

THE DIFFERENCE IN EVEN AND ODD LATIN RECTANGLES FOR SMALL CASES

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RÉSUMÉ. Les rectangles latins peuvent être classifiés «pairs selon les lignes» ou «impairs selon les lignes», selon les signes des permutations qui composent les lignes. Nous présentons ici une généralisation de la conjecture d’Alon-Tarsi sur les carrés latins, à savoir une conjecture sur la différence entre le nombre de rectangles latins pairs selon les lignes et le nombre de rectangles latins impairs selon les lignes, de taille donnée. Dans ce contexte, nous donnons, pour $k \leq 3$, des formules pour la différence entre le nombre de rectangles latins pairs et impairs selon les lignes, de taille $k \times n$ et ayant une première ligne normalisée.

ABSTRACT. Latin rectangles can be classified as row-odd or row-even according to the signs of the permutations that make up the rows. As a generalization of the Alon-Tarsi conjecture on latin squares, a conjecture on the difference between the number of row-odd and row-even latin rectangles of a certain size is presented here. In this context, formulas are given for the difference between row-odd and row-even $k \times n$ latin rectangles with normalized first row for the values $k \leq 3$.

1. Introduction. A *latin square* of order n can be defined as an $n \times n$ array such that any row represents a permutation of the numbers $1, 2, \dots, n$ in order, and the product $\alpha^{-1}\beta$ of any two distinct row permutations α, β is fixed-point free. A *latin rectangle* of size $k \times n$ is a $k \times n$ array with the same properties, and hence can be seen as a collection of k rows of a latin square of order n . A *reduced* latin square or rectangle [G] has the additional property that the first row is the identity permutation. A latin square or rectangle is called *row-even* if the product of the signs of its row permutations is $+1$, and *row-odd* if this product is -1 .

Recently, parities of latin squares have been studied in the context of invariant theory of superalgebras, graph theory, and permutation group theory. Most notably, relations between a conjecture on the number of row-even and row-odd latin squares (the Huang-White conjecture, stated below) and some conjectures about vectorspaces and bracket algebras were discussed in [HR]. For a general treatise on parities of latin squares and some background, the reader is referred to [J].

Let $S_{k,n}$ denote the difference between the number of even and odd reduced latin rectangles of size $k \times n$. In [HR], the following conjecture is stated.

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Conjecture (Huang-White, 1986). *For n even,*

$$S_{n,n} \neq 0.$$

In fact, this conjecture was posed about all latin squares, not just the reduced ones. For even k , however, any permutation of the columns does not change the parity of a $k \times n$ latin rectangle, and thus the difference between the number of row-odd and row-even latin rectangles of this size is given by $n!S_{k,n}$. For odd k , however, any odd permutation of the columns changes the signs of all k rows and hence changes the parity of the rectangle. So for odd k , the difference between the number of row-odd and row-even latin rectangles is always equal to zero. This fact motivated us to define the $S_{k,n}$ in terms of reduced rectangles.

In an attempt to tackle the Huang-White conjecture we propose the following, more general conjecture.

Conjecture. *For all $n > 4$, $1 \leq k \leq n$,*

$$S_{k,n} \neq 0.$$

The conjecture has been computationally verified for small values of k and n . The calculated values of $S_{k,n}$ are given in Table 1.1. The values for $n = 7$ are due to Alberto Marini [M].

n	k	row-even	row-odd	$S_{k,n}$
2	2	0	1	-1
3	2	2	0	2
3	3	2	0	2
4	2	3	6	-3
4	3	12	12	0
4	4	12	0	12
5	2	24	20	4
5	3	312	240	72
5	4	384	960	-576
5	5	384	960	-576
6	2	130	135	-5
6	3	10480	10800	-320
6	4	203040	190080	2960
6	5	576000	552960	23040
6	6	426240	702720	-276480
7	2			6
7	3			3600
7	4			126720
7	5			829440
7	6			265420800
7	7			265420800

TABLE 1.1

Here we present explicit formulas for all $S_{k,n}$ with $1 \leq k \leq 3$, namely

Theorems 1.0, 1.1 and 1.2. For all $n \geq 1$,

$$\begin{aligned} S_{1,n} &= 1 \\ S_{2,n} &= (-1)^{n-1}(n-1) \\ S_{3,n} &= (-1)^{n-1}(n-2) \sum_{\substack{m_j \geq 0 (j>0) \\ \sum_j (2j+1)m_j = n}} \frac{n!}{\prod_{j>0} (2j+1)^{m_j} m_j!} 2^{\sum_{j>0} m_j - 1} \\ &= (-1)^{n-1} \frac{(n-2)n!}{2} \langle t^n, e^{2(\sinh t - t)} \rangle, \end{aligned}$$

These theorems verify our conjecture for $k \leq 3$. The proof of Theorem 1.0 is trivial, as there is only one reduced latin rectangle of size $1 \times n$. The other two theorems can be proved using linear algebra, and these proofs are presented in the next section.

2. Some Formulas. The formulas for $S_{2,n}$ and $S_{3,n}$ will be proved by an analysis of the eigenvalues of certain matrices of zeros and ones.

Theorem 1.1. For all $n \geq 1$,

$$S_{2,n} = (-1)^{n-1}(n-1).$$

Proof. We give a proof of this theorem based on determinants, as this is the approach we will take to explain our formula for $S_{3,n}$ as well. Other simple proofs exist, most notably one based on exponential generating functions. The $2 \times n$ reduced latin rectangles are exactly the fixed point free permutations, i.e. the derangements in $\mathcal{S}_n$. The difference between the number of even and odd derangements is given by the determinant of the matrix $J - I$, where J is the all-one matrix and I the identity matrix of size $n \times n$. Namely, $\det(J - I) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\epsilon(\sigma)} \prod_{1 \leq i \leq n} \delta_{i, \sigma(i)}$, and any non-zero term of this sum corresponds to a derangement of size n and vice versa. It is easy to verify that $J - I$ has eigenvalues -1 and $n-1$, with multiplicities $n-1$ and 1 , respectively, so $\det(J - I) = (-1)^{n-1}(n-1)$. $\square$

The proof of Theorem 1.2 is likewise based on eigenvalues and determinants of 0-1 matrices. It makes use of the following two lemmas. It may be noted here that Lemma 2.1 is special case of a known result on the eigenvalues of circulant matrices. All through the following, let ξ_ℓ denote a primitive ℓ -th root of unity.

Lemma 2.1. Let A_ℓ ($\ell \geq 2$) be the 0-1 matrix of size $\ell \times \ell$ that has the following form:

$$A_\ell = \begin{pmatrix} 0 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 0 & 1 & \ddots & 1 \\ \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & 1 & 0 & 0 \\ 0 & 1 & \dots & \ddots & 1 & 0 \end{pmatrix}.$$

The matrix A_ℓ has ℓ distinct eigenvalues. The eigenvalues are the numbers $\ell - 2$ and $-1 - \xi_\ell^i$, $1 \leq i < \ell$. The eigenvectors are the vectors $\vec{u}_{\ell,i}$ ($0 \leq i < \ell$), where

$$\vec{u}_{\ell,i} = \begin{pmatrix} 1 \\ \xi_\ell^i \\ \xi_\ell^{2i} \\ \vdots \\ \xi_\ell^{(\ell-1)i} \end{pmatrix}.$$

The vector $\vec{u}_{\ell,0}$ corresponds to the eigenvalue $\ell - 2$ and, for each $1 \leq i < \ell$, the vector $\vec{u}_{\ell,i}$ corresponds to the eigenvalue $-1 - \xi_\ell^i$.

Proof. By verification. The j -th coefficient of $(A_\ell \cdot \vec{u}_{\ell,0})$ is the row-sum of the j -th row of A_ℓ , and hence $\ell - 2$. For $1 \leq i < \ell$, the j -th coefficient of $(A_\ell \cdot \vec{u}_{\ell,i})$ is $\sum_{m=0}^{\ell-1} \xi_\ell^{mi} - (\xi_\ell^{ji} + \xi_\ell^{(j+1)i}) = (-1 - \xi_\ell^i) \xi_\ell^{ij}$, which is $(-1 - \xi_\ell^i)$ times the j -th coefficient of $\vec{u}_{\ell,i}$. All vectors $\vec{u}_{\ell,i}$ with $1 \leq i < \ell$ are independent since they correspond to distinct eigenvalues, and all are part of the hyperplane $x_1 + x_2 + \dots + x_\ell = 0$. Hence they form a basis for this hyperplane. The all-one vector $\vec{u}_{\ell,0}$ does not belong to this hyperplane, and $(\vec{u}_{\ell,0}, \vec{u}_{\ell,1}, \dots, \vec{u}_{\ell,\ell-1})$ is a basis of $\mathbb{C}^\ell$. $\square$

Lemma 2.2. Fix $\ell_1, \ell_2, \dots, \ell_k \geq 2$. Let A be the matrix

$$A = \begin{pmatrix} A_{\ell_1} & & & & \\ & A_{\ell_2} & & & \\ & & \ddots & & \\ & & & 1 & \\ & 1 & & & \ddots \\ & & & & & A_{\ell_k} \end{pmatrix}$$

Then

$$\det A = \begin{cases} 0 & \text{if one of the } \ell_j \text{ is even,} \\ (-2)^{k-1}(\ell_1 + \dots + \ell_k - 2) & \text{if every } \ell_j \text{ is odd.} \end{cases}$$

Proof. We give the matrix A with respect to the following basis:

$$\begin{aligned} & (\vec{u}_{\ell_1,i}^T, \vec{0}, \dots, \vec{0})^T \quad \text{for } 0 \leq i < \ell_1, \\ & (\vec{0}, \vec{u}_{\ell_2,i}^T, \dots, \vec{0})^T \quad \text{for } 0 \leq i < \ell_2, \\ & \dots \\ & (\vec{0}, \dots, \vec{0}, \vec{u}_{\ell_k,i}^T)^T \quad \text{for } 0 \leq i < \ell_k. \end{aligned}$$

The new matrix is

$$A' = \begin{pmatrix} D_{\ell_1} & C_{\ell_2} & \dots & C_{\ell_k} \\ C_{\ell_1} & D_{\ell_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_{\ell_k} \\ C_{\ell_1} & \ddots & C_{\ell_{k-1}} & D_{\ell_k} \end{pmatrix},$$

where D_{ℓ_j} ($1 \leq j \leq k$) is the diagonalization of A_{ℓ_j} , i.e.

$$D_{\ell_j} = \begin{pmatrix} \ell_j - 2 & 0 & \dots & 0 \\ 0 & -1 - \xi_{\ell_j} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \dots & -1 - \xi_{\ell_j}^{\ell_j-1} \end{pmatrix},$$

and, for each $1 \leq j \leq k$,

$$C_{\ell_j} = \begin{pmatrix} \ell_j & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

If one of the entries $-1 - \xi_{\ell_j}^i$ is zero, then a column of A' is zero and hence $\det A' = \det A = 0$. This happens precisely when ℓ_j is even (and thus $\xi_{\ell_j}^{\ell_j/2} = -1$) for some j .

Assume that all ℓ_j are odd. If we develop the determinant of A' with respect to columns $2, \dots, \ell_1, \ell_1 + 2, \dots, \ell_1 + \ell_2, \ell_1 + \ell_2 + 2, \dots$, each of which contains precisely one non-zero entry, we obtain that $\det A' = \prod_{j=1}^k \prod_{i=1}^{\ell_j-1} (-1 - \xi_{\ell_j}^i) \det H$, where

$$H = \begin{pmatrix} \ell_1 - 2 & \ell_2 & \dots & \ell_k \\ \ell_1 & \ell_2 - 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ell_k \\ \ell_1 & \ddots & \dots & \ell_k - 2 \end{pmatrix}.$$

Now $\prod_{i=1}^{\ell_j-1} (-1 - \xi_{\ell_j}^i) = 1$, and the eigenvalues of H are -2 with multiplicity $k-1$ and $\ell_1 + \ell_2 + \dots + \ell_k - 2$ with multiplicity 1, so $\det A = \det H = (-2)^{k-1} (\ell_1 + \dots + \ell_k - 2)$.

Theorem 1.2. *For all $n \geq 1$,*

$$\begin{aligned} S_{3,n} &= (-1)^{n-1} (n-2) \sum_{\substack{m_j \geq 0 (j>0) \\ \sum_j (2j+1)m_j = n}} \frac{n!}{\prod_{j>0} (2j+1)^{m_j} m_j!} 2^{\sum_{j>0} m_j - 1} \\ &= (-1)^{n-1} \frac{(n-2) n!}{2} \langle t^n, e^{2(\sinh t - t)} \rangle, \end{aligned}$$

where $\langle t^n, f(t) \rangle$ denotes the coefficient of t^n in the expansion of f into a power series.

Proof. The statement is trivial for $n < 3$, so in the following we assume that $n \geq 3$. For any two permutations $\alpha, \beta \in \mathcal{S}_n$, the $3 \times n$ array that has as a first row the identity, and α and β as its second and third row, is a latin rectangle precisely when α , β , and $\alpha^{-1}\beta$ are derangements, and the parity of this latin rectangle is given by $\epsilon(\alpha)\epsilon(\beta)$. Let $\mathcal{D}_n$ be the set of derangements of size n , then we can give the following expression for $S_{3,n}$:

$$S_{3,n} = \sum_{\alpha \in \mathcal{D}_n} \sum_{\substack{\beta \in \mathcal{D}_n \\ \alpha^{-1}\beta \in \mathcal{D}_n}} \epsilon(\alpha)\epsilon(\beta).$$

The permutations of $\mathcal{S}_n$ can be divided into conjugacy classes according to their cycle type. A cycle type can be characterized by a vector $\vec{c} = (c_1, c_2, \dots, c_n)$. A permutation is of cycle type $\vec{c}$ if its cycle representation consists of c_i cycles of length i , $1 \leq i \leq n$. All permutations of a certain cycle type $\vec{c}$ have the same sign. For each cycle type $\vec{c}$, let $\alpha_{\vec{c}}$ denote the canonical permutation of cycle type $\vec{c}$. This means that

$$\alpha_{\vec{c}} = \underbrace{(1, 2)(3, 4) \dots (2c_2 - 1, 2c_2)}_{c_2 \text{ cycles of length 2}} \underbrace{(2c_2 + 1, 2c_2 + 2, 2c_2 + 3) \dots (..)}_{c_3 \text{ cycles of length 3}} \dots,$$

for any cycle type of a derangement. In other words, if we write $\alpha_{\vec{c}}$ as the product of cycles of increasing length, the numbers $1, 2, \dots, n$ appear in order. Now the conjugacy class of permutations of cycle type $\vec{c}$ can be represented as $\{\gamma^{-1}\alpha_{\vec{c}}\gamma \mid \gamma \in \mathcal{S}_n\}$. By a counting argument one obtains that the size of this conjugacy class is equal to $\frac{n!}{\prod_i i^{c_i} c_i!}$.

For any $3 \times n$ reduced latin rectangle (id, α, β) , where α is of cycle type $\vec{c}$, there exists a latin rectangle $(id, \alpha_{\vec{c}}, \beta')$, where $\alpha_{\vec{c}} = \gamma\alpha\gamma^{-1}$ and $\beta' = \gamma\beta\gamma^{-1}$ for some $\gamma \in \mathcal{S}_n$, and $\epsilon(\alpha) = \epsilon(\alpha_{\vec{c}})$ and $\epsilon(\beta) = \epsilon(\beta')$. This is because we can permute the columns of the latin rectangle according to γ and the symbols according to γ^{-1} , and again obtain a reduced latin rectangle. Hence we can modify our formula for $S_{3,n}$:

$$\begin{aligned}
S_{3,n} &= \sum_{\substack{\vec{c}=(c_1,\dots,c_n) \\ c_1=0}} \sum_{\alpha \text{ of type } \vec{c}} \sum_{\substack{\beta \in \mathcal{D}_n \\ \alpha^{-1}\beta \in \mathcal{D}_n}} \epsilon(\alpha)\epsilon(\beta) \\
&= \sum_{\substack{\vec{c}=(c_1,\dots,c_n) \\ c_1=0}} \sum_{\alpha \text{ of type } \vec{c}} \epsilon(\alpha) \sum_{\substack{\beta \in \mathcal{D}_n \\ \alpha^{-1}\beta \in \mathcal{D}_n}} \epsilon(\beta) \\
&= \sum_{\substack{\vec{c}=(c_1,\dots,c_n) \\ c_1=0}} (-1)^{\sum_j c_{2j}} \sum_{\alpha \text{ of type } \vec{c}} \sum_{\substack{\beta \in \mathcal{D}_n \\ \alpha^{-1}\beta \in \mathcal{D}_n}} \epsilon(\beta) \\
&= \sum_{\substack{\vec{c}=(c_1,\dots,c_n) \\ c_1=0}} (-1)^{\sum_j c_{2j}} \frac{n!}{\prod_i i^{c_i} c_i!} \sum_{\substack{\beta \in \mathcal{D}_n \\ \alpha_{\vec{c}}^{-1}\beta \in \mathcal{D}_n}} \epsilon(\beta).
\end{aligned}$$

Fix $\vec{c} = (c_1, \dots, c_n)$ such that $c_1 = 0$. Now

$$\sum_{\substack{\beta \in \mathcal{D}_n \\ \alpha^{-1}\beta \in \mathcal{D}_n}} \epsilon(\beta) = \det A,$$

where

$$A = \begin{pmatrix} A_2 & & & & & \\ & \ddots & & & & \\ & & A_2 & & & \\ & & & & & 1 \\ & & & & A_3 & \\ & & & & & \ddots \\ & & & & & & A_3 \\ & & & & & & & c_3 \\ & & 1 & & & & & & \\ & & & & & & & & \ddots & \end{pmatrix}.$$

The matrices A_ℓ are as defined in Lemma 2.1. From Lemma 2.2 it follows that $\det A = 0$ if any of the even c_i is non-zero. Suppose that all even c_i are zero. Define $m_j = c_{2j+1}$ ($1 \leq j \leq n/2$). According to Lemma 2.2,

$$\det A = (-2)^{\sum_j m_j - 1} \left(\sum_j (2j+1)m_j - 2 \right) = (-1)^{n-1} (n-2) 2^{\sum_j m_j - 1},$$

since $\sum_j m_j \equiv \sum_j (2j+1)m_j \equiv n \pmod{2}$. This gives us the first equality of the theorem. We also have

$$S_{3,n} = (-1)^{n-1} \frac{(n-2)n!}{2} \left\langle t^n, \prod_j \left(\sum_{m \geq 0} \frac{1}{m!} \left(\frac{2t^{2j+1}}{(2j+1)} \right)^m \right) \right\rangle,$$

and

$$\begin{aligned} \prod_j \left(\sum_{m \geq 0} \frac{1}{m!} \left(\frac{2t^{2j+1}}{(2j+1)!} \right)^m \right) &= \prod_j \exp \left(\frac{2t^{2j+1}}{(2j+1)!} \right) \\ &= \exp \left(2 \sum_j \frac{t^{2j+1}}{(2j+1)!} \right) \\ &= \exp(2(\sinh t - t)). \quad \square \end{aligned}$$

This concludes the verification of the conjecture for its smallest values. It should be noticed that the obvious generalization of the methods used here immediately becomes much more complex, due to the many different shapes that the 0-1 matrices to be handled can assume.

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Résumé substantiel en français. Un *rectangle latin* de taille $k \times n$ est une matrice de taille $k \times n$ dont chaque ligne représente une permutation des nombres $1, 2, \dots, n$ et telle que le produit de deux permutations de lignes distinctes soit sans point fixe. Un *rectangle latin réduit* [G] doit en outre posséder une première ligne égale à la permutation identité. Nous dirons qu'un rectangle latin est *pair selon les lignes* si le produit des signes des permutations de ligne vaut +1 et nous dirons qu'il est *impair selon les lignes* si ce produit vaut -1 ; lorsqu'il n'y a aucune ambiguïté possible, nous parlerons plus simplement de rectangles latins pairs et impairs.

Notons $S_{k,n}$ la différence entre le nombre de rectangles réduits pairs et impairs de taille $k \times n$. Dans [HR], nous trouvons la conjecture suivante.

Conjecture (Huang-White, 1986). *Pour n pair,*

$$S_{n,n} \neq 0.$$

Nous proposons ici la conjecture plus générale suivante.

Conjecture. *Pour tout $n > 4$, $1 \leq k \leq n$,*

$$S_{k,n} \neq 0.$$

Nous avons vérifié cette conjecture pour les petites valeurs de k et n (cf. Table 1.1) ; pour $n = 7$, les valeurs mentionnées sont dues à Marini [M].

Dans les théorèmes 1.0-1.2, nous présentons des formules explicites pour $S_{k,n}$ lorsque $k = 1, 2, 3$. Ces formules permettent de vérifier facilement notre conjecture dans ces trois cas.

La formule $S_{1,n} = 1$ est évidente. La formule $S_{2,n} = (-1)^{n-1}(n-1)$ se prouve comme suit. Les rectangles latins réduits de taille $2 \times n$ correspondent aux dérangements de S_n . La différence entre le nombre de dérangements pairs et le nombre de dérangements impairs est donnée par le déterminant de la matrice $J - I$, où I est la matrice identité et J est la matrice ne comportant que des 1, de taille $n \times n$. Il est facile de vérifier que la matrice $J - I$ admet -1 comme valeur propre d'ordre $n-1$, et $n-1$ comme valeur propre d'ordre 1, ce qui prouve la formule désirée.

La formule donnant $S_{3,n}$ nécessite une analyse matricielle plus fine. C'est pourquoi nous déterminons dans le Lemme 2.1 les valeurs propres et les vecteurs propres des matrices

$$A_\ell = \begin{pmatrix} 0 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 0 & 1 & \ddots & 1 \\ \vdots & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & 1 & 0 & 0 \\ 0 & 1 & \dots & \ddots & 1 & 0 \end{pmatrix}.$$

La démonstration est aisée, il s'agit d'une simple vérification. Nous utilisons ensuite ce résultat pour calculer le déterminant des matrices du type

$$A = \begin{pmatrix} A_{\ell_1} & & & \\ & A_{\ell_2} & & 1 \\ & & \ddots & \\ & 1 & & \ddots \\ & & & & A_{\ell_k} \end{pmatrix}.$$

En effet, lorsque l'on utilise les bases de diagonalisation des différents A_{ℓ_j} , on trouve que $\det A = \prod_{j=1}^k \prod_{i=1}^{\ell_j-1} (-1 - \xi_{\ell_j}^i) \det H$, où ξ_ℓ désigne une racine primitive ℓ -ième de l'unité et où

$$H = \begin{pmatrix} \ell_1 - 2 & \ell_2 & \dots & \ell_k \\ \ell_1 & \ell_2 - 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ell_k \\ \ell_1 & \ddots & \dots & \ell_k - 2 \end{pmatrix}.$$

Il est facile de voir que $\prod_{i=1}^{\ell_j-1} (-1 - \xi_{\ell_j}^i) = 0$ ou 1 selon la parité de ℓ_j , et que les valeurs propres de H sont -2 avec multiplicité $k-1$, et $\ell_1 + \ell_2 + \dots + \ell_k - 2$ avec multiplicité 1. Ceci montre le Lemme 2.2 et nous permet de déterminer $S_{3,n}$.

$$A = \begin{pmatrix} A_2 & & & & \\ & \ddots & & & \\ & & A_2 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} c_2$$

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