{ is isotropic, } \langle \partial_0, \partial_i \rangle = 0.$$

Along $\mathbf{P}$, $\langle \partial_0, \partial_{n+1} \rangle = 1$, and furthermore we have

$$\partial_0 \langle \partial_0, \partial_{n+1} \rangle = \langle D_{\partial_0} \partial_0, \partial_{n+1} \rangle + \langle \partial_0, D_{\partial_0} \partial_{n+1} \rangle = 0,$$

because $D_{\partial_0} \partial_0 = 0$ and $D_{\partial_0} \partial_{n+1} = D_{\partial_{n+1}} \partial_0 \in T^0$. This shows that $\langle \partial_0, \partial_{n+1} \rangle = 1$ everywhere.

Using Proposition 4, we get

$$\begin{aligned} \partial_0 \langle \partial_i, \partial_j \rangle &= \langle D_{\partial_0} \partial_i, \partial_j \rangle + \langle \partial_i, D_{\partial_0} \partial_j \rangle \\ &= \langle D_{\partial_i} \partial_0, \partial_j \rangle + \langle \partial_i, D_{\partial_j} \partial_0 \rangle \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \partial_0 \langle \partial_i, \partial_{n+1} \rangle &= \langle D_{\partial_0} \partial_i, \partial_{n+1} \rangle + \langle \partial_i, D_{\partial_0} \partial_{n+1} \rangle \\ &= \langle D_{\partial_i} \partial_0, \partial_{n+1} \rangle + \langle \partial_i, D_{\partial_{n+1}} \partial_0 \rangle \\ &= \partial_i \langle \partial_0, \partial_{n+1} \rangle - \langle \partial_0, D_{\partial_i} \partial_{n+1} \rangle + \langle \partial_i, D_{\partial_0} \partial_{n+1} \rangle \\ &= 0 \end{aligned}$$

Then $\langle \partial_i, \partial_j \rangle$'s and $\langle \partial_i, \partial_{n+1} \rangle$'s are independent of x_0 . $\square$

3.1. Particular cases. We may ameliorate the form of the metric $\langle \rangle$, if we have some further information about its restricted holonomy group $\mathbf{H}$. For this, we need some lemmas.

Lemma 6. Let x be a point in $W = W_{\gamma(t)}$ (see the notations in proof of Theorem 5). If we denote by H_x the restricted holonomy group corresponding to the metric $\langle \cdot, \cdot \rangle$, and by H'_x the (restricted) holonomy group corresponding to the Riemannian metric $g = g_{\gamma(t)}$ on W , then H'_x is a subgroup of the projection of H_x into $\mathbf{K}$ with respect to $\mathbf{AN}$.

Proof. Let T' be the orthogonal complement distribution of T^0 in T and let D' the Levi-Civita connection of g . It is then clear that

$$\text{on } W, \langle D_{\partial_i} \partial_j, \partial_k \rangle = \langle D'_{\partial_i} \partial_j, \partial_k \rangle.$$

Hence $D_{\partial_i} \partial_j - D'_{\partial_i} \partial_j \in T^0$. So

$$D_X Y - D'_X Y \in T^0, \text{ for all vector fields } X, Y \text{ on } W. \quad (3)$$

Now, let $\alpha : [0, 1] \rightarrow W$ be a curve in W . For $X_0 \in T'_x$, we denote $\tau(\alpha_t)X_0$ by X_t , where $\tau(\alpha_t)$ denotes the parallel transport along $\alpha_t : [0, t] \rightarrow W$, with respect to the connection D . We denote also the projection of X_t on $T'_{\alpha(t)}$ (resp. on $T^0_{\alpha(t)}$) by X'_t (resp. X^0_t). Along α , $D_{\dot{\alpha}} X_t = D_{\dot{\alpha}} X'_t + D_{\dot{\alpha}} X^0_t = 0$. Then $D_{\dot{\alpha}} X'_t \in T^0_{\alpha(t)}$. Applying [3], we have $D_{\dot{\alpha}} X'_t - D'_{\dot{\alpha}} X'_t \in T^0_{\alpha(t)}$. Consequently, $D'_{\dot{\alpha}} X'_t \in T^0_{\alpha(t)} \cap T'_{\alpha(t)} = 0$. Hence X'_t is parallel along α , with respect to the connection D' . Then, for every loop α in W , based at x , $\tau'(\alpha_t)X_0$ is the projection of $\tau(\alpha_t)X_0$ on T'_x , where $\tau'(\alpha_t)$ denotes the parallel transport along $\alpha_t : [0, t] \rightarrow W$, with respect to the connection D' . $\square$

Lemma 7. If ∂_i is parallel on each $W_{\gamma(t)}$ with respect to the connection D' , and $H_x(\partial_i x) \in \partial_i x + T^0_x$, for all $x \in W_{\gamma(t)}$, then $D_{\partial_i} \partial_{n+1} \in T^0$.

Proof. Along γ , we have $D_{\partial_i} \partial_{n+1} = 0$. Let then y_1 be a point of $\mathbf{P}$, not belonging to γ . Let $\alpha_1 : [t_0, t_0 + t] \rightarrow \mathbf{P}$ be the geodesic generated by $(\partial_{n+1})y_1$. Put $y_2 = \alpha_1(t_0 + t)$, $z_1 = \gamma(t_0)$ and $z_2 = \gamma(t_0 + t)$. Then there exists a curve α_2 (resp. α_3) in W_{z_1} (resp. W_{z_2}) joining y_1 to z_1 (resp. y_2 to z_2). To simplify the writing, we denote by γ the curve $\gamma : [t_0, t_0 + t] \rightarrow \mathbf{P}$, and by α the composed loop $\alpha_1^{-1} \cup \alpha_3^{-1} \cup \gamma \cup \alpha_2$ based at y_1 . We have $\tau(\alpha)(\partial_i y_1) - \partial_i y_1 \in T^0_{y_1}$. But $\tau(\alpha_2)(\partial_i y_1) - \partial_i z_1 \in T^0_{z_1}$ and $\tau(\alpha_3^{-1})(\partial_i z_2) - \partial_i y_2 \in T^0_{y_2}$. Consequently $\tau(\alpha_1^{-1})(\partial_i y_2) - \partial_i y_1 \in T^0_{y_1}$. $\square$

Lemma 8. If $\mathbf{KN}$ contains $\mathbf{H}$, then in formula (2) of Theorem 5, the function h is independent of x_0 .

Proof. In this case, there exists a parallel vector field in T^0 . Hence there exists a function ψ (with $\psi = 1$ at θ for example) such that $\psi \partial_0$ is parallel. On the other hand, $D_{\partial_0} \partial_0 = D_{\partial_i} \partial_0 = 0$ for all $i = 1, \dots, n$. Then ψ is independent of x_0 and x_i . So it is sufficient to see ψ along γ :

$$D_{\partial_{n+1}}(\psi \partial_0) = (\psi' + \psi \alpha) \partial_0$$

with $\alpha = \Gamma^0_{0, n+1}$ (Christoffel symbol). Hence ψ satisfies a differential equation of first order. But along γ , we have $D_{\partial_{n+1}} \partial_{n+1} = 0$. Then

$$\langle D_{\partial_{n+1}} \partial_0, \partial_{n+1} \rangle = \partial_{n+1} \langle \partial_0, \partial_{n+1} \rangle - \langle \partial_0, D_{\partial_{n+1}} \partial_{n+1} \rangle = 0$$

So $\alpha = 0$, then $\psi = 1$ along γ . $\square$

Now, if we denote by B_x the projection of H_x into $\mathbf{K}$ with respect to $\mathbf{AN}$, from Theorem 3, there exists an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}^{p_0} \oplus \mathbb{R}^{p_1} \oplus \cdots \oplus \mathbb{R}^{p_r}$ into B_x -invariant subspaces and a corresponding decomposition into the direct product $B_x = B_0 \times B_1 \times \cdots \times B_r$ of a normal subgroups, where B_i is trivial on $\mathbb{R}^{p_j}$ if $i \neq j$ and is irreducible on $\mathbb{R}^{p_i}$ for each $i = 1, \dots, r$. B_0 consists of the identity only. Using Lemma 6, we have then $H'_x = B_0 \times C_1 \times \cdots \times C_r$, where $C_i = B_i \cap H'_x$ (see Lemma 6 for notations). By virtue of Borel-Lichnerowicz's theorem [7] then, even if we restrict $W_{\gamma(x_{n+1})}$, it is the direct product $W_{\gamma(x_{n+1})} = W_0 \times W_1 \times \cdots \times W_r$ of open neighborhoods of $\gamma(x_{n+1})$ and the Riemannian metric $g_{x_{n+1}}$ is the direct sum

$$g_{x_{n+1}} = g_{x_{n+1}}^0 + g_{x_{n+1}}^1 + \cdots + g_{x_{n+1}}^r$$

of irreducible Riemannian metrics in the W_i 's, where $g_{x_{n+1}}^0$ is flat. Then for all $i = 1, \dots, p_0$, ∂_i is parallel on each $W_{\gamma(x_{n+1})}$ for the connection D' . Using Lemma 7, it is then clear that $g_{x_{n+1}}^0$ is independent of x_{n+1} . Summarizing all that, we get

Corollary 9. *Let $\mathbf{H}$ be the restricted holonomy group of the Lorentzian metric $\langle \rangle$. We assume that $\mathbf{G}$ contains $\mathbf{H}$. Then there exists some coordinate system $(x_0, x_1, \dots, x_{n+1})$ in which the metric $\langle \rangle$ has the form (2), moreover $g_{x_{n+1}}$ is the direct sum of irreducible Riemannian metrics*

$$g_{x_{n+1}} = \sum_{i=1}^{p_0} dx_i^2 + g_{x_{n+1}}^1 + \cdots + g_{x_{n+1}}^r. \quad (4)$$

In particular

- i) if $\mathbf{H} \subset \mathbf{AN}$, then $g_{x_{n+1}}$ is flat,
- ii) if $\mathbf{H} \subset \mathbf{KN}$, then the function h is independent of x_0 ,
- iii) if $\mathbf{H}$ verifies the second conclusion of Theorem 3, then $g_{x_{n+1}} = g'_{x_{n+1}} + \sum_{j=p+1}^n dx_j^2$, where $g'_{x_{n+1}}$ is a Riemannian metric.

Remark. $\mathbf{N}$ is the restricted holonomy of the indecomposable, non-irreducible Lorentzian locally symmetric spaces [4, 12]. But $\mathbf{N}$ is also the restricted holonomy of a Lorentzian manifold not necessarily locally symmetric. It suffices to take the metric

$$2dx_0dx_{n+1} - \sum_{i=1}^n dx_i^2 + h dx_{n+1}^2,$$

with h sufficiently generic.

4. Examples of metrics. In [4], the author and L. Berard Bergery have given some examples of metrics when the holonomy groups satisfies the conclusions of both the theorems 2 and 3, in dimensions ≤ 7 . In this paragraph, we give some new examples.

4.1. Example 1.

Theorem 10. *Let H be a Lie subgroup of $\mathbf{G}$, and denote by $\mathcal{H}$ its Lie algebra. We assume that the projection $\mathcal{B}$ of $\mathcal{H}$ into $\mathcal{K}$ is abelian ($\mathcal{C} = 0$) and completely reducible. Then there exists a Lorentzian metric whose holonomy group is H .*

Proof. The representation of $\mathcal{B}$ is the exterior direct sum representation of s irreducible representations and a trivial representation:

$$\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_s$$

Each $\mathcal{B}_i (\neq 0)$ is an abelian irreducible subalgebra of $so(p_i)$, where $p_1 + \cdots + p_s = n$, then it is 1-dimensional. Hence, up to conjugacy, $\mathcal{B}_i = so(2)$, and

$$J_i = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & J & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \in \mathcal{B}, \text{ for } i = 1, \dots, s,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then up to conjugacy in $\mathcal{K}$, $\mathcal{B}$ is a direct sum of s representations $so(2)$ and a trivial representation:

$$\mathcal{B} = so(2) \oplus \cdots \oplus so(2) \quad (s \text{ times}). \quad (5)$$

We may distinguish four cases:

4.1.1. Type 1: H contains $\mathbf{A}$.

Then $\mathcal{H}$ contains $\mathcal{N}$, and $\mathcal{H} = \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{N}$, where $\mathcal{B}$ satisfies (5). We observe that the subgroup $\mathbf{B}$ of $\mathbf{K}$ generated by $\mathcal{B}$ is the holonomy group of a Riemannian metric dt^2 on $\mathbb{R}^n$, for example:

$$dt^2 = (x_1^2 + x_2^2)(dx_1^2 + dx_2^2) + \cdots + (x_{2s-1}^2 + x_{2s}^2)(dx_{2s-1}^2 + dx_{2s}^2)$$

Using Ambrose-Singer's theorem [2, 3, p. 10–61], the Lorentzian metric

$$ds^2 = 2dx_0dx_{n+1} - dt^2 + k dx_{n+1}^2$$

admits H as holonomy group provided that k is sufficiently generic.

4.1.2. Type 2: H contains $\mathbf{N}$ and is contained in $\mathbf{KN}$.

As examples, we may take the same as above, but with k independent of x_0 .

4.1.3. Type 3: H contains $\mathbf{N}$ but H is not of type 1 or 2.

In this case, $\mathcal{H} = L \oplus \mathcal{N}$, where $L = \{(\varphi(A), A, 0) \mid A \in \mathcal{B}\}$ and $\varphi : \mathcal{B} \mapsto \mathcal{A}$ is a subjective linear map. Then there exists $\alpha \in \mathbb{R}^*$ such that $(1, \alpha J_i, 0) \in \mathcal{H}$. Up to conjugacy in $\mathcal{G}$, we can suppose $(1, \alpha J_1, 0) \in \mathcal{H}$. We consider the Lorentzian metric

$$ds^2 = 2dx_0dx_{n+1} - \sum_{i=1}^n dx_i^2 - 4\alpha x_1x_2dx_1dx_{n+1} - 4x_3x_4dx_3dx_{n+1} - \cdots - 4x_{2s-1}x_{2s}dx_{2s-1}dx_{n+1} - 2x_0x_1dx_{n+1}^2,$$

this admits H as holonomy group.

4.1.4. Type 4: H does not contain $\mathbf{N}$.

Then $\mathcal{K} \oplus \mathcal{N}$ contains $\mathcal{H}$ and there exists a decomposition $\mathcal{N} = \mathcal{N}_2 \oplus \mathcal{N}_1$ and $\mathcal{H} = L \oplus \mathcal{N}_1$, where L is the subalgebra of $\mathcal{B} \oplus \mathcal{N}_2$ defined by $L = \{(A, \varphi(A)) \in \mathcal{B} \oplus \mathcal{N}_2\}$ and $\varphi : \mathcal{B} \rightarrow \mathcal{N}_2$ is a surjective linear map. We let $\varphi(J_i) = (\lambda_{p+1}^i, \dots, \lambda_n^i)$ for $i = 1, 3, \dots, 2s-1$ and we consider the Lorentzian metric

$$ds^2 = 2dx_0dx_{n+1} - \sum_{i=1}^n dx_i^2 - 4x_1x_2dx_1dx_{n+1} - 4x_3x_4dx_3dx_{n+1} - \dots - 4x_{2s-1}x_{2s}dx_{2s-1}dx_{n+1} - 2 \left(\sum_{\substack{i=1,3,\dots,2s-1 \\ p+1 \leq j \leq n}} \lambda_j^i x_i x_j + \sum_{k=2s+1}^p x_k^2 \right) dx_{n+1}^2$$

Then a straightforward calculation shows that its (restricted) holonomy group is H . $\square$

4.2. Example 2. It is known that, if a subalgebra $\mathcal{H} \subset \mathcal{N} \oplus \mathcal{K}$ of $\mathcal{G}$ contains the ideal $\mathcal{N}$ and its projection $\mathcal{B}$ into $\mathcal{K}$ is a holonomy algebra of a Riemannian manifold, then $\mathcal{H}$ is also a holonomy algebra of a Lorentzian manifold (see [4]). But the converse is false. Indeed, the list of all possible restricted holonomy groups of irreducible pseudo-Riemannian manifolds established by M. Berger in [5, 6] does not contain the group $SO(3)$ for the case of dimension 5. Then it cannot even be the restricted holonomy group of a 5-dimensional Riemannian manifold. On the other hand, in dimension 7, the Lorentzian metric

$$ds^2 = 2dx_0dx_6 - \sum_{i=1}^5 dx_i^2 + \sum_{i=1}^5 u_i dx_i dx_6 \quad (6)$$

with $u_2 = u_4 = 0$, $u_1 = -x_3^2 - 4x_4^2 - x_5^2$, $u_3 = -2\sqrt{3}x_2x_3 - 2x_4x_5$, and $u_5 = 2\sqrt{3}x_2x_5 + 2x_3x_4$, has as its holonomy group the semidirect product $\varphi(SO(3)).N$ with $\varphi : SO(3) \rightarrow SO(5)$ being an irreducible representation.

Proof. An irreducible representation of $so(3)$ in $so(5)$ is the subalgebra of $so(5)$ spanned by the following matrices:

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 1 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and $C = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -\sqrt{3} \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \sqrt{3} & 0 & 0 & 0 \end{pmatrix}.$

A computation of the non zero Christoffel symbols then gives:

$$\begin{aligned}\Gamma_{ij}^0 &= \frac{1}{2}(\partial_i u_j + \partial_j u_i) \\ \Gamma_{i6}^0 &= \frac{1}{2} \sum_{k=1}^5 u_k (\partial_k u_i - \partial_i u_k) \\ \Gamma_{i6}^j &= \frac{1}{2}(-\partial_i u_i + \partial_j u_i)\end{aligned}$$

for $i, j = 1, \dots, 5$. The non zero components of the curvature R are:

$$\begin{aligned}R_{ijk}^0 &= \frac{1}{2}(-\partial_k \partial_i u_j + \partial_j \partial_k u_i) \\ R_{i6k}^0 &= -\frac{1}{4} \sum_{k=1}^5 (3\partial_i u_k + \partial_k u_i)(\partial_k u_j - \partial_j u_k) - \sum_{k=1}^5 u_k (\partial_k \partial_i u_j - \partial_i \partial_j u_k) \\ R_{i6j}^k &= \frac{1}{2}(\partial_i \partial_j u_k - \partial_k \partial_i u_j) .\end{aligned}$$

Then the curvature evaluated at zero is as follow:

$$\begin{aligned}R(e_3, e_6) &= (0, A, 0), \quad R(e_4, e_6) = (0, B, 0), \quad R(e_5, e_6) = (0, C, 0) \\ R(e_1, e_3) &= -R(e_2, e_3) = -(0, 0, E_3) \\ R(e_1, e_4) &= 2R(e_3, e_5) = -4(0, 0, E_4) \\ R(e_1, e_5) &= -\frac{1}{\sqrt{3}}R(e_2, e_5) = -(0, 0, E_5),\end{aligned}$$

with $(E_1, \dots, E_5)$ the canonical basis of $\mathbb{R}^5$. Furthermore, we have

$$\begin{aligned}[(0, A, 0), (0, 0, E_4)] &= -2(0, 0, E_1), \\ [(0, A, 0), (0, 0, E_3)] &= (0, 0, -E_1 + \sqrt{3}E_2) .\end{aligned}$$

Hence the holonomy algebra of ds^2 contains $so(3) \oplus \mathcal{N}$. With a trivial computation we verify that $so(3) \oplus \mathcal{N}$ contains the covariant derivatives of R evaluated at 0. $\square$

5. Einstein Lorentzian manifolds and holonomy groups. J. F. Schell [14, 16] showed that there is no 4-dimensional Einstein Lorentzian manifold with a 3-dimensional holonomy group. Then a general formulation is the following :

Theorem 11. *With the same hypothesis as in Theorem 10, there is no Einstein Lorentzian manifold, with holonomy group of type 3 or of type 4 (see 4.1).*

Proof. Suppose that $(M, < >)$ is an Einstein Lorentzian manifold, then there exists a real constant λ such that $r = \lambda < >$, where r is the Ricci curvature tensor associated to $< >$. With the same notations as in § 2 and § 4, if the holonomy algebra $\mathcal{H}$ of $(M, < >)$ is of type 3, then up to conjugacy in $\mathcal{G}$, it is spanned by the following matrices relative to the basis $(e_0, e_1, \dots, e_n, e_{n+1})$:

$$A_1 = (1, \alpha J_1, 0), A_2 = (0, J_2, 0), \dots, A_s = (1, J_s, 0) \quad \text{and} \quad \mathcal{N} .$$

We look first to the curvature R of $(M, \langle \rangle)$ at θ . For $X, Y \in T_\theta M$, $R(X, Y)e_0 \in \mathbb{R}e_0$, hence

$$R(e_i, e_0) = 0, \quad \text{for } i = 1, \dots, n \quad (7)$$

and by the algebraic Bianchi identity we get

$$\langle R(e_i, e_j)e_0, e_{n+1} \rangle = 0, \quad \text{for } i, j = 1, \dots, n. \quad (8)$$

By hypothesis

$$-\alpha \langle R(X, Y)e_0, e_{n+1} \rangle = \langle R(X, Y)e_1, e_2 \rangle, \quad \text{for } X, Y \in T_\theta M,$$

hence

$$R(e_1, e_2) = -\alpha R(e_0, e_{n+1}) \quad (9)$$

By definition,

$$r(X, Y) = \langle R(X, e_0)Y, e_{n+1} \rangle + \langle R(X, e_{n+1})Y, e_0 \rangle - \sum_{i=1}^n \langle R(X, e_i)Y, e_i \rangle.$$

So (7), (8) and (9) imply, $r(e_0, e_{n+1}) = \langle R(e_0, e_{n+1})e_{n+1}, e_0 \rangle = 0$. Since $\langle e_0, e_{n+1} \rangle \neq 0$, we have $\lambda = 0$, and $(M, \langle \rangle)$ satisfies the Einstein condition $r = 0$. $R(e_1, e_i) = 0$ and $R(e_2, e_i) = 0$ for $i \geq 3$, hence

$$r(e_1, e_{n+1}) = \langle R(e_1, e_{n+1})e_{n+1}, e_0 \rangle - \langle R(e_1, e_2)e_{n+1}, e_2 \rangle = 0$$

and

$$r(e_2, e_{n+1}) = \langle R(e_2, e_{n+1})e_{n+1}, e_0 \rangle + \langle R(e_1, e_2)e_{n+1}, e_1 \rangle = 0$$

Applying (8) we obtain

$$\langle R(e_1, e_{n+1})e_1, e_2 \rangle = \langle R(e_2, e_{n+1})e_1, e_2 \rangle = 0.$$

Furthermore we have

$$\langle R(e_i, e_{n+1})e_1, e_2 \rangle = \langle R(e_1, e_2)e_i, e_{n+1} \rangle = 0, \quad \text{for } i \geq 3.$$

All this together implies that $R(X, Y)$ belongs to $\text{span}\{A_2, \dots, A_s\} \oplus \mathcal{N}$ for $X, Y \in T_\theta M$. We consider now the subspaces $E = \mathbb{R}e_0 \oplus \text{span}\{e_1, e_2\}$ and $E^\perp = \mathbb{R}e_0 \oplus \text{span}\{e_3, \dots, e_s\}$. They are $\mathcal{H}$ -invariant, so there exists two parallel distributions of spaces on M generated by parallel transport of E and $E^\perp$. The same calculation as above may be done on the curvature at each point. Using the Ambrose-Singer Theorem's, we deduce that the algebra $\text{span}\{A_2, \dots, A_s\} \oplus \mathcal{N}$ contains the holonomy algebra $\mathcal{H}$, and we deduce a contradiction. Now if $\mathcal{H}$ is of type 4, then

$$\mathcal{H} = L \oplus \mathcal{N}_1, \text{ where } L = \{(0, A, \varphi(A)) \in \mathcal{B} \oplus \mathcal{N}_2\}. \quad (10)$$

Hence $R(e_0, X) = 0$ and $r(e_0, X) = \lambda \langle e_0, X \rangle = 0$ for all $X \in T_\theta M$. Consequently $(M, \langle \rangle)$ satisfies the Einstein condition $r = 0$. From (10), we have $R(e_i, e_j) = 0$ and $R(e_j, e_{j'}) = 0$ for $i = 1, \dots, p$ and $j, j' = p+1, \dots, n$. Then

$$\langle R(e_j, e_{n+1})e_i, e_{i+1} \rangle = \langle R(e_i, e_{i+1})e_j, e_{n+1} \rangle = 0.$$

Hence

$$R(e_j, e_{n+1}) \in \mathcal{N}_2 \text{ and } R(e_i, e_{i+1}) \in \mathcal{B} \oplus \mathcal{N}_2$$

Furthermore,

$$r(e_i, e_{i+1}) = \langle R(e_i, e_{i+1})e_{n+1}, e_{i+1} \rangle = 0,$$

and

$$r(e_{i+1}, e_{n+1}) = \langle R(e_{i+1}, e_i)e_{n+1}, e_i \rangle = 0 \text{ for } i = 1, 3, \dots, 2s-1.$$

Consequently, $R(e_i, e_{n+1})$ and $R(e_i, e_{i+1}) \in \mathcal{N}_2$. Summarizing all that, we get that $R(X, Y) \in \mathcal{B} \oplus \mathcal{N}_2$. And by the similar argument as in type 3, we deduce a contradiction. $\square$

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Résumé substantiel en français. Soit $(M, \langle \cdot, \cdot \rangle)$ une variété pseudo-riemannienne connexe de dimension m , alors elle possède une connexion canonique D (dite de Levi-Civita) et par conséquent une notion de transport parallèle des vecteurs tangents, le long de tout chemin. Si l'on considère, pour un point x quelconque de M les transports parallèles le long de tout lacet homotope à x , on obtient un sous-groupe du groupe orthogonal de l'espace tangent $T_x M$ en x . Ce sous-groupe est appelé groupe d'holonomie restreint (ou représentation d'holonomie restreinte) et on le notera $\mathbf{H}$. À une conjugaison près, $\mathbf{H}$ ne dépend pas de x , et définit donc un invariant fondamental de la variété.

Définitions. On dit que $(M, \langle \cdot, \cdot \rangle)$ est *irréductible*, si $\mathbf{H}$ ne laisse invariant aucun sous espace propre de $T_x M$; *indécomposable*, si $\mathbf{H}$ ne laisse invariant aucun sous espace propre non dégénéré de $T_x M$ (dans ce cas on dit que $\mathbf{H}$ est indécomposable).

H. Wu a généralisé le théorème de De Rham, et il a démontré dans [17] que la représentation d'holonomie (restreinte) d'une variété pseudo-riemannienne est somme directe d'une représentation triviale et d'un nombre fini de représentations indécomposables. D'autre part tous les groupes d'holonomie restreints d'une variété pseudo-riemannienne (en particulier d'une variété riemannienne) sont classifiés (voir [1, 3, 5, 6, 8, 9, 10, 13]). Le problème fondamental de la théorie est de trouver la liste des groupes d'holonomie possibles. D'après ce qui précède, ce problème est donc réduit aux cas des variétés pseudo-riemanniennes indécomposables non irréductibles. D'où les deux questions suivantes:

- 1) Trouver tout les sous-groupes connexes du groupe orthogonal $O(p, q)$, qui sont indécomposables non irréductibles.
- 2) Pour chaque groupe classifié H dans 1), existe-il effectivement des métriques dont l'holonomie est H ?

Jusqu'à maintenant la réponse à ces deux questions n'est pas connue, à l'exception du cas lorentzien symétrique [12], ou du cas pseudo-riemannienne symétrique en dimensions ≤ 7 [11]. L'auteur et L. Berard Bergery ont donné la solution à la première question, dans le cas lorentzien. Plus précisément, ils ont démontré les deux théorèmes suivants :

Théorème 1 [4]. Soit $\mathcal{H}$ l'algèbre de Lie d'un sous-groupe H de $SO(1, 1+n)$ qui laisse invariant une droite isotrope $\mathbb{R}e_0$ et qui ne laisse invariant aucun sous espace propre non dégénéré de $\mathbb{R}^m$ ($m = n+2$). Alors $\mathcal{H}$ est dans $\mathcal{G}$ et, ou bien $\mathcal{H}$ contient $\mathcal{N}$, ou bien il existe une décomposition non triviale $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, avec $n = p+q$, une sous-algèbre non triviale Λ de $so(p)$, une sous-algèbre semi-simple $\mathcal{C}$ de $so(p)$ (éventuellement 0), qui commute avec Λ et une application linéaire surjective $\varphi : \Lambda \rightarrow \mathbb{R}^q$ telle que, à une conjugaison près dans $\mathcal{G}$, $\mathcal{H}$ est une sous-algèbre de $\mathcal{G}$, formée des matrices blocs (1). ($\mathcal{G}$ et $\mathcal{N}$ sont données dans §2.)

Théorème 2 [4]. Soit $\mathbf{H}$ le groupe d'holonomie restreint d'une variété lorentzienne $(M, < >)$ connexe, avec M de dimension $m = n+2$ et $\mathcal{H}$ l'algèbre de Lie de $\mathbf{H}$. On suppose que $\mathbf{H}$ laisse invariant une droite isotrope $\mathbb{R}e_0$. Alors la projection $\mathcal{B}$ de $\mathcal{H}$ sur $\mathcal{K}$ parallèlement à $\mathcal{A} \oplus \mathcal{N}$ est complètement réductible. ($\mathcal{A}$ et $\mathcal{K}$ sont données dans §2.)

Donc, dans le cas lorentzien, la première question est complètement résolue. Dans cet article, on donne une réponse partielle à la seconde question en quatre étapes :

La première consiste à lire de la structure du groupe d'holonomie des informations sur la variété lorentzienne. Autrement dit, étant donné H un sous groupe de G (G est le sous groupe connexe de $O(1, n+1)$ dont l'algèbre de Lie est $\mathcal{G}$), classifié dans les théorèmes 1 et 2, alors localement, la métrique dont l'holonomie est H a la forme « canonique » (2). L'outil essentiel est l'existence de deux distributions parallèles intégrables, orthogonales de dimensions respectivement 1 et $(n+1)$ telle que la première est isotrope. On étudie ensuite des cas particulier de H . Puisque la projection $\mathcal{B}$ de H sur $\mathcal{K}$ doit être complètement réductible, on démontre que la métrique $g_{x_{n+1}}$ est somme de métriques riemanniennes irréductibles qui dépendent de la coordonnée x_{n+1} (voir (4)). En particulier,

- i) si $H \subset \mathbf{AN}$, alors $g_{x_{n+1}}$ est plate,
- ii) si $H \subset \mathbf{KN}$, alors la fonction h ne dépend pas de x_0 ,
- iii) si H vérifie la seconde conclusion du Théorème 1, alors

$$g_{x_{n+1}} = g'_{x_{n+1}} + \sum_{j=p+1}^n dx_j^2,$$

avec $g'_{x_{n+1}}$ une métrique riemannienne (où $\mathbf{A}$ (resp. $\mathbf{N}$, $\mathbf{K}$) est le sous-groupe connexe de $O(1, n+1)$ dont l'algèbre de Lie est $\mathcal{A}$ (resp. $\mathcal{N}$, $\mathcal{K}$)).

Dans la deuxième étape, on cherche des exemples de métriques dont le groupe d'holonomie est donné.

Les premiers exemples. Lorsque la projection $\mathcal{B}$ de l'algèbre de Lie $\mathcal{H}$ de H sur $\mathcal{K}$ est commutative ($\mathcal{C} = 0$). D'après le théorème 2, $\mathcal{B}$ doit être complètement réductible. Comme elle est dans $so(n)$, elle ne peut être que somme directe d'une représentation triviale et d'un nombre fini de représentation de $so(2)$ (voir (5)). On distingue quatre cas de H (voir 4.1.1 - 4.1.4) et en utilisant l'étape précédente, on trouve pour chaque cas une métrique correspondante.

Autre exemple. Dans [4], l'auteur et L. Berard Bergery ont démontré que si H est dans $\mathbf{K.N}$, et contient $\mathbf{N}$, et si la projection $\mathcal{B}$ de H sur $\mathbf{K}$ est l'holonomie d'une métrique

riemannienne g en dimension n , alors H est l'holonomie de la métrique lorentzienne

$$ds^2 = 2 dx_0 dx_{n+1} - g + h dx_{n+1}^2, \text{ en dimension } n + 2,$$

avec h une fonction suffisamment générique. Cet exemple montre que la réciproque est fausse. Les listes des groupes d'holonomie possibles, établies par M. Berger [5, 6] ne contient pas le groupe $SO(3)$ en cas de dimension 5. Donc il ne peut pas être l'holonomie d'une variété riemannienne. Mais en dimension 7, on peut démontrer que la métrique lorentzienne [6] a pour holonomie le produit semi-direct $\varphi(SO(3)) N$, avec $\varphi : SO(3) \rightarrow SO(5)$ une représentation irréductible bien choisie.

La dernière étape généralise un résultat de J. F. Shell [14, 16] qui a montré qu'une variété lorentzienne d'Einstein de dimension 4, n'admet pas d'holonomie de dimension 3. A l'aide de l'identité de Bianchi algébrique et le Théorème d'Ambrose-Singer [2], on montre qu'il n'existe pas de variété lorentzienne d'Einstein de type 3 ou de type 4 (voir 4.1 pour les notations).

REFERENCES

1. D. V. Alekseevskii, *Riemannian manifolds with exceptional holonomy groups*, Funktsional Anal. i Prilozhen **2** (1968), no. 2, 1–10; English translation, Funct. Anal. Appl. **2** (1968), 97–105.
2. W. Ambrose and I. M. Singer, *A theorem on holonomy*, Amer. Math. Soc. **79** (1953), 428–443.
3. A. L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin-Heidelberg-New York, 1987.
4. L. Berard Bergery and A. Ikemakhen, *On the holonomy of Lorentzian Manifolds*, Proc. Sympos. Pure Math. **54** (1993); Part 2, 27–40.
5. M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330.
6. ———, *Les espaces symétriques non compacts*, Ann. Sci. École Norm. Sup. **74** (1957), 85–177.
7. A. Borel and A. Lichnerowicz, *Groupes d'holonomie des variétés riemanniennes*, C. R. Acad. Sci. Paris **234** (1952), 1835–1837.
8. R. B. Brown and A. Gray, *Riemannian manifolds with holonomy group $Spin(9)$* , Differential Geometry (in honor of Kentaro Yano), Kinokuniya, Tokyo, 1972, pp. 41–59.
9. R. L. Bryant, *Metrics with exceptional holonomy*, Ann. of Math. (2) **126** (1987), 525–576.
10. R. L. Bryant and S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850.
11. M. Cahen and M. Parker, *Pseudo-Riemannian symmetric spaces*, Mem. Amer. Math. Soc. **229** (1980), 829–850.
12. M. Cahen and N. Wallach, *Lorentzian symmetric spaces*, Bull. Amer. Math. Soc. **79** (1970), 829–850.
13. E. Cartan, *Les groupes d'holonomie des espaces généralisés*, Acta Math. **48** (1926), 1–42.
14. G. S. Hall, *Space-times and holonomy groups*, Differential geometry and its applications (Opava, 1992), Math. Publ. 1, Silesian Univ. Opava, Opava, 1993, pp. 201–210.
15. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 1, Interscience Wiley, New York, 1963.; vol. 2, 1969.
16. J. F. Schell, *Classification of 4 dimensional Riemannian spaces*, J. Math. Phys. **2** (1961), 202–206.

17. H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8** (1964), 291–311.

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