

**INTEGRITY, INTEGRAL CLOSEDNESS AND FINITENESS
OVER THEIR CENTERS OF THE COORDINATE ALGEBRAS
OF QUANTUM GROUPS AT p^ν -TH ROOTS OF UNITY**

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RÉSUMÉ. Nous montrons que les algèbres de coordonnées duales des algèbres enveloppantes quantiques de Lusztig sont intègres aux racines p^ν -ièmes de 1 (pour p premier ne divisant pas les coefficients de la matrice de Cartan). Nous en déduisons le même résultat pour un paramètre transcendant. Nous montrons aussi qu'après une complétion p -adique convenable, les algèbres de coordonnées aux racines p^ν -ièmes de 1 sont finies sur leurs centres et intégralement fermées (au sens admis par Kac, de Concini et Procesi). Dans l'appendice (écrit avec N. Andruskiewitsch), nous construisons un morphisme de Frobenius dans le cas non simplement connexe.

ABSTRACT. We prove that the coordinate algebras dual to the Lusztig quantum enveloping algebras at p^ν -th roots of 1 are integral domains (p a prime number not dividing the coefficients of the Cartan matrix). From this, we deduce the same result when the parameter is transcendental. We also prove that after a suitable p -adic completion, the coordinate algebras at p^ν -th roots of 1 are finite over their centers and integrally closed (in the sense of de Concini, Kac and Procesi). In the appendix (written jointly with N. Andruskiewitsch), a Frobenius morphism is constructed in the non-simply connected case.

Introduction. It is certainly desirable to have a geometrical description of quantum groups. Such a description, for the center of the enveloping algebra, was given in [dCK], [dCKP].

The present work is a contribution to the dual viewpoint: we prove the integrity of the coordinate algebras, when the parameter is a p^ν -th root of 1, over the field of all $p^{\nu'}$ -th roots of 1 ($\nu, \nu' \in \mathbf{N}$, p a prime number).

We show in passing how this result implies the integrity of the coordinate algebras in the transcendental case; again the base field $\mathbf{Q}(v)$ is not algebraically closed. This illustrates a general idea that we owe to [dCKP]: the study at roots of unity can give information on the transcendental case.

We then show that after a suitable completion of the coordinate algebras at p^ν -th roots of 1 (the base rings or fields are now arbitrary finite extensions of the cyclotomic

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extensions of $\mathbf{Z}_p$ and $\mathbf{Q}_p$) they are integrally closed, in the sense of [dCKP]; this implies that the same is true for their centers.

We then prove that the completed algebras contain a finitely generated dense sub-algebra (this gives meaning to the statement that in this case, coordinate algebras are function algebras over the simply connected quantum group); in addition, they are finite modules over the “Frobenius part” of their centers, in the case where the base ring is the cyclotomic extension of $\mathbf{Z}_p$. Both results use the Chevalley classification of isogenies in characteristic p .

This “Frobenius part” is a completion of a finite type commutative algebra over $\mathbf{Q}_p$; it is shown to be noetherian. In fact, it is an “affinoid” algebra (in the sense of [BGR]). Thus, commutative algebra arguments can be used to analyze the structure of the completed algebras on $\mathbf{Q}_p$.

All these results make use of base changes, more precisely of the specialization to characteristic p discovered by Lusztig in [L1], and subsequently exploited by many authors (cf. e.g. [APW]). The Lusztig-Rosso theory of representations in the case of a transcendental parameter is also used in several occasions.

The algebraic results of [dCK], [dCKP] needed to analyze the quantum algebras by algebraic geometrical means are thus available in the setting. Analysis of the sl_2 case, comparison with the [dCK], [dCKP] results and discussions with C. Procesi lead us to expect the following picture for the center. The center should be generated by two parts, the “Frobenius part” Z_F described here (Z_F is isomorphic to a completion of the ring of algebraic functions on the semi-simple group over $\mathbf{Q}_p$), and another part Z_S linked to the structure of symplectic leaves. This other part should, contrary to the case of Z_F , use only little of the information of the Cartan matrix A : Z_S should consist in the n -th tensor power ($n = rkA$) of the Z_S of each sl_2 -component of the algebraic group. This should be linked with the works of Vaksman and Soibelman.

Finally, we would like to stress that the morphism $\mathcal{U}_q(A) \rightarrow U_{B_l} \otimes_{B_l} \mathbf{C}$ from the “non-restricted” enveloping algebras at roots of 1 (studied by de Concini, Kac and Procesi) to the “restricted” ones (studied by Lusztig and in this work) makes possible a pairing between $\mathcal{U}_q(A)$ and the O_l defined below.

1. Generalities on Hopf algebras. We felt it useful to begin with some basic facts about Hopf algebra theory.

1.1. The restricted dual algebra.

1.1.1. Noetherian base. Let R be a commutative ring, $\mathbf{A}$ a R -Hopf algebra (that is, with the antipode being bijective). Then $\mathbf{A}^*$ has an R -algebra structure: the product in $\mathbf{A}^*$ is given by $(\phi\phi')(x) = \langle \phi \otimes \phi', \Delta x \rangle$ for $\phi, \phi' \in \mathbf{A}^*$ and $x \in \mathbf{A}$; the unity of $\mathbf{A}^*$ is the counit ε of $\mathbf{A}$. Let $\mathcal{F}_{\mathbf{A}}$ be the set of finite codimensional (i.e., the quotient $\mathbf{A}/I$ is finitely generated) two-sided ideals of $\mathbf{A}$. The *restricted dual* of $\mathbf{A}$ is defined to be

$$\mathbf{A}^\circ = \{f \in \mathbf{A}^* \mid \exists I \in \mathcal{F}_{\mathbf{A}}, f|_I = 0\}.$$

Let V be an R -module, $\rho : \mathbf{A} \rightarrow \text{End}_R(V)$ an $\mathbf{A}$ -module structure on V . Let ϕ_ρ (we will also use ϕ_V): $V^* \otimes V \rightarrow \mathbf{A}^*$ be the linear map defined by $\phi_\rho(\mu \otimes v)(x) = \langle \mu, \rho(x)v \rangle$.

Proposition. *If R is noetherian,*

$$\mathbf{A}^\circ = \bigcup_{\rho \in \hat{\mathbf{A}}_f} \text{Im } \phi_\rho,$$

where $\hat{\mathbf{A}}_f$ is the set of $\mathbf{A}$ -modules that are R -finitely generated.

Proof. If V is an $\mathbf{A}$ -module, and $f \in \text{Im } \phi_V$, then $f|_{\text{Ker } \rho_V} = 0$, and $\text{Ker } \rho_V$ is a two-sided ideal of $\mathbf{A}$. Let us now assume that V is R -finitely generated, with generators $v_1, \dots, v_n$. The mapping $\text{End}_R(V) \rightarrow V^n$ defined by $T \mapsto (T(v_1), \dots, T(v_n))$, is an injection. By noetherianity, $\text{End}_R(V)$ is finitely generated over R , and so is $\text{Im } \phi_V$. An inverse image of a family of generators of $\text{Im } \phi_V$ generates a R -submodule of $\mathbf{A}$, such that its sum with $\text{Ker } \phi_V$ is $\mathbf{A}$.

Let f be an element of $\mathbf{A}^\circ$, and $I \in \mathcal{F}_\mathbf{A}$ be such that $f|_I = 0$. $W := \mathbf{A}/I$ is finitely generated over R . The left action of $\mathbf{A}$ induces an $\mathbf{A}$ -module structure on W . Let us remark that $f \in \text{Hom}_R(W, R) = W^*$, and denote by $\bar{x}$ the class of $x \in \mathbf{A}$ in W . Then for any $x \in \mathbf{A}$, we have $\phi_W(f \otimes \bar{1})(x) = \langle f, x \cdot \bar{1} \rangle = \langle f, \bar{x} \rangle = f(x)$, hence $f = \phi_W(f \otimes \bar{1}) \in \text{Im } \phi_W$. $\square$

Corollary. *If R is noetherian, $\mathbf{A}^\circ$ is an R -subalgebra of $\mathbf{A}^*$.*

Proof. If $\lambda \in R$, $\lambda \text{Im } \phi_\rho \subset \text{Im } \phi_\rho$; $\text{Im } \phi_\rho + \text{Im } \phi_{\rho'} \subset \text{Im } \phi_{\rho \oplus \rho'} \in \mathbf{A}^\circ$, since $\hat{\mathbf{A}}_f$ is stable under direct sums. Let us recall that the coproduct in $\mathbf{A}$ allows to define an $\mathbf{A}$ -module structure on the tensor product of two $\mathbf{A}$ -modules, and that $\phi_\rho(\mu \otimes v)\phi_{\rho'}(\mu' \otimes v') = \phi_{\rho \oplus \rho'}[(\mu \otimes \mu') \otimes (v \otimes v')]$; hence, $\text{Im } \phi_\rho \text{Im } \phi_{\rho'} \subset \text{Im } \phi_{\rho \oplus \rho'} \subset \mathbf{A}^\circ$, since $\hat{\mathbf{A}}_f$ is stable under tensor products. $\square$

Remark. If S denotes the antipode of $\mathbf{A}$, ${}^tS$ is bijective; if (ρ, V) is any $\mathbf{A}$ -module, the contragredient module to (ρ, V) is (ρ^d, V^*) where $\rho^d := {}^t \circ \rho \circ S$. As ${}^tS[\phi_\rho(\mu \otimes v)] = \phi_{\rho^d}(\bar{v} \otimes \mu)$, ($\bar{v}$ denotes the canonical image of v in V^{**}) ${}^tS(\text{Im } \phi_\rho) \subset \text{Im } \phi_{\rho^d}$. When R is noetherian, $\hat{\mathbf{A}}_f$ is stable under taking the contragredients (since if V is generated by $v_1, \dots, v_n$, the mapping $V^* \rightarrow R^n$, $\phi \mapsto (\phi(v_1), \dots, \phi(v_n))$ is injective), so $\mathbf{A}^\circ$ is ${}^tS$ -stable.

1.1.2. Noetherian integral base. Let $\mathcal{G}_\mathbf{A}$ be the subset of $\mathcal{F}_\mathbf{A}$ consisting of those ideals I such that $\mathbf{A}/I$ is R -torsion-free. If R is an integral domain, the map $\mathcal{F}_\mathbf{A} \rightarrow \mathcal{G}_\mathbf{A} : I \mapsto I' = \{x \in \mathbf{A} / \exists \lambda \in R, \lambda \neq 0 / \lambda x \in I\}$ shows that $\mathcal{F}_\mathbf{A}$ may be replaced by $\mathcal{G}_\mathbf{A}$ in the definition of $\mathbf{A}^\circ$.

We then have the following version of proposition 1.1.1:

Proposition. *If R is noetherian integral domain,*

$$\mathbf{A}^\circ = \bigcup_{\rho \in \hat{\mathbf{A}}_{f,tf}} \text{Im } \phi_\rho,$$

where $\hat{\mathbf{A}}_{f,tf}$ is the set of $\mathbf{A}$ -modules that are R -finitely generated and torsion-free.

Proof. Let us improve in the first part of proof 1.1.1, by showing that if V is R -finitely generated and torsion-free, $\text{Ker } \rho \in \mathcal{G}_\mathbf{A}$. If $x \in \mathbf{A}$ and $\lambda \in R - \{0\}$ are such that

$\lambda x \in \text{Ker } \rho$, one has: $\forall v \in V$, $\lambda \rho(x)v = \rho(\lambda x)v = 0$, and as V is torsion-free, $\rho(x)v = 0$, and $x \in \text{Ker } \rho$, so $\mathbf{A}/\text{Ker } \rho$ is torsion-free.

Let us now improve the second part of it, and show that the W in it may be taken torsion-free. We can do the construction of W again using an ideal I' in $\mathcal{G}_{\mathbf{A}}$. The R -module $W' := \mathbf{A}/I'$ allows the same computation, and is torsion-free. $\square$

Remarks. 1) If R is Dedekind, $\hat{\mathbf{A}}_{f,tf}$ is stable under tensor products, since in that case, any torsion-free module is projective (see e.g. [CE, p. 133, prop. 4.1]). For the same reason, one then has $\mathbf{A}^\circ = \sum_{\rho \in \hat{\mathbf{A}}_{f,f}} \text{Im } \phi_\rho$, where $\hat{\mathbf{A}}_{f,f}$ is the set of R -free and finite dimensional $\mathbf{A}$ -modules.

2) If R is an integral domain, $\mathbf{A}^\circ$ is R -torsion-free (as a submodule of the dual of an R -module) and divisible in $\mathbf{A}^*$: if $x \in \mathbf{A}^*$, $\lambda \in R - \{0\}$, $\lambda x \in \mathbf{A}^\circ$, then $x \in \mathbf{A}^\circ$. This is because x has to vanish on the same ideal than λx .

1.2. The case of a principal ideal ring. Assume R to be a principal ideal ring (for example any field, or the rings $\mathbf{Z}$ or $\mathbf{Q}[v, v^{-1}]$). Then it is a Dedekind ring, and has the property that each finite torsion-free module on it is free.

$\mathbf{A}$ will be assumed R -free.

Now let (ρ, V) be an $\mathbf{A}$ -module, V being R -finite and free, and denote by $\text{Coeff}(\mathbf{A}, V)$ the subalgebra of $\mathbf{A}^\circ$ generated by $\phi_\rho(V^* \otimes V)$. Denote by $T(V^* \otimes V)$ the tensor algebra on $V^* \otimes V$ and denote also by ϕ_ρ the induced morphism from $T(V^* \otimes V)$ to $\mathbf{A}^\circ$. The coproduct Δ_V on $\text{Coeff}(\mathbf{A}, V)$ is defined in the following way: let e_i, e^i be dual bases of V, V^* . The linear mapping from $V^* \otimes V$ to $\text{Coeff}(\mathbf{A}, V) \otimes \text{Coeff}(\mathbf{A}, V)$ defined by $\mu \otimes v \mapsto \sum_i \phi_\rho(\mu \otimes e_i) \otimes \phi_\rho(e^i \otimes v)$ induces an algebra morphism from $T(V^* \otimes V)$ to $\text{Coeff}(\mathbf{A}, V) \otimes \text{Coeff}(\mathbf{A}, V)$, denoted $\bar{\Delta}_V$.

Remark that $\text{Coeff}(\mathbf{A}, V) \otimes \text{Coeff}(\mathbf{A}, V)$ injects into $(\mathbf{A} \otimes \mathbf{A})^*$; indeed, let $F_n := \phi_\rho(T_n(V^* \otimes V))$; it is a (R -finite and free, because torsion-free) submodule of $\mathbf{A}^*$. Hence, the canonical map $F_n \otimes F_n \rightarrow (F_n^* \otimes F_n^*)^*$ is an isomorphism. Note also that since $\mathbf{A} \rightarrow F_n^*$ is surjective, so is $\mathbf{A} \otimes \mathbf{A} \rightarrow F_n^* \otimes F_n^*$, so that $(F_n^* \otimes F_n^*)^* \rightarrow (\mathbf{A} \otimes \mathbf{A})^*$ is injective. Finally, $\text{Coeff}(\mathbf{A}, V) \otimes \text{Coeff}(\mathbf{A}, V)$ is the union of the images of the $F_n \otimes F_n$, by injective mappings (because F_n and $\text{Coeff}(\mathbf{A}, V)$ are torsion-free, hence flat, since R is principal). This proves our claim. (I am indebted to P. Cartier for these arguments.)

Then $\langle \bar{\Delta}_V(t), x \otimes y \rangle = \langle \phi_\rho(t), xy \rangle, \forall x, y \in \mathbf{A}, \forall t \in T(V^* \otimes V)$. Hence $\bar{\Delta}_V$ factorizes through an algebra morphism $\Delta_V : \text{Coeff}(\mathbf{A}, V) \rightarrow \text{Coeff}(\mathbf{A}, V) \otimes \text{Coeff}(\mathbf{A}, V)$; we deduce (all these morphisms are compatible thanks to the common embedding in $(\mathbf{A} \otimes \mathbf{A})^*$) a coproduct $\mathbf{A}^\circ \rightarrow \mathbf{A}^\circ \otimes \mathbf{A}^\circ$. This makes $\mathbf{A}^\circ$ a bialgebra, and $\text{Coeff}(\mathbf{A}, V)$ a subbialgebra of it. Remark that $\mathbf{A}^\circ$ is ${}^tS$ -stable; ${}^tS$ is checked to be the antipode of $\mathbf{A}^\circ$, which is now a Hopf algebra.

If S^2 is *inner* in $\mathbf{A}$, the subalgebra $\text{Coeff}(\mathbf{A}, V)$ generated by $\phi_\rho(V^* \otimes V) + \phi_{\rho^d}(V \otimes V^*)$ is a Hopf subalgebra of $\mathbf{A}^\circ$ (cf. [A]).

1.3. Base change. Let $R \rightarrow R'$ be a morphism of commutative rings, and let $X_{R'}$ denote $X \otimes_R R'$ for any R -module X . The operation $\otimes_R 1$ induces a R -Hopf algebra morphism $\mathbf{A} \rightarrow \mathbf{A}_{R'}$, and an R -algebra morphism $\otimes 1 : \text{Hom}_{R-\text{mod}}(\mathbf{A}, R) \rightarrow \text{Hom}_{R'-\text{mod}}(\mathbf{A}_{R'}, R')$. The image of $\mathbf{A}^\circ$ by $\otimes 1$ is a subalgebra of $(\mathbf{A}_{R'})^\circ$. $(\otimes 1)|_{\mathbf{A}^\circ} : \mathbf{A}^\circ \rightarrow (\mathbf{A}_{R'})^\circ$ factors through an algebra morphism from $\mathbf{A}^\circ \otimes_R 1(\subset \mathbf{A}^\circ \otimes_R R')$ to $(\mathbf{A}_{R'})^\circ$.

If $\mathbf{A}$ is R -free, and I denotes the kernel of $R \rightarrow R'$, the kernel of this last morphism is $\text{Hom}(\mathbf{A}, I) = I\mathbf{A}^*$.

1.3.1. Taking quotients. Assume that R is a domain, I is a principal ideal in R , and $R \rightarrow R'$ is the projection map $R \rightarrow R/I$. Then the above described factor morphism is an isomorphism. Put more simply, $(\mathbf{A}^\circ)_{R'}$ is a subalgebra of $(\mathbf{A}^*)_{R'}$. Indeed, as $\otimes$ is right-exact, for any R -module M , $M \otimes_R R' = M/IM$; so we have to see that $I\mathbf{A}^* \cap \mathbf{A}^\circ = I\mathbf{A}^\circ$; and this is because by its very definition, $\mathbf{A}^\circ$ is divisible in $\mathbf{A}^*$.

Remark. If $R \rightarrow R'$ is surjective, and $\mathbf{A}^\circ$ is generated by some $\text{Im}(\phi_{\rho'})$ for representations (ρ', V') which admit an R -form (ρ, V) with V torsion-free; if in addition for any torsion-free R -module M , $(M^*)_{R'} \rightarrow (M_{R'})^*$ is surjective; then $\mathbf{A}^\circ \rightarrow \mathbf{A}_{R'}^\circ$ is surjective.

1.3.2. The flat case. Let us assume $R \rightarrow R'$ to be flat, and $\mathbf{A}$ to be free as R -module. Then $\mathbf{A}^\circ \otimes_R R'$ can be viewed as a subalgebra of $(\mathbf{A}_{R'})^*$. The image of $\mathbf{A}^\circ \rightarrow \mathbf{A}^\circ \otimes_R R'$ is sent to a subset of $(\mathbf{A}_{R'})^\circ$: indeed, if I is finite-codimensional in $\mathbf{A}$ so will be $I \otimes_R R'$ in $\mathbf{A}_{R'}$. So, the identification above views $\mathbf{A}^\circ \otimes_R R'$ as a subalgebra of $(\mathbf{A}_{R'})^\circ$.

1.3.3. Localization. Let us assume that R is a domain, and let S be a multiplicative part in it.

If $\mathbf{A}$ is R -free, we have the injections $\mathbf{A} \rightarrow \mathbf{A}_{R_S}$, $\mathbf{A}^* \rightarrow (\mathbf{A}_{R_S})^*$, and $\mathbf{A}^\circ \rightarrow (\mathbf{A}_{R_S})^\circ$; in addition, $\mathbf{A}^* \otimes_{R_S}$ injects into $(\mathbf{A}_{R_S})^*$.

For any S -torsion free R -module V , the R -module morphism $V \rightarrow V_{R_S}$ is injective; the same for $\text{Hom}(V, R) \rightarrow \text{Hom}(V, R_S)$. Finally, if V is finite, $\text{Hom}(V, R_S) = \text{Hom}(V, R) \otimes_{R_S} R_S \simeq \text{Hom}(V_{R_S}, R_S)$.

Localizing is left-exact, hence $\mathbf{A}^\circ \otimes_R R_S$ may be identified with its image in $(\mathbf{A}_{R_S})^*$; in it, $\text{Im } \phi_\rho \otimes_R R_S$ is isomorphic to $\text{Im } \phi_{\rho_{R_S}}$; assume R to be noetherian. Then $\mathbf{A}^\circ \otimes_R R_S \simeq \bigcup_{\rho \in \hat{\mathbf{A}}_f} \text{Im } \phi_{\rho_{R_S}}$ is included in $(\mathbf{A}_{R_S})^\circ$. This identification induces a bialgebra (or Hopf algebra if S^2 is inner in $\mathbf{A}$) isomorphism between $\text{Coeff}(\mathbf{A}, V) \otimes_R R_S$ and $\text{Coeff}(\mathbf{A}_{R_S}, V_{R_S})$, V being a S -torsion free R -finite $\mathbf{A}$ -module.

If in addition, any finite R_S -module W which is an $\mathbf{A}_{R_S}$ -module is the localization of some finite $\mathbf{A}$ -module V which is R -finite (we then call V an R -finite form of W), then $\mathbf{A}^\circ \otimes_R R_S = (\mathbf{A}_{R_S})^\circ$.

1.3.4. Field extention. Let us assume that $k = R \rightarrow R' = K$ is a field extension; the k -vector space morphisms $\mathbf{A} \rightarrow \mathbf{A}_K$, $\mathbf{A}^* \rightarrow \mathbf{A}_K^*$ are injective. In addition, because $\otimes_k K$ is left exact, $\mathbf{A}^\circ$ injects into $\mathbf{A}_K^\circ$. Let V be a k -finite dimensional representation of $\mathbf{A}$. Still by left exactness, $\text{Im } \phi_V \otimes K \simeq \text{Im } \phi_{V_K}$. It follows that the Hopf algebras $\text{Coeff}(\mathbf{A}, V) \otimes K$ and $\text{Coeff}(\mathbf{A}_K, V_K)$ are isomorphic.

We deduce: $\mathbf{A}^\circ \otimes_k K \subset (\mathbf{A}_K)^\circ$. If in addition, every K -finite $\mathbf{A}_K$ -module has a k -finite k -form, then $\mathbf{A}^\circ \otimes K$ is Hopf algebraically isomorphic to $\mathbf{A}_K^\circ$. This is in particular the case if K is a finite extension of k (then each K -finite module is also k -finite).

2. Integrity of the Coordinate Algebras on Quantum Groups. After a short review of the Lusztig definition of quantum groups, we define the coordinate algebras on them when the parameter is a root of unity. We then prove their integrity.

2.1. Quantized enveloping algebras. We adopt the notations of [L1], except for the symbol ν . Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$, $\mathcal{A}' = \mathbf{Q}(v)$ be its fraction field. Let $\mathbf{U}$ be the $\mathcal{A}'$ -Hopf algebra generated by E_i, F_i, K_i , and the relations (a1) to (a5), (b), (c1) and (c2) in [L1]. Let us define $E_i^{(N)} = \frac{E_i^N}{[N]_{d_i}!}$, $F_i^{(N)} = \frac{F_i^N}{[N]_{d_i}!}$ to be the quantum divided powers. Let U be the $\mathcal{A}$ -subalgebra of $\mathbf{U}$ generated by $E_i^{(N)}, F_i^{(N)}, K_i, K_i^{-1}$. Then U is a $\mathcal{A}$ -Hopf algebra. It is a free $\mathcal{A}$ -module (see [L1]).

Let $\mathcal{A} \rightarrow \Gamma$ be any morphism of commutative rings; we shall write X_Γ for $X \otimes_{\mathcal{A}} \Gamma$, for any $\mathcal{A}$ -module X . Then U_Γ is a Γ -Hopf algebra.

Among other results, Lusztig has shown that the quotient $U_{\mathbf{Z}}/(K_i - 1)$ ($U_{\mathbf{Z}}$ corresponding to the algebra morphism $\mathcal{A} \rightarrow \mathbf{Z}, v \mapsto 1$) is isomorphic to the Kostant $\mathbf{Z}$ -form $\overline{U}_{\mathbf{Z}}$, and that if p is prime, the quotient $U_{\mathbf{F}_p}/(K_i - 1)$ is isomorphic to $\overline{U}_{\mathbf{F}_p} := \overline{U}_{\mathbf{Z}} \otimes \mathbf{F}_p$.

Let l be any positive integer, and let ϕ_l denote the l -th cyclotomic polynomial. Let us denote by $\mathcal{B}_l$ the ring $\mathbf{Z}[v]/(\phi_l)$, and let π_l denote the quotient morphism from $\mathcal{A}$ to $\mathcal{B}_l$; $\mathcal{B}_l$ is the ring of integers of its quotient field $B_l = \mathbf{Q}[v]/(\phi_l)$. If l' is a multiple of l , the ring morphism $\varphi_{ll'} : \mathcal{B}_l = \mathbf{Z}[v]/(\phi_l(v)) \rightarrow \mathcal{B}_{l'} = \mathbf{Z}[v]/(\phi_{l'}(v))$ defined by $\varphi_{ll'}(v) = (v')^{l'/l}$ is flat (because B_l is $\mathcal{B}_{l'}$ -free; an integral base for $\mathcal{B}_l$ being $1, v, v^2, \dots, v^{\phi(l)-1}$, cf. [Hs]) and induces a field extension (also denoted $\varphi_{ll'}$) $\mathcal{B}_l \rightarrow \mathcal{B}_{l'}$.

The element $v - 1 \in \mathcal{B}_l$ is a unit if l is not a prime power; if $l = p^\nu$, $v - 1$ has norm p and $\mathcal{B}_l/(v - 1)$ is isomorphic to $\mathbf{F}_p$, the field with p elements.

Denoting by φ_l the morphism from $\mathcal{B}_l$ to $\mathbf{F}_p$ defined by $\varphi_l(v) = 1$, we remark that if l' is a prime power multiple of l , $\varphi_{l'} \circ \varphi_{ll'} = \varphi_l$. The tensor products $\otimes_{\varphi_{ll'}}$ over $\varphi_{ll'}$ will also be denoted by $\otimes_{ll'}$ instead of $\otimes_{\varphi_{ll'}}$.

2.2. The coordinate algebras at l -th roots of unity.

2.2.1. The basic construction. Let l be a positive integer, and let $\mathcal{I}$ be the set of all two-sided finite codimensional ideals of $U_{\mathcal{B}_l}$ (such that $U_{\mathcal{B}_l}/I$ is torsion-free if one wishes) which contain all the $K_i^l - 1$, $1 \leq i \leq n$. Let then $\mathcal{O}_l$ be the subspace of $U_{\mathcal{B}_l}^\circ$ consisting of the $\mathcal{B}_l$ -forms on $U_{\mathcal{B}_l}$ vanishing on some $I \in \mathcal{I}$. We will call $\mathcal{O}_l$ the *quantum coordinate algebra* for the Cartan matrix A at l -th roots of 1. It enjoys the following properties:

Proposition.

- i) $\mathcal{O}_l$ is a subalgebra of $U_{\mathcal{B}_l}^\circ$
- ii) $\mathcal{O}_l$ is divisible in $U_{\mathcal{B}_l}^*$
- iii) if l is an odd prime power, $f \in \mathcal{O}_l$, $1 \leq i \leq n$, $x \in U_{\mathcal{B}_l}$, $f((K_i - 1)x)$ and $f(x(K_i - 1))$ belong to $(v - 1)\mathcal{B}_l$.

Proof. i) Recall that the left and right actions of $U_{\mathcal{B}_l}$ on itself by multiplication induce by duality respectively right and left actions on $U_{\mathcal{B}_l}^*$. They are checked to be actions on $U_{\mathcal{B}_l}^\circ$ and denoted R and L . If $x \in U_{\mathcal{B}_l}$ and $\Delta x = \sum x^{(1)} \otimes x^{(2)}$, if f and g belong to $U_{\mathcal{B}_l}^*$, $L(x)(fg) = \sum L(x^{(1)})(f)L(x^{(2)})(g)$, and $R(x)(fg) = \sum R(x^{(1)})(f)R(x^{(2)})(g)$. Then $\mathcal{O}_l = \{f \in U_{\mathcal{B}_l}^\circ / \forall i \in [1, n], L(K_i^l)f = R(K_i^l)f = f\}$. As the K_i^l are group-like, the above remark shows that $\mathcal{O}_l$ is a subalgebra of $U_{\mathcal{B}_l}^\circ$.

ii) follows from the definition.

iii) Recall that $\begin{bmatrix} K_i \\ 1 \end{bmatrix} = \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}$ is in $U_{\mathcal{B}_l}$; hence, $K_i^2 = 1 + (v^{d_i} - v^{-d_i})K_i \begin{bmatrix} K_i \\ 1 \end{bmatrix} \in 1 + (v-1)U_{\mathcal{B}_l}$. Then $K_i^{l+1} = (K_i^2)^{\frac{l+1}{2}} \in 1 + (v-1)U_{\mathcal{B}_l}$. It follows that $f(K_i x) - f(x) = f(K_i^{l+1} x) - f(x) \in (v-1)\mathcal{B}_l$. In the same way, $f(xK_i) - f(x) \in (v-1)\mathcal{B}_l$. $\square$

Now let us give the following description of $\mathcal{O}_l$, l being arbitrary:

Lemma.

$$\mathcal{O}_l = \bigcup_{\rho \in \hat{U}_{\mathcal{B}_l}^{f,c}} \text{Im } \phi_\rho,$$

where $\hat{U}_{\mathcal{B}_l}^{f,c}$ denotes the set of $U_{\mathcal{B}_l}$ -modules which are $\mathcal{B}_l$ -finite and torsion-free and such that the central elements K_i^l , $1 \leq i \leq n$, act by the identity.

Proof. Let $f = \phi_\rho(\mu \otimes v)$, $\rho \in \hat{U}_{\mathcal{B}_l}^{f,c}$. Then $\langle \mu, (K_i^l - 1)x \cdot v \rangle = \langle \mu, x(K_i^l - 1) \cdot v \rangle$ (by centrality of K_i^l) = 0 (by assumption on ρ).

In the other way, let $f \in \mathcal{O}_l$. Let I be an ideal in $\mathcal{G}_{U_{\mathcal{B}_l}}$ (this object is defined in 1.1.2), containing the $K_i^l - 1$'s, associated to f . Let us apply construction 1.1.2. The action of $K_i^l - 1$ on $U_{\mathcal{B}_l}/I$ is zero since $K_i^l - 1 \in I$. Hence $W \in \hat{U}_{\mathcal{B}_l}^{f,c}$. $\square$

2.2.2. Ring extensions. Let K be a field extension of B_l , $\mathfrak{p}$ a prime ideal of its ring of integers $\mathbf{o}_K$ above $(v-1)$ (i.e. such that $\mathfrak{p} \cap B_l = (v-1)$). By flatness of $\mathbf{o}_K$ over $\mathcal{B}_l$, $\mathcal{O}_l \otimes \mathbf{o}_K$ and $U_{\mathcal{B}_l}^\circ \otimes \mathbf{o}_K$ are identified with subsets of $\text{Hom}_{\mathcal{B}_l}(U_{\mathcal{B}_l}, \mathbf{o}_K)$. These subsets consist exactly of all $\mathbf{o}_K$ -forms vanishing on some $I \in \mathcal{I}$ (resp. $I \in \mathcal{G}_{U_{\mathcal{B}_l}}$). Indeed, an intersection of finitely many finite codimensional $\mathcal{B}_l$ -submodules in $U_{\mathcal{B}_l}$ is still finite codimensional (by noetherianity); the same for an intersection of two-sided ideals; and the same if the ideals contain the $K_i^l - 1$'s.

Proposition 2.2.1 is immediately generalized to $\mathcal{O}_l \otimes \mathbf{o}_K$:

Proposition.

- i) $\mathcal{O}_l \otimes \mathbf{o}_K$ is a subalgebra of $U_{\mathcal{B}_l}^\circ \otimes \mathbf{o}_K$
- ii) $\mathcal{O}_l \otimes \mathbf{o}_K$ is divisible in $\text{Hom}_{\mathcal{B}_l}(U_{\mathcal{B}_l}, \mathbf{o}_K)$
- iii) if l is an odd prime power, $f \in \mathcal{O}_l \otimes \mathbf{o}_K$, $1 \leq i \leq n$, $x \in U_{\mathcal{B}_l}$, $f((K_i - 1)x)$ and $f(x(K_i - 1))$ belong to $(v-1)$.

On the other hand, 1.3.2 allows to identify $(U_{\mathcal{B}_l})^\circ \otimes \mathbf{o}_K$ with a subalgebra of $(U_{\mathcal{B}_l} \otimes \mathbf{o}_K)^\circ$. In addition:

Lemma. This identification allows to view, in $(U_{\mathcal{B}_l} \otimes \mathbf{o}_K)^*$, $\mathcal{O}_l \otimes \mathbf{o}_K$ as $\bigcup_{\rho \in \hat{U}_{\mathcal{B}_l}^{f,c}} \text{Im } \phi_{\rho \otimes 1}$, $\rho \otimes 1$ denoting the representation of $U_{\mathcal{B}_l} \otimes \mathbf{o}_K$ obtained from ρ .

Proof. The flatness of $\mathbf{o}_K$ over $\mathcal{B}_l$ allows to identify $\mathcal{O}_l \otimes \mathbf{o}_K$, and each $\text{Im } \phi_\rho \otimes \mathbf{o}_K$ in it, with a subset of $U_{\mathcal{B}_l}^* \otimes \mathbf{o}_K$, which is then, by freeness of $U_{\mathcal{B}_l}$, a subalgebra of $(U_{\mathcal{B}_l} \otimes \mathbf{o}_K)^*$. Fix any (ρ, V) in $\hat{U}_{\mathcal{B}_l}^{f,c}$. The image of $\text{Im } \phi_\rho \otimes \mathbf{o}_K$ then consists of a sum of forms sending $x \otimes b'$, $x \in U_{\mathcal{B}_l}$, $b' \in \mathbf{o}_K$ to $\langle \mu, \rho(x)v \rangle b' b''$, $b'' \in \mathbf{o}_K$ and $\mu \otimes v \in V^* \otimes V$ being fixed. This form is $\phi_{\rho \otimes 1}(\beta'' \mu \otimes v)$.

In the other way, any form in $\text{Im } \phi_{\rho \otimes 1}$ is a sum of forms of the type $\phi_{\rho \otimes 1}((\mu \otimes b') \otimes (v \otimes b''))$, $b', b'' \in \mathbf{o}_K$, $\mu \otimes v \in V^* \otimes V$, equal to $\phi_{\rho \otimes 1}(b' b'' \mu \otimes v)$, and hence of the type above. $\square$

2.2.3. Localization. Let $\mathcal{I}'$ be the set of all two-sided ideals in U_{B_l} (recall that B_l is the cyclotomic field) with finite codimension, containing the $K_i^l - 1$, $1 \leq i \leq n$. Let O'_l be the subspace of $U_{B_l}^\circ$ of the B_l -forms vanishing on some $I \in \mathcal{I}'$. O'_l is clearly the union of $\text{Im } \phi_\rho$, for (ρ, V) all B_l -finite U_{B_l} -modules where the $K_i^l - 1$'s act by zero. By freeness of U_{B_l} , $U_{B_l}^*$ can be viewed as a subalgebra of $U_{B_l}^*$. Then: $O_l = U_{B_l}^* \cap O'_l$. Let O_l stand for $O_l \otimes_{B_l} B_l$. We have proved:

Lemma. O_l can be viewed as a subalgebra of O'_l .

By 1.3, $U_{B_l}^\circ$ is a Hopf algebra; let us prove that

Lemma. O_l is a Hopf subalgebra of $U_{B_l}^\circ$.

Proof. First remark that O_l is the subalgebra of $U_{B_l}^\circ$ consisting of the elements invariant by the K_i^l . Then recall that if $f \in U_{B_l}^\circ$, Δf is the element of $U_{B_l}^\circ \otimes U_{B_l}^\circ$ defined by $\langle \Delta f, x \otimes y \rangle = \langle f, xy \rangle$ for any $x, y \in U_{B_l}$. It follows from $(K_i^l x)y = (x K_i^l)y = x(K_i^l y) = x(y K_i^l) = K_i^l(xy)$ that $(L(K_i^l) \otimes 1)\Delta f = (1 \otimes L(K_i^l))\Delta f = \Delta f$, and similarly for the right action. Finally we make repeated use of the following

Lemma. If k is a field and $a : V \rightarrow W$ is a morphism of k -vector spaces, $\text{Ker}(a \otimes 1) \cap \text{Ker}(1 \otimes a) = \text{Ker } a \otimes \text{Ker } a$.

Proof. Tensoring the exact sequence $\text{Ker } a \rightarrow V \rightarrow \text{Im } V \rightarrow 0$ by V yields the exact sequence $\text{Ker } a \otimes V \rightarrow V \otimes V \rightarrow \text{Im } V \otimes V \rightarrow 0$ and the equality $\text{Ker}(a \otimes 1) = \text{Ker } a \otimes V$. Similarly $\text{Ker}(1 \otimes a) = V \otimes \text{Ker } a$. Let then $K \subset V$ be any inclusion of vector spaces. Then $(V \otimes K) \cap (K \otimes V) = K \otimes K$, as is checked using bases. $\square$

Let V be any finite dimensional U_{B_l} -module, where each K_i^l acts by identity. Then

Proposition. $\text{Coeff}(U_{B_l}, V)$ is a Hopf subalgebra of O'_l .

Proof. See the appendix. $\square$

In fact, O_l is equal to $\cup_{\rho \in \hat{U}_{B_l, B_l}} \text{Im } \phi_\rho$, $\hat{U}_{B_l, B_l}$ denoting the set of B_l -finite U_{B_l} -modules where the K_i^l 's act by 1 and having a B_l -finite form. O_l is as well equal to the union of all coefficient algebras for the modules in $\hat{U}_{B_l, B_l}$. Combining the three lemmas and the proposition above, we get:

Proposition. O_l is a Hopf subalgebra of O'_l .

Remarks. 1) The results of 1.1.1 alone are enough to construct a Hopf algebra structure on O_l .

2) It is not clear if O_l and O'_l are equal. This problem amounts to the following one: has any B_l -finite U_{B_l} -module a B_l -finite form?

For the rest of this work, we will focus on the analysis of O_l , O_l and certain of their completions.

3) Associativity of $\otimes$ shows that if l' is a multiple of l , $O_l \otimes_{l'} B_{l'} = (O_l \otimes B_{l'}) \otimes B_{l'}$. The strategy will then be the following: we will use base change to $\mathbf{F}_p$ to prove results

on $\mathcal{O}_l \otimes \mathcal{B}_{l'}$, and then localization to deduce properties of $\mathcal{O}_l \otimes \mathcal{B}_{l'}$. This last algebra, having a Hopf algebra structure, is expected to be easier to analyse.

2.3. Integrity of the coordinate algebras at p^{ν} -th roots of unity. Let us assume that $l = p^{\nu}$, $l' = p^{\nu'}$, with the integers $\nu \leq \nu'$ and p prime $\neq 2$. Let us consider the $\mathcal{B}_{l'}$ -algebra $\mathcal{O}_l \otimes \mathcal{B}_{l'}$, and consider the change of rings $\mathcal{B}_{l'} \rightarrow \mathbf{F}_p$ defined by $v' \mapsto 1$. First remark that tensoring by $\mathbf{F}_p$ transforms the sequence of inclusions $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \rightarrow U_{\mathcal{B}_l}^{\circ} \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \rightarrow U_{\mathcal{B}_l}^* \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ into $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} / (v' - 1) \mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \rightarrow U_{\mathcal{B}_l}^{\circ} \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} / (v' - 1) U_{\mathcal{B}_l}^{\circ} \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \rightarrow U_{\mathcal{B}_l}^* \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} / (v' - 1) U_{\mathcal{B}_l}^* \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$, which is still a sequence of inclusions since each of the initial spaces was divisible in $U_{\mathcal{B}_l}^* \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ (as is checked writing the definitions of these spaces in $\text{Hom}_{\mathcal{B}_l}(U_{\mathcal{B}_l}, \mathcal{B}_{l'})$).

Then, by $\mathcal{B}_l$ -freeness of $U_{\mathcal{B}_l}$, we have $U_{\mathcal{B}_l}^* \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \otimes \mathbf{F}_p = U_{\mathbf{F}_p}^*$.

Recall that on the $\mathbf{F}_p$ side, we have two concurring algebras $U_{\mathbf{F}_p}$ and $\overline{U}_{\mathbf{F}_p} = U_{\mathbf{F}_p} / (K_i - 1)$. Let us now clarify the relations between the duals of these algebras.

Lemma. *The canonical injection $\overline{U}_{\mathbf{F}_p}^* \rightarrow U_{\mathbf{F}_p}^*$ is an algebra morphism; its image is the subset of $U_{\mathbf{F}_p}^*$ consisting of the f such that $\forall x \in U_{\mathbf{F}_p}$, $f((K_i - 1)x) = f(x(K_i - 1)) = 0$.*

Proof. The first assertion is due to the fact that $I := (K_i - 1)$ is a Hopf ideal, i.e. $\Delta I \subset I \otimes U_{\mathbf{F}_p} + U_{\mathbf{F}_p} \otimes I$, $S(I) = I$; this induces a Hopf algebra structure on $\overline{U}_{\mathbf{F}_p}$ (which is checked to be the usual one); the projection is a Hopf algebra morphism. The second assertion is easy linear algebra. $\square$

Let us still denote by $\overline{U}_{\mathbf{F}_p}^*$ its image in $U_{\mathbf{F}_p}^*$. Then by proposition 2.2.1 iii), since $(\otimes_{\mathbf{F}_p} 1)(l'(v)) = 1$, and by the second part of lemma 2.2.3, $(\otimes_{\mathbf{F}_p} 1)(\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}) = (\otimes_{\mathbf{F}_p} 1)(\mathcal{O}_l) \subset \overline{U}_{\mathbf{F}_p}^*$. Finally, the kernel of $\otimes_{\mathbf{F}_p} 1 : \mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \rightarrow \overline{U}_{\mathbf{F}_p}^*$ is $(v' - 1)U_{\mathcal{B}_l}^* \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} \cap \mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$, which is equal to $(v' - 1)\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$. So, $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'} / (v' - 1)\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ embeds in $\overline{U}_{\mathbf{F}_p}^*$, hence $([K])$ is an integral domain. We are now ready to prove:

Theorem.

- i) If $l = p^{\nu}$, $l' = p^{\nu'}$, $\nu \leq \nu'$ positive integers, p a prime integer $\neq 2$, the $\mathcal{B}_{l'}$ -algebra $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ is an integral domain.
- ii) With the same assumptions, the $\mathcal{B}_{l'}$ -algebra $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ is an integral domain.

Proof. i) First remark that $\cap_{n \geq 0} (v' - 1)^n \mathcal{O}_l \otimes \mathcal{B}_{l'} \subset \cap_{n \geq 0} (v' - 1)^n \text{Hom}_{\mathcal{B}_l}(U_{\mathcal{B}_l}, \mathcal{B}_{l'}) = \text{Hom}_{\mathcal{B}_l}(U_{\mathcal{B}_l}, \cap_{n \geq 0} (v' - 1)^n \mathcal{B}_{l'}) = 0$ since $v' - 1$ being of norm p in $\mathcal{B}_{l'}$, the norm of any element in $\cap_{n \geq 0} (v' - 1)^n \mathcal{B}_{l'}$ should be multiple of any p^n and hence zero.

Now assume that a, b are two non-zero elements in $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ such that $ab = 0$. By the remark above, there exist a', b' in $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ and integers k, l such that $a = (v' - 1)^k a'$, $b = (v' - 1)^l b'$, and $a', b' \notin (v' - 1)\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$. Then $(v' - 1)^{k+l} a' b' = 0$, and by the torsion-freeness of $\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$ (a consequence of its inclusion in the torsion-free $\text{Hom}_{\mathcal{B}_l}(U_{\mathcal{B}_l}, \mathcal{B}_{l'})$), $a' b' = 0$.

Let $\overline{a'}, \overline{b'}, \overline{a' b'}$ be the images of $a', b', a' b'$ in $\overline{U}_{\mathbf{F}_p}$. Then $\overline{a'} \cdot \overline{b'} = \overline{a' b'} = 0$. Now recall ([K]) that $\overline{U}_{\mathbf{F}_p}^*$ is an integral domain, since it is an algebra of formal series. Hence one of the $\overline{a'}, \overline{b'}$ should vanish, so one of the a', b' should belong to $(v' - 1)\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}$, contradiction.

ii) follows since localization of the base preserves the integrity. $\square$

Remarks. 1) If l' is not a prime power, the technique used above does not work since in that case, $v - 1$ is a unit in $\mathcal{B}_{l'}$.

2) Let l be arbitrary. An intermediate algebra $\mathcal{O}_l^\infty$ between $\mathcal{O}_l$ and $U_{\mathcal{B}_l}^\circ$ can be defined by the requirement that the ideal I contain the family $(K_i^l - 1)^{\alpha_1}, \dots, (K_i^l - 1)^{\alpha_n}$, the α_i 's being sufficiently large. $\mathcal{O}_l^\infty \otimes B_l$ is a Hopf subalgebra of $U_{B_l}^\circ$ (being a union of coefficient algebras; see 3.2). It arises in the following way: the change of rings $\mathcal{A}_0 \rightarrow B_l$ induces an algebra morphism $U_{\mathcal{A}_0}^* \rightarrow U_{B_l}^*$; the image of the Lusztig algebra $\mathcal{O}$ by this morphism is contained in $\mathcal{O}_l^\infty \otimes B_l$. The algebra $\mathcal{O}$ is divisible in $(U_{\mathcal{A}_0})^*$, and it follows that $\mathcal{O} \otimes B_l$ is a subalgebra of $\mathcal{O}_l^\infty \otimes B_l$. We will show in 4.1 that if l is an odd prime power, up to some p -adic completion this algebra is equal to $\mathcal{O}_l$. Let us prove its integrity. For this, it is enough (by an argument like in the theorem above) to show that $\otimes 1 : U_{\mathcal{B}_l}^\circ \rightarrow U_{\mathbf{F}_p}^\circ$ sends $\mathcal{O}_l^\infty$ to $\overline{U}_{\mathbf{F}_p}^\circ$. For each i, α , we develop $(K_i^l - 1)^\alpha$ and group together the even and odd powers of K_i . Using proof 2.1, $(K_i^l - 1)^\alpha \in K_i^l(2^\alpha + (v - 1)U_{\mathcal{B}_l}) - (2^\alpha + (v - 1)U_{\mathcal{B}_l})$ follows. Let f belong to $\mathcal{O}_l^\infty$. Its image $\bar{f}$ in $U_{\mathbf{F}_p}^*$ is such that $\bar{f}(2^\alpha(K_i^l - 1)x) = 0$, for any $x \in U_{\mathbf{F}_p}$. As 2 is invertible in $\mathbf{F}_p$, it follows: $\bar{f}(2^\alpha((K_i^l - 1)x) = 0$, for any $x \in U_{\mathbf{F}_p}$. In addition, as $K_i^{l+1} \in 1 + (v - 1)U_{\mathcal{B}_l}$, we also have $\bar{f}(K_i^{l+1}x) = \bar{f}(x)$ for any $x \in U_{\mathbf{F}_p}$. K_i^l being invertible in $U_{\mathcal{B}_l}$, it follows that $f(K_i x) = f(x)$ for any $x \in U_{\mathbf{F}_p}^*$.

2.4. The coordinate algebras for transcendental parameter. Recall that Lusztig has defined ([L1]) an $\mathcal{A}'$ -algebra $\mathbf{O} \subset \mathbf{U}^\circ$ as the subspace consisting of the forms on $\mathbf{U}$ that vanish on some two-sided finite codimensional ideal in $\mathbf{U}$ containing the products $\prod_{-N \leq k \leq N} (K_i - v^{d_i k})$ for all $1 \leq i \leq n$, for N big enough, and a $\mathcal{A}_0$ -Hopf subalgebra of it by $\mathcal{O} = \mathbf{O} \cap U_{\mathcal{A}_0}^*$ ($\mathcal{A}_0 = \mathbf{Q}[v, v^{-1}]$). He also proved that $\mathcal{O} \otimes_{\mathcal{A}_0} \mathcal{A}' = \mathbf{O}$.

By $\mathcal{A}$ -freeness of U , U^* can be seen as a subspace of $\mathbf{U}^*$. Let us then define $\mathcal{O}$ to be $U^* \cap \mathbf{O}$. $\mathcal{O}$ is clearly a divisible subalgebra in U^* .

Let us first show that:

Lemma. *The $\mathcal{A}'$ -algebra $\mathcal{O} \otimes_{\mathcal{A}} \mathcal{A}'$ is isomorphic to $\mathbf{O}$.*

Proof. By the final remark of 1.3.3, we need to see that if M is any $\mathcal{A}'$ -finite $\mathbf{U}$ -module where the eigenvalues of the K_i are contained in $v^{d_i \mathbf{Z}}$, it has a $\mathcal{A}$ -finite form. By [Ro], [L1], M is completely reducible. So let us assume M to be simple. It is then a highest weight module (*ibid.*). Let ξ be a highest weight vector. Then, $M = \mathbf{U}\xi$, and $\mathbf{U}^+\xi = \mathcal{A}'\xi$. Denote by N the U -submodule equal to $U\xi$. Using the Lusztig $\mathcal{A}$ -basis of U , we obtain: $N = \sum_{k_1, \dots, k_n \geq 0} \mathcal{A} F_{\alpha_1}^{(k_1)} \dots F_{\alpha_r}^{(k_r)} \xi$. (Here, $(\alpha_1, \dots, \alpha_r)$ is an admissible ordering of the positive roots; $r = \text{card } R^+$.) Indeed, the K_i^k 's and $\begin{bmatrix} K_i \\ t \end{bmatrix}$'s act on ξ by product by an element of $\mathcal{A}$.

Remark now that for the k_i 's big enough, $F_{\alpha_1}^{(k_1)} \dots F_{\alpha_r}^{(k_r)} \xi$ has to vanish. This is because M^* is also a highest weight representation of $\mathbf{U}$.

This proves that N is $\mathcal{A}$ -finitely generated. Clearly, $N \otimes_{\mathcal{A}} \mathcal{A}' = M$, and we are done. $\square$

(Note that the same argument gives the result of Lusztig $\mathcal{O} \otimes_{\mathcal{A}_0} \mathcal{A}' = \mathbf{O}$, and also that $\mathcal{O} \otimes_{\mathcal{A}} \mathcal{A}_0 = \mathcal{O}$).

Remarks. 1) The module N constructed in the proof above is $\mathcal{A}$ -free: indeed, the sum $N = \sum_{k_1, \dots, k_n \geq 0} \mathcal{A} F_{\alpha_1}^{(k_1)} \cdots F_{\alpha_r}^{(k_r)} \xi$ is direct since it is so in M . It is also $\mathcal{A}$ -finite dimensional. We will need this remark.

2) The base change $\mathcal{A} \rightarrow \mathcal{B}_l$ sends $\mathcal{O}$ into $\mathcal{O}_l$.

3) T. Levasseur has made the following observation: U^* has zero divisors. Indeed, the element η sending 1 to 1 and the other elements of the Lusztig basis to zero is such that $\eta^2 = \eta$, but $\eta \neq \varepsilon$. The same observation holds for $U_{\mathcal{B}_l}^*$. The earlier results on $\mathcal{O}_l$ are thus really needed in the proof above.

4) T. Levasseur and T. Stafford have proved the integrity for an arbitrary parameter over an algebraically closed field, in the special case of SL_n .

2.5. Integrity of the coordinate algebras for transcendental parameter. Let us prove first the integrity of $\mathcal{O}$.

Theorem. $\mathcal{O}$ is an integral domain.

Proof. Let us consider the commutative square of algebras, p being a prime number:

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & U_{\mathcal{B}_p}^* \\ \downarrow & & \downarrow \\ U_{\mathbf{Z}}^* & \longrightarrow & U_{\mathbf{F}_p}^* \end{array}$$

by $\mathcal{A}$ -freeness of U , the kernels of $U_{\mathbf{Z}}^* \rightarrow U_{\mathbf{F}_p}^*$ and of $U_{\mathcal{B}_p}^* \rightarrow U_{\mathbf{F}_p}^*$ are respectively $pU_{\mathbf{Z}}^*$ and $(v-1)U_{\mathcal{B}_p}^*$. Similarly, the kernels of $\mathcal{O} \rightarrow U_{\mathcal{B}_p}^*$ and $\mathcal{O} \rightarrow U_{\mathbf{Z}}^*$ are respectively $\mathcal{O} \cap \phi_p U^*$ and $\mathcal{O} \cap (v-1)U^*$, that is (as $\mathcal{O}$ is divisible in U^*) $\phi_p \mathcal{O}$ and $(v-1)\mathcal{O}$.

Note that $\cap_{n \geq 0} (v-1)^n \mathcal{O} \subset \cap_{n \geq 0} (v-1)^n U^* = \text{Hom}_{\mathcal{A}}(U, \cap_{n \geq 0} (v-1)^n \mathcal{A})$; as no non-zero polynomial has all its derivatives at 1 vanishing, $\cap_{n \geq 0} (v-1)^n \mathcal{O} = 0$.

Assume a and b are non-zero elements in $\mathcal{O}$, such that $ab = 0$. Then there exists integers $k, l \geq 0$ such that $a = (v-1)^k a'$, $b = (v-1)^l b'$, a' and b' in $\mathcal{O} \setminus (v-1)\mathcal{O}$. Furthermore, $\mathcal{O}$ is included in the torsion-free module $\text{Hom}(U, \mathcal{A})$ and is also torsion-free. Hence, $a'b' = 0$.

Remark that for p big enough, the images $a'_{\mathcal{B}_p}$ and $b'_{\mathcal{B}_p}$ of a' and b' in $U_{\mathcal{B}_p}^*$ are actually in $\mathcal{O}_p$: we know that $L(\prod_{- [p/2] \leq k \leq [p/2]} (K_i - v^{d_i k})) a' = 0$ provided $[p/2] > N_{a'}$, and if p is prime to the d_i 's (d_i is 1, 2, or 3) this relation gives (if $p \neq 2$) $L(K_i^p - 1) a'_{\mathcal{B}_p} = 0$. The same reasoning holds for the R -action, so for $p > \max(3, 2N_{a'} + 1)$, $a'_{\mathcal{B}_p} \in \mathcal{O}_p$. Let us now assume $p > \max(3, 2N_{a'} + 1, 2N_{b'} + 1)$; we have $a'_{\mathcal{B}_p}, b'_{\mathcal{B}_p} \in \mathcal{O}_p$.

Now the images $a'_{\mathbf{F}_p}, b'_{\mathbf{F}_p}$ of a', b' in $U_{\mathbf{F}_p}^*$ belong to $\overline{U_{\mathbf{F}_p}^*}$. As $\overline{U_{\mathbf{F}_p}^*}$ is a domain, $a'_{\mathbf{F}_p}$ or $b'_{\mathbf{F}_p}$ should vanish. More than that, one of the elements (a' or b') should be such that his reduction mod p vanishes for infinitely many primes p . Assume a' has this property, and call P this infinite set of primes. Denote by $a'_{\mathbf{Z}}$ its image in $U_{\mathbf{Z}}^*$; then $a'_{\mathbf{Z}} \in \cap_{p \in P} p U_{\mathbf{Z}}^* = \text{Hom}(U_{\mathbf{Z}}, p\mathbf{Z}) = \text{Hom}(U_{\mathbf{Z}}, \cap_{p \in P} p\mathbf{Z}) = 0$ since no non-zero integer is multiple of an infinite number of primes. Since $a'_{\mathbf{Z}} = 0$, $a' \in (v-1)U^*$; then, $a' \in \mathcal{O} \cap (v-1)U^* = (v-1)\mathcal{O}$ since $\mathcal{O}$ is divisible in U^* , contradiction. $\square$

We deduce from this:

Corollary. *The algebras O and $\mathbf{O}$ are also integral domains over their respective base rings $\mathbf{Q}[v, v^{-1}]$ and $\mathbf{Q}(v)$.*

Proof. For $\mathbf{O}$, it is again the fact that localizing in the base preserves the integrity.

Note that O , being included in $U_{\mathcal{A}_0}^*$, is $\mathcal{A}_0$ -torsion-free. Recall that after [L1, 7.3], $\mathbf{O} = O \otimes_{\mathcal{A}} \mathcal{A}'$. $\mathcal{A}'$ being the fraction field of $\mathcal{A}_0$, the morphism $O \rightarrow \mathbf{O}$, $x \mapsto x \otimes 1$ is an injection. Hence O is also a domain. $\square$

3. Integral closedness of the coordinate algebras at p^ν -th roots of 1. Let again K be a finite extension of $\mathcal{B}_l$, $l = p^\nu$ and $\mathfrak{p}$ be a prime ideal of its ring of integers $\mathfrak{o}_K$ above $(v-1)$. The quotient $\mathfrak{o}_K/\mathfrak{p}$ is isomorphic to a finite field $\mathbf{F}_q$ (q a power of p).

3.1. Hopf algebra structure on the mod p reduction of $\mathcal{O}_l \otimes \mathfrak{o}_K$. Let us begin with the following.

Lemma. *Let M be a finitely generated torsion-free $\mathcal{B}_l$ -module, $l = p^\nu$. Then the mapping $(M^*)_{\mathbf{F}_p} \rightarrow (M_{\mathbf{F}_p})^*$ is a $\mathbf{F}_p$ -vector spaces isomorphism.*

Proof. This mapping is written $M^*/(v-1)M^* \rightarrow \text{Hom}_{\mathcal{B}_l}(M/(v-1)M, \mathcal{B}_l/(v-1)\mathcal{B}_l)$ and so is certainly injective. Recall that because $\mathcal{B}_l$ is Dedekind, M is projective ([CE]). Let N be a $\mathcal{B}_l$ -module such that $M \oplus N = F$ is $\mathcal{B}_l$ -finite and free. The direct sum of the injections corresponding to M and N gives the injection $(M^* \oplus N^*)_{\mathbf{F}_p} \rightarrow (M_{\mathbf{F}_p} \oplus N_{\mathbf{F}_p})^*$, which identifies itself with the mapping $(F^*)_{\mathbf{F}_p} \rightarrow (F_{\mathbf{F}_p})^*$. As this last mapping is a $\mathbf{F}_p$ -vector space isomorphism, so have to be the other ones. $\square$

Recall that by 1.2, $\overline{U}_{\mathbf{F}_p}^\circ$ has a Hopf algebra structure, and that $\overline{U}_{\mathbf{F}_p}^\circ \otimes \mathbf{F}_q$ is identified with a Hopf subalgebra of $\overline{U}_{\mathbf{F}_q}^\circ$.

We can now prove:

Proposition. *For any $l = p^\nu$, the algebra $(\mathcal{O}_l \otimes \mathfrak{o}_K) \otimes \mathbf{F}_q$ is a Hopf subalgebra of $\overline{U}_{\mathbf{F}_p}^\circ$.*

Proof. First remark that $(\mathcal{O}_l \otimes \mathfrak{o}_K) \otimes \mathbf{F}_q$ is isomorphic to $(\mathcal{O}_l \otimes \mathbf{F}_p) \otimes \mathbf{F}_q$, so the result deserves proof only for $K = \mathcal{B}_l$. Let f belong to $\mathcal{O}_l$. By lemma 2.2.1, as an element of $U_{\mathcal{B}_l}^* \otimes \mathcal{B}_l$, f is in $\text{Im } \phi_\rho$, for (ρ, V) some $\mathcal{B}_l$ -finite and torsion-free $U_{\mathcal{B}_l}$ -module. Let $\mu \otimes v \in V^* \otimes V$ be such that $f = \phi_\rho(\mu \otimes v)$. The mod p reduction of f has coproduct $\Delta(f_{\mathbf{F}_p}) = \sum_i \phi_{\rho_{\mathbf{F}_p}}(\mu_{\mathbf{F}_p} \otimes e_i) \otimes \phi_{\rho_{\mathbf{F}_p}}(e^i \otimes v_{\mathbf{F}_p})$ where (e_i) is a basis for $V_{\mathbf{F}_p}$ and (e^i) is its dual basis. Both $\mu_{\mathbf{F}_p} \otimes e_i$ and $e^i \otimes v_{\mathbf{F}_p}$ belong to $(V^* \otimes V)_{\mathbf{F}_p}$: for the first one it is clear, and for the second one it is a consequence of the above lemma.

Let us check that $\mathcal{O}_l$ is ${}^t(S \otimes 1)$ -stable: in addition to the arguments of lemma 2.2, we see that if the actions of the K_i^l 's on some $U_{\mathcal{B}_l}$ -module is by 1 so will they be on the contragredient module.

Hence, $(\mathcal{O}_l)_{\mathbf{F}_p}$ will also be ${}^tS$ -stable. $\square$

3.2. Integral closedness results. Let us recall that according to [dCKP], a domain Ω with center Z over a base ring R is said to be integrally closed iff any algebra X such that $\Omega \subset X \subset \frac{1}{z}\Omega$ ($\subset Q(Z) \otimes_Z \Omega$), for some $z \neq 0$ in Z is equal to Ω . In that case, Z is integrally closed as well, and so is the center of $\Omega_R = \Omega \otimes_R Q(R)$, since it is equal to $Z \otimes_R Q(R)$. (This notion coincides with the usual one in case Ω is commutative and noetherian.)

Let us recall that $\otimes_{\mathbf{F}_q} 1$ maps $\mathcal{O}_l \otimes \mathbf{o}_K$ into $\overline{U}_{\mathbf{F}_q}^\circ$. Denote by Ω the image of this morphism. By proposition 3.1, Ω is a Hopf subalgebra of $\overline{U}_{\mathbf{F}_q}^\circ$; it is equal to the union of $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes \mathbf{F}_q$, ρ' running over some set of finite dimensional representations of $\overline{U}_{\mathbf{F}_p}$.

Let k be an algebraic closure of $\mathbf{F}_p$. Then $\Omega \otimes k$ is the union of all $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes k$, ρ' as above, and is included in $\overline{U}_{\mathbf{F}_p}^* \otimes k$ (field extensions are exact). We first prove:

Lemma. *If ρ' is a $\mathbf{F}_p$ -finite $\overline{U}_{\mathbf{F}_p}$ -module, $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes k$ is integrally closed.*

Proof. $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes k$ is a finitely generated commutative Hopf algebra, and is a domain: the last result in [K] implies that $\overline{U}_{\mathbf{F}_p}^*$ is isomorphic to $\mathbf{F}_p[[\gamma_1, \dots, \gamma_n]]$; then $\overline{U}_{\mathbf{F}_p} \otimes k$ is isomorphic to a subalgebra of $k[[\gamma_1, \dots, \gamma_n]]$. Hence, its spectrum Γ is an affine variety; as the base field is algebraically closed, we can apply [Di, p.128]: the set of normal points of Γ contains an open set in Γ ; remark also that Γ is a group, and that as this set has to be Γ -invariant, every point is normal; hence Γ is normal and $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes k$ is integrally closed. $\square$

We can now state:

Lemma. *$\Omega \otimes k$ is integrally closed.*

Proof. Let $f \in Q(W \otimes k)$ be a solution of a monic equation $f^n + \sum a_i f^i = 0$, $a_i \in \Omega \otimes k$ (*). Let us express f as g/h , ρ' , ρ'' , ρ_i be $\overline{U}_{\mathbf{F}_p}$ -modules respectively attached to g , h , a_i . Let $\bar{\rho}$ be the $\overline{U}_{\mathbf{F}_p}$ -module equal to $(\sum \rho'^i \otimes \rho''^{n-i}) \otimes (\varepsilon + \sum \rho_i)$. (*) is written in $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes k$, $g^n + \sum a_i g^i h^{n-i} = 0$; by lemma above, $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \rho') \otimes k$ is integrally closed, so f belongs to it. Moreover, the class of the mod p reductions of $\rho \otimes 1$'s, $\rho \in \hat{U}_{\mathcal{B}_l} \otimes \mathcal{B}_l'$ is closed under direct sums and tensor products, and contains the trivial modules, so $\bar{\rho}$ is in this class. Finally, $f \in \Omega \otimes k$. $\square$

This implies that Ω itself is integrally closed: let $f \in Q(\Omega)$ be a solution of a monic equation $f^n + \sum a_i f^i = 0$, $a_i \in \Omega$ (**). Write $f = g/h$, $g, h \neq 0$ belonging to Ω . Let us write it in $\Omega \otimes k$: we get $g^n + \sum a_i g^i h^{n-i} = 0$, and this implies that f belongs to $\Omega \otimes k$. Let us write $f = f^0 \otimes 1 + \sum f^i \otimes k_i$, with $f^i \in \Omega$ and $(1, k_i)$ a $\mathbf{F}_q$ -free and finite family in k . We deduce $g = \sum h f^i \otimes k_i$; so, $h f^0 = g$, and the other $h f^i$ vanish. By integrity, this implies that the f^i 's vanish, and that $f = f^0$; so f belongs to Ω .

Let us recall that $\mathcal{O}_l$ verifies $\cap_{n \geq 0} (v-1)^n \mathcal{O}_l = 0$, and if $v : \mathcal{O}_l \rightarrow \mathbf{N}$ is defined by $v(x) = \sup \{n \in \mathbf{N}, x \in (v-1)^n \mathcal{O}_l\}$ if $x \neq 0$, $v(0) = +\infty$, then $|x| := (\frac{1}{l})^{v(x)}$ is an algebra norm on $\mathcal{O}_l$. Let $\hat{\mathcal{O}}_l$ be the completion of $\mathcal{O}_l$ for this norm. It is an algebra over the p -adic ring $\hat{\mathcal{B}}_l$ (the $(v-1)$ -adic completion of $\mathcal{B}_l$); $\mathcal{O}'$ injects into $\hat{\mathcal{O}}'$.

Let K' be a finite extension of $\hat{\mathcal{B}}_l$ (the fraction field of $\hat{\mathcal{B}}_l$) and $\mathbf{o}'$ be its ring of integers. Let π be a uniformizing parameter for K' ; then $\mathbf{o}'/(\pi)$ is isomorphic to a finite field $\mathbf{F}_q$.

We pose $\hat{\mathcal{O}}' = \mathcal{O}_l \otimes \mathbf{o}'$.

As a topological algebra, $\hat{\mathcal{O}}'$ is identified with the completion of the tensor product $\mathcal{O}_l \otimes \mathbf{o}'$, given the valuation $v'(f) = \sup \{n \in \mathbf{N}, \lambda \in \pi^n \mathcal{O}_l \otimes \mathbf{o}'\}$ if $f \neq 0$, $v(0) = +\infty$, and the norm $\|f\| := (\frac{1}{l})^{rv'(f)}$, r being the ramification index of the extension $\hat{\mathcal{B}}_l \rightarrow K'$.

Lemma. $\hat{\mathcal{O}}'$ is an integral domain.

Proof. To repeat the argument in Theorem 2.3, with π instead of $v' - 1$, we have to check the following facts:

a) $\cap_{n \geq 0} \pi^n \hat{\mathcal{O}}' = 0$: a Cauchy sequence (x_n) representing an element in this intersection can be put in the form $\pi^N(y_n) + (z_n)$, (y_n) Cauchy, $v(z_n) \rightarrow +\infty$, for any fixed N ; hence $v(x_n) \rightarrow \infty$;

b) the $\hat{\mathcal{O}}'/\pi\hat{\mathcal{O}}'$ is an integral domain; we know that $\hat{\mathcal{O}}'/\pi\hat{\mathcal{O}}' = (\hat{\mathcal{O}}_l \otimes \mathbf{o}') \otimes \mathbf{F}_q = \hat{\mathcal{O}}_l \otimes \mathbf{F}_q$. Let us show that the canonical algebra morphism $\mathcal{O}_l/(v-1)\mathcal{O}_l \rightarrow \hat{\mathcal{O}}_l/(v-1)\hat{\mathcal{O}}_l$ is bijective:

α) injectivity: if $x \in \mathcal{O}_l$, $(x) = (v-1)(y_n) + (z_n)$, (y_n) Cauchy, $v(z_n) \rightarrow +\infty$, then for n large enough, $v(z_n) \geq 1$, and for such n the equality gives $v(x) \geq 1$;

β) surjectivity: if (x_n) is Cauchy, for $n \geq n_0$, we have $v(x_n - x_{n_0}) \geq 1$ and the sequence $(y_n) := \left(\frac{x_n - x_{n_0}}{v-1}\right)$ is also Cauchy; then $(x_n) = (x) + (v-1)(y_n)$.

We can thus consider $\hat{\mathcal{O}}_l \otimes \mathbf{F}_q$ as a subalgebra of $\overline{U}_{\mathbf{F}_p}^\circ \otimes \mathbf{F}_q$, and this last algebra is in turn included in $\overline{U}_{\mathbf{F}_q}^*$, which is integral (it is a formal series algebra). $\square$

Remark. The subalgebra of $\text{Hom}(U_{\mathcal{B}_l}, \hat{\mathcal{B}}_l, \hat{\mathcal{B}}_{l'})$ consisting of the K_i^l -invariant forms vanishing on a finite-codimensional ideal is identified with $\mathcal{O}' \otimes \hat{\mathcal{B}}_{l'}$. As $\hat{\mathcal{B}}_l$ is a principal noetherian domain, 1.2 shows that this algebra has a Hopf algebra structure; moreover, it is dense in $\hat{\mathcal{O}}'$. This endows $\hat{\mathcal{O}}'$ with a structure of topological Hopf algebra: $\hat{\mathcal{O}}' \hat{\otimes}_{\mathcal{B}_{l'}} \hat{\mathcal{O}}'$ is the completion of $\mathcal{O}' \otimes_{\mathcal{B}_{l'}} \mathcal{O}'$ for the norm given by the embedding in $\text{Hom}(U_{\mathcal{B}_l}, \mathcal{B}_{l'}) \otimes_{\mathcal{B}_{l'}} \text{Hom}(U_{\mathcal{B}_l}, \mathcal{B}_{l'})$. $\Delta_{\mathcal{O}' \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}}$ is continuous for this norm. Together with b) in the proof above, this gives another proof of proposition 3.1.

We are now ready to prove:

Theorem. The algebra $\hat{\mathcal{O}}'$ is integrally closed.

Proof. Recall that $\otimes_{\mathbf{F}_q} 1$ maps $\hat{\mathcal{O}}'$ onto Ω , with kernel $\pi\hat{\mathcal{O}}'$. Let Z denote the center of $\hat{\mathcal{O}}'$. It is a domain after the lemma above. Let X be a subalgebra of $\hat{\mathcal{O}}' \otimes_Z Q(Z)$ and $z \in Z$ be a non-zero element such that the inclusion $\hat{\mathcal{O}}' \subset X \subset \left(\frac{1}{z}\right)\hat{\mathcal{O}}'$ holds in $\hat{\mathcal{O}}' \otimes_Z Q(Z)$.

As $\cap_{n \geq 0} \pi^n \hat{\mathcal{O}}' = 0$, we can assume the mod p reduction $z_{\mathbf{F}_p}$ of z to be non-zero (write $z = \pi^n z'$, with $z'_{\mathbf{F}_p} \neq 0$; then $\hat{\mathcal{O}}' \subset X \subset \left(\frac{1}{z'}\right)\hat{\mathcal{O}}'$ holds). In that case, $\hat{\mathcal{O}}'$ is π -divisible in $\left(\frac{1}{z}\right)\hat{\mathcal{O}}'$ (and hence also in X), i.e. $(x \in \hat{\mathcal{O}}')$ and $\left(\frac{1}{\pi}x \in \left(\frac{1}{z}\right)\hat{\mathcal{O}}'\right) \Rightarrow \left(\frac{1}{\pi}x \in \frac{1}{\pi}\hat{\mathcal{O}}'\right)$.

We can then assume X to be π -divisible in $\left(\frac{1}{z}\right)\hat{\mathcal{O}}'$, i.e. $(x \in X)$ and $\left(\frac{1}{\pi}x \in \left(\frac{1}{z}\right)\hat{\mathcal{O}}'\right) \Rightarrow \left(\frac{1}{\pi}x \in X\right)$: indeed, $X' = \{x \in \left(\frac{1}{z}\right)\hat{\mathcal{O}}' / \exists k \geq 0, (\pi)^k x \in X\}$ is also a subalgebra of $\left(\frac{1}{z}\right)\hat{\mathcal{O}}'$: first, if a, b belong to X' , $\pi^{k+k'}ab$ will belong to X ; let us show that $ab \in \left(\frac{1}{z}\right)\hat{\mathcal{O}}'$; let m be the smallest ≥ 0 integer such that $\alpha := \pi^m zab \in \hat{\mathcal{O}}'$. The mod p reduction $\alpha_{\mathbf{F}_q}$ will be non-zero. Since $z^2ab \in \hat{\mathcal{O}}'$, $z\alpha \in \pi^m \hat{\mathcal{O}}'$ which contradicts the integrality of Ω if $m \neq 0$.

Let us now consider the mapping $\iota : \left(\frac{1}{z}\right)\hat{\mathcal{O}}' \rightarrow Q(\Omega)$, $\frac{a}{z} \mapsto \frac{a_{\mathbf{F}_q}}{z_{\mathbf{F}_q}}$; its kernel is equal to $\left(\frac{\pi}{z}\right)\hat{\mathcal{O}}'$; hence it can be identified with $\otimes_{\mathbf{F}_q} 1$. By the divisibility properties shown above, for the sequence of inclusions $\hat{\mathcal{O}}' \subset X \subset \left(\frac{1}{z}\right)\hat{\mathcal{O}}'$, $\otimes_{\mathbf{F}_p} 1$ is left-exact and we get

a sequence of inclusions $\Omega \rightarrow X \otimes \mathbf{F}_q \rightarrow \left(\frac{1}{z\mathbf{F}_q}\right) \Omega$. By the lemma above, this implies that $\iota(X) = \Omega$, so that $X \subset \hat{\mathcal{O}}' + \frac{\pi}{z} \hat{\mathcal{O}}'$. Let x be any element of X , and let us decompose it $x = a + \frac{\pi}{z} b$, $a, b \in \hat{\mathcal{O}}'$. a belongs to X and by the π -divisibility of X in $\left(\frac{1}{z}\right) \hat{\mathcal{O}}'$, $\left(\frac{1}{z}\right) b$ belongs to X . The reasoning above applies to $\left(\frac{1}{z}\right) b$, and by induction we construct a sequence $a_n, b_n \in \hat{\mathcal{O}}'$ with $x = a_n + \frac{\pi^n}{z} b_n$. Hence $zX \subset \hat{\mathcal{O}}'$ is included in the closure of $z\mathcal{O}'$. To end the proof, we have now to prove the following

Lemma. *For each element z in $\hat{\mathcal{O}}'$, the ideal $z\hat{\mathcal{O}}'$ is closed in $\hat{\mathcal{O}}'$.*

Proof. Assume z to be non-zero. Let $x \in \hat{\mathcal{O}}'$ be the limit of a sequence (zf_n) of elements of $z\hat{\mathcal{O}}'$. The equality $v(ab) = v(a) + v(b)$ holds for a, b in $\hat{\mathcal{O}}'$ (by integrity of $\hat{\mathcal{O}}'/\pi\hat{\mathcal{O}}'$). The equalities $|z(f_n - f_m)| = |z| |f_n - f_m|$ then imply ($z \neq 0$) that (f_n) is Cauchy. By continuity of the product: $x = z \lim f_n$. $\square$

Now we have the double inclusion $z\hat{\mathcal{O}}' \subset zX$ (by hypothesis) $\subset z\hat{\mathcal{O}}'$ (by the lemma above); the equality $X = \hat{\mathcal{O}}'$ follows. $\square$

Denote by $\hat{B}_{l'}$ the completion of $B_{l'}$ for the norm deduced from $||$ on $B_{l'}$. Then $\hat{B}_{l'}$ is the localization of $\hat{B}_{l'}$ in $(v' - 1)$. Let $\mathcal{O}'$ stand for $\mathcal{O}_l \otimes B_{l'}$. It has a natural norm extending that of $\mathcal{O}'$; let $\hat{\mathcal{O}}'$ denote its completion for it. It is obtained from its closed subalgebra $\hat{\mathcal{O}}'$ by localizing in $(v' - 1)$.

Corollary.

- i) *The center $Z(\hat{\mathcal{O}}')$ of $\hat{\mathcal{O}}'$ is integrally closed;*
- ii) *if $Z(\hat{\mathcal{O}}_l)$ denotes the center of $\hat{\mathcal{O}}_l$, the tensor product $Z(\hat{\mathcal{O}}_l) \otimes K'$ is integrally closed.*

Proof. i) is a consequence of the theorem above and of the remark of [dCKP] on the relation between integral closedness of algebras and their centers.

For commutative algebras, localizing in the base preserves the integral closedness, hence ii). $\square$

Remark. It is not clear if the center of $\mathcal{O}_l$ itself is integrally closed; we will see in the next section that the most precise results concern the algebras $\hat{\mathcal{O}}_l$ and $\hat{\mathcal{O}}_l$. They are topological algebras over their respective base rings and fields $\hat{B}_{l'}$ and $\hat{B}_{l'}$ ($l = p^{\nu'}$), which are cyclotomic extensions of the p -adic ring and field $\mathbf{Z}_p$ and $\mathbf{Q}_p$. Note that the inclusions $\mathcal{O}_l \otimes_{B_l} \hat{B}_l \subset \hat{\mathcal{O}}_l$ and $\mathcal{O}_l \otimes_{B_l} \hat{B}_l \subset \hat{\mathcal{O}}_l$ are proper.

4. Finiteness of the coordinate algebras over their centers.

4.1. Density of a finite type algebra in $\hat{\mathcal{O}}_l$. Let again p be arbitrary. We first prove:

Proposition. *Let us assume the Cartan matrix A has no components of type B_k, C_k, F_4 if $p = 2$, and no component of type G_2 if $p = 3$.*

- i) *Let k be an algebraic closure of $\mathbf{F}_p$. The injection of $\overline{U}_Z^\circ \otimes k$ into $(\overline{U}_k)^\circ$ is an isomorphism.*
- ii) *The injection of $\overline{U}_Z^\circ \otimes \mathbf{F}_p$ into $(\overline{U}_{\mathbf{F}_p})^\circ$ is an isomorphism.*

Proof. i) The injection of $\overline{U}_Z^* \otimes k$ into $(\overline{U}_k)^*$ induces an injection of $\overline{U}_Z^\circ \otimes k$ into $(\overline{U}_k)^\circ$. Let π be a $\mathbf{Z}$ -finite $\overline{U}_Z$ -module such that $\overline{U}_Z^\circ = \text{Coeff}(\overline{U}_Z, \pi)$ (see [K]). The

image of $\overline{U}_k^\circ \otimes k$ in $(\overline{U}_k)^\circ$ is then equal to $\text{Coeff}(\overline{U}_k, \pi_k)$ (by finite generation of these algebras). Assume $(\overline{U}_k)^\circ$ and $\text{Coeff}(\overline{U}_k, \pi_k)$ are different and let $f \in (\overline{U}_k)^\circ$ belong to the complement of $\text{Coeff}(\overline{U}_k, \pi_k)$. Then f is a coefficient of some k -finite $\overline{U}_k$ -module, π' , and set $\varpi = \pi_k \oplus \pi'$. We then have $\text{Coeff}(\overline{U}_k, \pi_k) \subset \text{Coeff}(\overline{U}_k, \varpi) \subset (\overline{U}_k)^\circ$, the first inclusion being strict.

Let G_π, G_ϖ denote the respective spectra of the first two algebras: $\text{Coeff}(\overline{U}_k, \pi_k) = k[G_\pi]$, $\text{Coeff}(\overline{U}_k, \varpi) = k[G_\varpi]$. G_π and G_ϖ are algebraic k -groups; they are connected since the algebras $k[G_\pi]$ and $k[G_\varpi]$ are included in $(\overline{U}_k)^*$; this last ring is equal to $k[[\gamma'_1, \dots, \gamma'_k]]$ repeating the argument of [K, prop. 3], γ'_i standing for the image of $\gamma_i \in \overline{U}_k^*$ in $(\overline{U}_k)^*$ by $\otimes_{\mathbf{Z}} 1$, and thus has no zero divisors.

Let I_π, I_ϖ denote the augmentation ideals of $k[G_\pi], k[G_\varpi]$; and let $k[G_\pi]^\wedge, k[G_\varpi]^\wedge$ denote their I_π (resp. I_ϖ)-adic completions. The inclusions $I_\pi \subset I_\varpi$ induces a continuous mapping $i : k[G_\pi]^\wedge \rightarrow k[G_\varpi]^\wedge$. Let us prove that i is injective: if (f_n) is Cauchy in $k[G_\pi]^\wedge$ and $f_n \in I_\varpi^N$ for any N provided n is large enough, we deduce $f_n \in I^N$ for the same n , I being the augmentation ideal of $(\overline{U}_k)^*$. $T \in \overline{U}_k$ being fixed, the duality between T and f_n will give zero as soon as N is greater than the order of T . It follows that $\lim f_n = f \in k[G_\pi]^\wedge$ has image zero in $(\overline{U}_k)^*$. On the other hand, it is known that $k[G_\pi]^\wedge = \text{Dist}(G_\pi)^*$; and by [Ja, (3), p. 194], $\text{Dist}(G_\pi) = \overline{U}_k$; hence $f = 0$ and the injectivity.

Let us now consider the sequence of inclusions $k[G_\varpi] \subset (\overline{U}_k)^* \subset \text{Dist}(G_\varpi)^*$. The inclusion $I_\varpi \subset I$ gives rise to the continuous mapping $j : k[G_\pi]^\wedge \rightarrow (\overline{U}_k)^*$. The same argument than above gives the injectivity of this mapping.

The composition $j \circ i$ is the identity; i and j being injective, they are k -vector space isomorphisms. Hence, so are the dual mappings i^* and j^* . In addition, $i^* : \text{Dist}(G_\varpi) \rightarrow \text{Dist}(G_\pi)$ is a Hopf algebra morphism. Hence, i^* induces a Hopf algebra isomorphism between $\text{Dist}(G_\varpi)$ and $\text{Dist}(G_\pi) = \overline{U}_k$.

By [Di, cor. 1, p. 27], the image of the morphism $f : G_\varpi \rightarrow G_\pi$ is dense in G_π . Let N be a connected abelian normal subgroup of G_ϖ . The Zariski closure of its image $\overline{f(N)}$ in G_ϖ is then an abelian, normal (by Zariski continuity of the adjoint action) and connected subgroup of G_π . G_π being semi-simple, $\overline{f(N)} = \{e\} \in G_\varpi$. Hence $N \subset f^{-1}(e) \in G_\pi$.

Note that $f^{-1}(e)$ injects in the maximal spectrum of the algebra $k[G_\varpi]/(I_\pi)$; this algebra is the algebra of functions on the fibre $(G_\varpi)_e$. As $f : G_\varpi \rightarrow G_\pi$ is a group morphism, the fibres $(G_\varpi)_x$ are all isomorphic for $x \in f(G_\varpi)$. By [Ht, 3.22 (e), p. 95] (following [CC]), for x in an open dense subset C_e of G_π the fibre $(G_\varpi)_x$ is zero-dimensional (the isomorphism of distribution algebras shows that G_π and G_ϖ have the same dimension since on any algebraic group, the completion of the local ring at identity is the dual of the distribution algebra). Choosing x in $C_e \cap f(G_\varpi)$, we get that the fibre $(G_\varpi)_e$ is zero-dimensional; hence, $k[G_\varpi]/(I_\pi)$ is finite over k , and $f^{-1}(e)$ is finite. The connectedness assumption on N then implies that $N = \{e\}$.

It follows that G_ϖ is semi-simple. Moreover, the image of $f : G_\varpi \rightarrow G_\pi$ is a constructible subset of G_π ([Hu2, p. 33]), and is closed (as any constructible subgroup, [Hu2, prop. A, p. 54]). We already noted that this image is dense, hence f is surjective. We also remarked that $\text{Ker } f$ is finite, hence f is an isogeny of semi-simple groups.

We can apply Chevalley's classification of such isogenies [C, exp. 22-23]: by assumption on A , all isogenies of semi-simple groups with Cartan matrix A are central; G_π being a connected, simply connected and semi-simple group over k , f is an isomorphism. Hence, the inclusion of Hopf algebras $k[G_\pi] \subset k[G_\varpi]$ is an isomorphism; it follows that $(\overline{U}_k)^\circ = \overline{U}_Z^\circ \otimes_Z k$.

ii) Assume the injection of $\overline{U}_Z^\circ \otimes_Z \mathbf{F}_p$ into $(\overline{U}_{\mathbf{F}_p})^\circ$ is not surjective; then $\otimes_{\mathbf{F}_p} k$ transforms it into a non-surjective injection of $\overline{U}_Z^\circ \otimes_Z k$ into $(\overline{U}_{\mathbf{F}_p})^\circ \otimes k$. This last space is identified with a subspace of $(\overline{U}_k)^\circ$. The composition of these two injections is the injection studied in i). By i), we know that it is a bijection, hence $\overline{U}_Z^\circ \otimes_Z \mathbf{F}_p \hookrightarrow (\overline{U}_{\mathbf{F}_p})^\circ$ has to be surjective. $\square$

We denote by π a $\mathbf{Z}$ -finite $\overline{U}_Z$ -module such that $\text{Coeff}(\overline{U}_Z, \pi) = \overline{U}_Z^\circ$. By the Rosso-Lusztig theory, and remark 2.4.1, there exists an $\mathcal{A}$ -finite and free U -module $\tilde{\pi}$ "quantizing" π in the following sense: consider the base change $\mathcal{A} \rightarrow \mathbf{Z}$, $v \mapsto 1$. The K_i 's act by identity on the U_Z -module $\tilde{\pi} \otimes_{\mathbf{Z}[v, v^{-1}]} \mathbf{Z}$ (the morphism $\mathbf{Z}[v, v^{-1}] \rightarrow \mathbf{Z}$ is defined by $v \mapsto 1$); the $\overline{U}_Z$ -module thus obtained is isomorphic to π .

The proposition above allows to prove:

Proposition. *Assume that the Cartan matrix A satisfies the hypothesis in the proposition above.*

Then the $\hat{\mathcal{B}}_l$ -algebra $\hat{\mathcal{O}}_l$ is isomorphic to the $|\cdot|$ -completion in $U_{\mathcal{B}_l}^*$ of $\text{Coeff}(U_{\mathcal{B}_l}, \tilde{\pi}_{\mathcal{B}_l})$.

Proof. Consider the mapping $\otimes_{\mathcal{B}_l} 1 : \mathcal{O}_l \rightarrow \overline{U}_{\mathbf{F}_p}^\circ$; we first prove that it sends $\text{Coeff}(U_{\mathcal{B}_l}, \tilde{\pi}_{\mathcal{B}_l})$ onto $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \pi_{\mathbf{F}_p})$ (which is the whole $\overline{U}_{\mathbf{F}_p}^\circ$ by the above proposition). By the properties of $\tilde{\pi}$, the $U_{\mathbf{F}_p}$ -module $(\tilde{\pi}_{\mathcal{B}_l})_{\mathbf{F}_p}$ descends to a $\overline{U}_{\mathbf{F}_p}$ -module isomorphic to $\pi_{\mathbf{F}_p}$. $\tilde{\pi}_{\mathcal{B}_l}$ is a free $\mathcal{B}_l$ -module, so the $\mathbf{F}_p$ -vector spaces $(\tilde{\pi}_{\mathcal{B}_l}^*)_{\mathbf{F}_p}$ and $(\pi^*)_{\mathbf{F}_p}$ are isomorphic; they are as well isomorphic as $\overline{U}_{\mathbf{F}_p}$ -modules. Finally, $\otimes_{\mathcal{B}_l} 1$ sends $\tilde{\pi}_{\mathcal{B}_l} \otimes_{\mathcal{B}_l} (\tilde{\pi}_{\mathcal{B}_l})^*$ and $(\tilde{\pi}_{\mathcal{B}_l})^* \otimes_{\mathcal{B}_l} \tilde{\pi}_{\mathcal{B}_l}$ onto $\pi_{\mathbf{F}_p} \otimes_{\mathbf{F}_p} (\pi_{\mathbf{F}_p})^*$ and $(\pi_{\mathbf{F}_p})^* \otimes_{\mathbf{F}_p} \pi_{\mathbf{F}_p}$. This proves the desired surjectivity.

Let now f belong to $\mathcal{O}_l$. Let g be a preimage of $(\otimes_{\mathcal{B}_l} 1)(f)$ in $\text{Coeff}(U_{\mathcal{B}_l}, \tilde{\pi}_{\mathcal{B}_l})$ by $\otimes_{\mathcal{B}_l} 1$. The difference $f - g$ belongs to the kernel of $\otimes_{\mathcal{B}_l} 1 : \mathcal{O}_l \rightarrow \overline{U}_{\mathbf{F}_p}^\circ$, that is $(v - 1)\mathcal{O}_l$. We repeat this construction with $\frac{1}{v-1}(f - g)$ instead of f . After n iterations, we obtain $f \in \text{Coeff}(U_{\mathcal{B}_l}, \tilde{\pi}_{\mathcal{B}_l}) + (v - 1)^n \mathcal{O}_l$. This proves the proposition. $\square$

Remarks. 1) The second proposition allows to show: $\mathcal{O}_l \otimes \mathbf{F}_p = \overline{U}_Z^\circ \otimes \mathbf{F}_p$. Indeed, we remarked that $\mathcal{O}_l \otimes \mathbf{F}_p = \hat{\mathcal{O}}_l \otimes \mathbf{F}_p$; and the density result shows that $\hat{\mathcal{O}}_l \otimes \mathbf{F}_p = \text{Coeff}(U_{\mathcal{B}_l}, \tilde{\pi}_{\mathcal{B}_l}) \otimes \mathbf{F}_p$. This last ring has been proved to be equal to $\text{Coeff}(\overline{U}_{\mathbf{F}_p}, \pi_{\mathbf{F}_p})$, which is easily proved to be equal to $\text{Coeff}(\overline{U}_Z, \pi) \otimes \mathbf{F}_p$.

2) The p -adic completions of $\mathcal{O}_l^\infty$ and $\mathcal{O}_l$ are equal. Indeed, $\mathcal{O}_l$ is a divisible subalgebra of $\mathcal{O}_l^\infty$, the resulting injection $\mathcal{O}_l \otimes \mathbf{F}_p \rightarrow \mathcal{O}_l^\infty \otimes \mathbf{F}_p$ is actually an isomorphism (by the remark above and remark 2.3.2).

4.2. The Frobenius mapping for the coordinate algebras. Let l be a positive integer, prime to the entries of the Cartan matrix. Recall that Lusztig has defined a Hopf algebra morphism $F : U_{\mathcal{B}_l} \rightarrow \overline{U}_Z \otimes \mathcal{B}_l$, by the formulas $F(E_i^{(N)}) = e_i^{(N/l)}$, $F(F_i^{(N)}) = f_i^{(N/l)}$,

$F(K_i) = 1$, with the convention that $e^{(r)} = 0$ whenever $r \in \mathbf{Q} - \mathbf{Z}$. We remark that F is surjective.

Hence, the dual mapping $F^* : \overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l \rightarrow U_{\mathcal{B}_l}^*$ is an injective morphism of $\mathcal{B}_l$ -algebras.

Lemma. $F^*(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \subset \mathcal{O}_l$.

Proof. Let f belong to $\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l$ and I be a two-sided finite codimensional ideal of $\overline{U}_{\mathbf{Z}}$ on which f vanishes.

The inverse image $F^{-1}(I \otimes \mathcal{B}_l)$ is a two-sided ideal of $U_{\mathcal{B}_l}$; if $I + \mathbf{Z}_{e_1} + \cdots + \mathbf{Z}_{e_n} = \overline{U}_{\mathbf{Z}}$, then denoting by $e'_1, \dots, e'_n$ the preimages of $e_1, \dots, e_n$ by F , we have $F^{-1}(I \otimes \mathcal{B}_l) + \mathcal{B}_l e'_1 + \cdots + \mathcal{B}_l e'_n = U_{\mathcal{B}_l}$, hence $F^{-1}(I)$ is finite codimensional in $U_{\mathcal{B}_l}$. Finally, $F(K_i^l - 1) = 0$, so the $K_i^l - 1$'s belong to $F^{-1}(I)$.

Recall that $F^*(f) = f \circ F$, so that $F^*(f)$ vanishes on $F^{-1}(I)$. $\square$

Let us identify $l\mathbf{Z}^n$ with its image in $U_{\mathcal{B}_l}^*$, by $(k_1, \dots, k_n) \mapsto K_i^{k_1} \cdots K_n^{k_n}$ and $U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$ denote the subalgebra of $U_{\mathcal{B}_l}^*$ composed of the elements left (or right) invariant by $l\mathbf{Z}^n$. Remark that $F^*(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \subset U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$, since $\forall f \in \overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l, x \in U_{\mathcal{B}_l}, 1 \leq i \leq n$, $F^*(f)(K_i^l x) = f(F(K_i^l)F(x)) = f(F(x))$ (since $F(K_i) = 1$) $= F^*(f)(x)$.

We then prove:

Proposition. $F^*(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l)$ is central in $U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$.

Proof. For $f \in F^*(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l)$, $g \in U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$, $x \in U_{\mathcal{B}_l}$, we have $\langle F^*(f)g, x \rangle = \langle f \otimes g, (F \otimes 1)\Delta x \rangle$, $\langle gF^*(f), x \rangle = \langle g \otimes f, (1 \otimes F)\Delta x \rangle$; let $\sigma : U_{\mathcal{B}_l, l\mathbf{Z}^n}^* \otimes \mathcal{B}_l \rightarrow (\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \otimes_{\mathcal{B}_l} U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$ denote the operator exchanging the factors. We claim that $\langle f \otimes g, (F \otimes 1)\Delta x - \sigma(1 \otimes F)\Delta x \rangle = 0$.

This follows from

Lemma. For any x in $U_{\mathcal{B}_l}$, $(F \otimes 1)\Delta x - \sigma(1 \otimes F)\Delta x \in (\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \otimes_{\mathcal{B}_l} (K_i^l - 1)$, if $(K_i^l - 1)$ is the ideal generated by the $K_i^l - 1$'s in $U_{\mathcal{B}_l}$.

Proof. Since both $(F \otimes 1)\Delta$ and $\sigma(1 \otimes F)\Delta$ are algebra morphisms from $U_{\mathcal{B}_l}$ to $(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \otimes_{\mathcal{B}_l} U_{\mathcal{B}_l}$, and $(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \otimes (K_i^l - 1)$ is a two-sided ideal in $(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l) \otimes_{\mathcal{B}_l} U_{\mathcal{B}_l}$, it is enough to check it on generators of $U_{\mathcal{B}_l}$; choose $E_i^{(N)}, F_i^{(N)}, K_i$. It is clear for K_i since it is group-like. Recall that $\Delta(E_i^{(N)}) = \sum_{b=0}^N v^{d_i b(N-b)} E_i^{(N-b)} K_i^b \otimes E_i^{(b)}$; then $(F \otimes 1)(\Delta(E_i^{(N)})) = \sum_{b=0}^N v^{\frac{(N-b)}{l}} E_i^{(N-b)} \otimes E_i^{(b)}$ and $(F \otimes 1)(\Delta'(E_i^{(N)})) = \sum_{b=0}^N v^{\frac{b}{l}} E_i^{(N-b)} \otimes E_i^{(b)} K_i^b$. (Here Δ' id Δ followed by the permutation of the two factors). As each $K_i^{ls} - 1, s \geq 0$ is in the ideal $(K_i^l - 1)$, the result follows. A similar reasoning holds for the F_i 's. $\square$

As $U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$ is exactly the annihilator of $(K_i^l - 1)$, the result follows. $\square$

Remarks. 1) In [L1], Lusztig showed that when l is prime, base change to $\mathbf{F}_p$ gives the usual Frobenius morphism on the distribution algebra of a simple group over $\mathbf{F}_p$, hence the terminology. (More generally, if $l = p^\nu$, base change to $\mathbf{F}_p$ gives the ν -th power of Frobenius.)

2) A “non-simply connected” version of the Frobenius mapping is to be given in the Appendix.

4.3. Finiteness of $\hat{\mathcal{O}}_l$ over its center. The Hopf algebra morphism $F : \overline{U}_{\mathbf{F}_p} \rightarrow \overline{U}_{\mathbf{F}_p}$ defined by $F(e_i^{(k)}) = e_i^{(k/p)}$ and $F(f_i^{(k)}) = f_i^{(k/p)}$ (conventions of 4.1) induces an algebra morphism on $\overline{U}_{\mathbf{F}_p}^*$ equal to elevation to the p -th power (as is seen using [Ha, Theorem 1], and doing some computations on the additive and multiplicative groups). It restricts to $\overline{U}_{\mathbf{F}_p}^\circ$, since it is a subalgebra of $\overline{U}_{\mathbf{F}_p}^*$. All these morphisms bear the name of Frobenius.

Let us now show:

Lemma. *Assume the Cartan matrix A to be like in proposition 4.1.1. Let ν be a nonnegative integer. Then $\overline{U}_{\mathbf{F}_p}^\circ$ is a finite module over $(F^\nu)^*(\overline{U}_{\mathbf{F}_p}^\circ)$.*

Proof. Let $\xi_1, \dots, \xi_n$ be generators for $\overline{U}_{\mathbf{F}_p}^\circ$ (see prop. 4.1.1). Then $(\xi_1^{k_1} \cdots \xi_n^{k_n})_{0 \leq k_i \leq l-1}$ is obviously a generating family of $\overline{U}_{\mathbf{F}_p}^\circ$ over $(F^\nu)^*(\overline{U}_{\mathbf{F}_p}^\circ)$. $\square$

Recall that $\overline{U}_{\mathbf{Z}}^\circ \subset \overline{U}_{\mathbf{Z}}^*$; the mapping $v_p : \overline{U}_{\mathbf{Z}}^* \rightarrow \mathbf{N}$ such that $v_p(f)$ is the largest integer such that $f \in p^k \overline{U}_{\mathbf{Z}}^*$ defines a valuation on $\overline{U}_{\mathbf{Z}}^*$; topological equivalent norms are deduced from it by defining $|f| = (\frac{1}{l})^{v_p(f)}$ ($l = p^\nu$). We denote by $\mathcal{O}_{\mathbf{Z}}$ the algebra $\overline{U}_{\mathbf{Z}}^\circ$, and by $\hat{\mathcal{O}}_{\mathbf{Z}}$ its completion for $|\cdot|$.

Then, we have

Lemma. *Assume p to be prime to the entries of the Cartan matrix. The dual of the Lusztig-Frobenius morphism induces a continuous injective algebra morphism from $\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l$ to $\hat{\mathcal{O}}_l$. We denote it by F . Then $F(\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l)$ is a subalgebra of the center $Z(\hat{\mathcal{O}}_l)$ of $\hat{\mathcal{O}}_l$.*

Proof. The injection $F^* : (\overline{U}_{\mathbf{Z}} \otimes \mathcal{B}_l)^* \rightarrow U_{\mathcal{B}_l}^*$ is seen to be norm-preserving. It restricts to a norm-preserving mapping $\overline{U}_{\mathbf{Z}}^\circ \otimes \mathcal{B}_l \rightarrow \mathcal{O}_l$; this mapping induces a norm-preserving mapping from $\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l$ to $\hat{\mathcal{O}}_l$. This proves the injectivity.

The multiplication is continuous, hence the second part of the assertion. $\square$

Let us now denote by $\hat{\mathcal{O}}_{\mathbf{Q}}$ the algebra $\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathbf{Q}$. It has a unique norm extending $|\cdot|$; F extends to a continuous injective algebra morphism from $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes \mathcal{B}_l$ to $\hat{\mathcal{O}}_l$. F and $|\cdot|$ still denote these newly defined objects.

We can now state:

Proposition. *Assume p to be prime to the entries of the Cartan matrix. $\hat{\mathcal{O}}_l$ is a module of finite type over $F(\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l)$.*

Proof. Let $\tilde{\xi}_1, \dots, \tilde{\xi}_n$ be preimages of $\xi_1, \dots, \xi_n$ in the lemma above. We are in the situation of lemma 4.3.1 since the assumption on A is now: $p \neq 2$, A has no G_2 component if $p = 3$. Hence we know that for each f in $\hat{\mathcal{O}}_l$, there exist elements $g_{i_1 \dots i_n} \in \hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l$ and $f^1 \in \hat{\mathcal{O}}_l$ such that: $f = \sum_{0 \leq i_k \leq p-1} F(g_{i_1 \dots i_n}) \tilde{\xi}_1^{i_1} \cdots \tilde{\xi}_n^{i_n} + (v-1)f^1$. We repeat the argument for f^1 , etc... Setting $g'_{i_1 \dots i_n} = g_{i_1 \dots i_n} + (v-1)g_{i_1 \dots i_n}^1 + \cdots$ (the serie obviously converges), we get $f = \sum_{0 \leq i_k \leq p-1} F(g'_{i_1 \dots i_n}) \tilde{\xi}_1^{i_1} \cdots \tilde{\xi}_n^{i_n}$. Then $(\xi_1^{k_1} \cdots \xi_n^{k_n})_{0 \leq k_i \leq p-1}$ is a generating family of $\hat{\mathcal{O}}_l$ over $F(\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l)$. $\square$

Remark. The fact that $\overline{U}_{\mathbf{F}_p}$ is a finite module over $F^\nu(\overline{U}_{\mathbf{F}_p})$ implies that $U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$ is a finite module over $F^*(\overline{U}_{\mathbf{Z}} \otimes \mathcal{B}_l)$. A direct proof of this fact, for all l , should not be difficult.

The following proposition shows that tools of commutative algebra could be used in the future to analyse the algebras $\hat{\mathcal{O}}_l$.

Proposition. *Let p be an arbitrary prime number, $l = p^\nu$. Then $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l$ is a noetherian algebra.*

Proof. Let I be an ideal of $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l$; let $J = I \cap (\hat{\mathcal{O}}_{\mathbf{Z}} \otimes B_l)$; it is a divisible ideal of $\hat{\mathcal{O}}_{\mathbf{Z}} \otimes B_l$. $(\otimes_{\mathbf{F}_p} 1)$ sends it to an ideal J' of $\bar{U}_{\mathbf{Z}}^\circ \otimes \mathbf{F}_p$ (by surjectivity of $\otimes_{\mathbf{F}_p} 1$). This last algebra is noetherian, so J' is finitely generated: let $\lambda'_1, \dots, \lambda'_s$ be generators for J' . Let $\lambda_1, \dots, \lambda_s$ be preimages in J of $\lambda'_1, \dots, \lambda'_s$ by $\otimes_{\mathbf{F}_p} 1$.

Fix f in J . There exist $g'_1, \dots, g'_s$ in $\bar{U}_{\mathbf{Z}}^\circ \otimes \mathbf{F}_p$ such that $f = g'_1 \lambda'_1 + \dots + g'_s \lambda'_s$. Let $g_1, \dots, g_s$ be preimages of $g'_1, \dots, g'_s$ by $\otimes_{\mathbf{F}_p} 1$. Then we have: $(\otimes_{\mathbf{F}_p} 1)(f - (g_1 \lambda_1 + \dots + g_s \lambda_s)) = 0$; so $f - (g_1 \lambda_1 + \dots + g_s \lambda_s)$ belongs to J and to $(v-1)\hat{\mathcal{O}}_{\mathbf{Z}}$. Set $f^1 = \frac{1}{v-1}(f - (g_1 \lambda_1 + \dots + g_s \lambda_s))$; J being divisible, $f^{(1)}$ belongs to J .

We apply the same construction to $f^{(1)}$, etc. The series $g_k + (v-1)g_k^{(1)} + \dots + (v-1)^t g_k^{(t)} + \dots$ converge. Let g''_k be their sums. Then $f - (g''_1 \lambda_1 + \dots + g''_s \lambda_s)$ belong to all the $(v-1)^k \hat{\mathcal{O}}_{\mathbf{Z}} \otimes B_l$'s, $k \geq 0$ hence is zero. Thus $J = (\hat{\mathcal{O}}_{\mathbf{Z}} \otimes B_l) \lambda_1 + \dots + (\hat{\mathcal{O}}_{\mathbf{Z}} \otimes B_l) \lambda_s$.

Finally, we remark that multiplying by some non-zero element of B_l transforms any element of J into an element of I ; hence $I = (\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l) \lambda_1 + \dots + (\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l) \lambda_s$. $\square$

Thus:

Corollary. *The center $Z(\hat{\mathcal{O}}_l)$ is a module of finite type over $F((\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l))$.*

Proof. By the two previous propositions and Hilbert's basis theorem. $\square$

Remarks. 1) Here appears the need to continue the analysis over $\hat{B}_l$; we do not know if $\hat{\mathcal{O}}_{\mathbf{Z}} \otimes B_l$ is noetherian or not.

2) The argument in the above proposition is standard in "affinoid" algebras theory.

3) All the ideals of $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l$ are closed (see [T]).

4.4. $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l, B_l)$. Remark that the set of continuous algebra morphisms from $\hat{\mathcal{O}}_{\mathbf{Q}}$ to $\hat{B}_l$ (which we will denote by $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l, B_l)$) is a subset of the maximal spectrum of $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l$; let us give it the Zariski topology.

Denote by G the $\mathbf{Z}$ -group defined by $\bar{U}_{\mathbf{Z}}^\circ$. Our purpose is to show:

Proposition. *$\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Q}} \otimes B_l, B_l)$ is homeomorphic to $G(\hat{B}_l)$, endowed with the Zariski topology.*

Proof. We first remark the sequence of inclusions $\mathcal{O}_{\mathbf{Z}} \hookrightarrow \hat{\mathcal{O}}_{\mathbf{Z}} \hookrightarrow \hat{\mathcal{O}}_{\mathbf{Q}}$. It induces a continuous mapping $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Q}}, \hat{B}_l) \rightarrow \text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{B}_l)$. This mapping is clearly injective; its image is contained in $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{B}_l)$. Indeed, if $f \in \hat{\mathcal{O}}_{\mathbf{Z}}$ mapped to $\lambda \in B_l - \hat{B}_l$, we would have $|\lambda| > 1$; the sequence f^n would be bounded and the sequence of images λ^n would not. $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{B}_l)$ also injects continuously in $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Q}}, \hat{B}_l)$ (by localization). We have thus constructed a homeomorphism between $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Q}}, \hat{B}_l)$ and $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{B}_l)$.

The restriction mapping from $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{B}_l)$ to $\text{Hom}(\mathcal{O}_{\mathbf{Z}}, \hat{B}_l)$ is injective (by density) and surjective (since each element in the latter space is automatically norm-decreasing). It is also continuous.

The inverse mapping is also continuous: fix $f \in \hat{\mathcal{O}}_{\mathbf{Z}} \otimes \hat{\mathcal{B}}_l$. Let us write $f = f_0 + (v-1)f_1 + \dots$, the f_i 's belonging to $\mathcal{O}_{\mathbf{Z}} \otimes \hat{\mathcal{B}}_l$. The inverse image of the closed subset of $\text{Hom}_{\text{cont}}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{\mathcal{B}}_l)$ defined by $\{\chi \mid \chi(f) = 0\}$ is equal to the infinite intersection $\bigcap_{n \geq 0} \{\chi \mid \chi(f_n) = 0\} \subset \text{Hom}(\hat{\mathcal{O}}_{\mathbf{Z}}, \hat{\mathcal{B}}_l) = G(\hat{\mathcal{B}}_l)$. This last space being noetherian, this infinite intersection of closed subsets is closed. $\square$

Remarks. 1) $G(\hat{\mathcal{B}}_l)$ is maximal compact subgroup of $G(\hat{\mathcal{B}}_l)$ in the p -adic topology. A generalization of the situation above may be related to a p -adic generalization of the Gel'fand correspondance (between commutative unital C^* -algebras and compact topological spaces).

2) $\hat{\mathcal{O}}_{\mathbf{Z}}$ is the algebra of “power bounded elements” of $\hat{\mathcal{O}}_{\mathbf{Q}}$ (with the terminology of [BGR]).

4.5. The sheaf of algebras (A_g). We will use the following result.

Lemma. *Under the assumptions of proposition 4.1.1 on A , if g is an algebra homomorphism from $\overline{U}_{\mathbf{F}_p}^{\circ}$ to $\mathbf{F}_p$, and I_g denotes the maximal ideal associated to it, then*

$$\dim_{\mathbf{F}_p} \overline{U}_{\mathbf{F}_p}^{\circ} / ((F^{\nu})^*(I_g)) = l^d$$

where $l = p^{\nu}$, $d = n + 2 \text{ card } R_+$.

Proof. Multiplication by g^{-1} gives an isomorphic $\mathbf{F}_p$ -vector space, hence we assume $g = e$.

Let I denote the augmentation ideal of $\mathcal{O}_{\mathbf{Z}} = \overline{U}_{\mathbf{Z}}^{\circ}$. The I -adic completion of $\mathcal{O}_{\mathbf{Z}}$ is the formal series ring $\mathbf{Z}[[\gamma_1, \dots, \gamma_d]]$. Hence the ring $\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p$ injects in $\mathbf{Z}[[\gamma_1, \dots, \gamma_d]] \otimes \mathbf{F}_p = \mathbf{F}_p[[\gamma_1, \dots, \gamma_d]]$. This injection will be denoted by ι ; it sends the augmentation ideal I_e of $\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p$ into the one of $\mathbf{F}_p[[\gamma_1, \dots, \gamma_d]]$ (which we denote by J). The I_e -adic completion of $\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p$ then injects into $\mathbf{F}_p[[\gamma_1, \dots, \gamma_d]]$ and this injection is seen to be an isomorphism.

ι induces a mapping $\iota' : (\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p) / I_e^{\nu} \rightarrow \mathbf{F}_p[[\gamma_1, \dots, \gamma_d]] / J^{\nu}$. It is clearly surjective; let us prove it is injective. Let f belong to $(\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p) \cap \iota'^{-1}(J^{\nu})$. Let n be the greatest integer such that $f \in I^n$. If $n < \nu$, ι sends f into a non-zero element in J^n ; contradiction.

We use the hypothesis of proposition 4.1.1 to identify $\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p$ with $\overline{U}_{\mathbf{F}_p}^{\circ}$. $\square$

Let us recall that the Frobenius morphism makes $\hat{\mathcal{O}}_l \otimes_{\mathcal{U}'} \mathcal{B}_{l'}$ a finite module over $\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_{l'}$. In the same way, $\hat{\mathcal{O}}_l \otimes_{\mathcal{U}'} B_{l'}$ is a finite module over $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes_{\mathcal{U}'} B_{l'}$. This allows to consider the quantum algebra $\hat{\mathcal{O}}_l \otimes_{\mathcal{U}'} \mathcal{B}_{l'}$ as a sheaf of finite dimensional algebras over $G(\hat{\mathcal{B}}_{l'})$ (this last space being given the Zariski topology).

For each $\hat{\mathcal{B}}_{l'}$ -point of G , we can then define the algebras $\hat{A}_g$ and $\hat{A}_g$ by

$$\hat{A}_g = \hat{\mathcal{O}}_l \otimes_{\mathcal{U}'} B_{l'} / (F(I_g)), \quad \hat{A}_g = \hat{\mathcal{O}}_l \otimes_{\mathcal{U}'} \mathcal{B}_{l'} / (F(I_g)).$$

These are respectively $\hat{\mathcal{B}}_{l'}$ - and $\hat{B}_{l'}$ -algebras of finite type.

Proposition. *Assume p to be prime to the entries of the Cartan matrix, and $l = p^{\nu}$, $l' = p^{\nu'}$, $\nu \leq \nu'$. Then:*

- i) *as a $\hat{\mathcal{B}}_{l'}$ -module, $\hat{A}_g$ can be generated by l^d elements.*
- ii) *$\dim_{\hat{B}_{l'}} \hat{A}_g \leq l^d$.*

Proof. i) $\hat{\mathcal{A}}_g \otimes \mathbf{F}_p$ is identified with $\hat{\mathcal{O}}_{\mathcal{B}_{l'}}/(v-1)\hat{\mathcal{O}}_{\mathcal{B}_{l'}} + (F(I_g))$, i.e. with $\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p/(F(I_{g'}))$, g' denoting the $\mathbf{F}_p$ -point of G corresponding to g , and $I_{g'}$ the associated maximal ideal. The above lemma then shows that the $\mathbf{F}_p$ -dimension of $\hat{\mathcal{A}}_g/(v-1)\hat{\mathcal{A}}_g$ is equal to l^d . A preimage of a $\mathbf{F}_p$ -base of $\hat{\mathcal{A}}_g/(v-1)\hat{\mathcal{A}}_g$ in $\hat{\mathcal{A}}$ is seen to generate $\hat{\mathcal{A}}_g$ over $\mathcal{B}_{l'}$.

ii) follows from i) and the observation $\hat{\mathcal{A}}_g = \hat{\mathcal{A}}_g \otimes \mathcal{B}_{l'}$. $\square$

The same proof shows an analogous result in the non-completed case:

Proposition. *If $g \in G(\mathcal{B}_l)$, and $A_g = \mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'}/(F(I_g))$; if $l = p^\nu$, $l' = p^{\nu'}$, $\nu \leq \nu'$ and p is prime to the entries of the Cartan matrix, then $\dim_{\mathcal{B}_{l'}} A_g \leq l^d$.*

Proof. The proof above shows that $\dim_{\mathbf{F}_p}(\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'})/(F(I_g)) \leq l^d$. We then use the following lemma:

Lemma. *Let M be a finitely generated $\mathcal{B}_{l'}$ -module, then $\dim_{\mathcal{B}_{l'}} M \otimes \mathcal{B}_{l'} \leq \dim_{\mathbf{F}_p} M \otimes \mathbf{F}_p$.*

Proof. Let us first recall that $\mathcal{B}_{l'}$ is a Dedekind ring (see [Mi]); hence, by Steinitz's theorem, the quotient of M by its torsion submodule M^t is isomorphic to a direct sum of non-zero ideals $I_1, \dots, I_s$ of $\mathcal{B}_{l'}$. Then, $\dim_{\mathcal{B}_{l'}} M \otimes \mathcal{B}_{l'} = s$. On the other hand, tensoring with the mapping $\varphi : \mathcal{B}_{l'} \rightarrow \mathbf{F}_p$ is right exact, hence $M \otimes_{\varphi} \mathbf{F}_p$ surjects to the direct sum of the $I_i \otimes_{\varphi} \mathbf{F}_p$'s. It remains to see that for each non-zero ideal I of $\mathcal{B}_{l'}$, $I \otimes_{\varphi} \mathbf{F}_p \neq 0$. But this space is isomorphic, as a $\mathbf{F}_p$ -module, to $I/(v-1)I$. Let J be the ideal of $\mathcal{B}_{l'}$ generated by $v-1$. $(v-1)I$ is isomorphic to IJ . Furthermore, J is the only ideal sharing this property since neither I nor $(v-1)I$ are zero and $\mathcal{B}_{l'}$ is Dedekind. Hence $I = (v-1)I$ would imply $J = \mathcal{B}_{l'}$, a contradiction. An other argument: $I = (v-1)^n I$, $\forall n \geq 0$, hence $I \subset \bigcup_{n \geq 0} (v-1)^n \mathcal{B}_{l'} = 0$ (because the norm of $(v-1)$ is p), contradiction. $\square$

The additional observation that $A_g = [(\mathcal{O}_l \otimes_{\mathcal{B}_{l'}} \mathcal{B}_{l'})/(F(I_g))] \otimes \mathcal{B}_{l'}$ proves the proposition. $\square$

Remarks. 1) It can be shown that $\mathcal{O}_{\mathbf{Z}} \otimes \mathbf{F}_p/(F^{*\nu}(I_e))$ can be identified with the Hopf algebra kernel of $F^\nu : \overline{U}_{\mathbf{F}_p} \rightarrow \overline{U}_{\mathbf{F}_p}$.

2) There is a Hopf algebra duality between the Lusztig finite dimensional Hopf algebra $\mathbf{u}$ (the Hopf kernel of the Lusztig-Frobenius morphism $U_{B_l} \rightarrow \overline{U}_{\mathbf{Q}} \otimes B_l$) and $\mathcal{O}_{B_l}/(F(I))$.

4.6. Summary of the results and open questions. The best results are about $\hat{\mathcal{O}}_l$. It is a topological Hopf algebra, integral and integrally closed when tensored with any $\mathcal{B}_{l'}$ (l, l' are prime powers, l divides l'). The finite type algebra $\text{Coeff}(U_{B_l}, \tilde{\pi}_{B_l})$ is a dense subalgebra. $\hat{\mathcal{O}}_l$ is finite over its center; this center is in turn finite over its ‘‘Frobenius part’’.

For $\hat{\mathcal{O}}_l$ the same results hold; the last (finiteness of the center over its ‘‘Frobenius part’’) is unknown.

The center $Z(\text{Coeff}(U_{B_l}, \tilde{\pi}_{B_l}))$ is a subalgebra of $Z(\hat{\mathcal{O}}_l)$; it is not known if it is dense. Same observation for $Z(\mathcal{O}_l)$ and $Z(\hat{\mathcal{O}}_l)$; and for the integral forms.

We know that O_l is a Hopf algebra, integral when tensored with any $B_{l'}$. It is unknown if it is equal to its subalgebra $\text{Coeff}(U_{B_l}, \tilde{\pi}_{B_l})$, if it is integrally closed, or finite over its center. We do not know if its center is finite over its “Frobenius part”.

Same observations for $\mathcal{O}_l$, except that we do not know if it is a Hopf algebra.

We do not know if any U_{B_l} -module has a finite B_l -form; this would simplify the definition of O_l .

It is unknown if the bounds on the dimensions of the algebras $\hat{A}_g$ given in 4.5 are the best possible or not. Their structures are expected to depend only on the symplectic leaves of G .

Appendix (in collaboration with Nicolás Andruskiewitsch). Let V be a finite-dimensional representation of U_{B_l} , where the K_i^l act by 1. The following proposition was mentioned in the text:

Proposition. *Let l be an arbitrary integer. Then $\text{Coeff}(U_{B_l}, V)$ is a Hopf subalgebra of $O_l^!$.*

Proof. By 1.2, it is enough to see that S^2 is inner in $U_{B_l}^1$. In fact,

$$\forall x \in U, \quad S^2(x) = K_{2\delta} x K_{2\delta}^{-1}$$

where 2δ is the sum of all positive roots and $K_{2\delta}$ is its image by the monoid morphism from the lattice of roots to the multiplicative U^0 sending α_i to K_i , $-\alpha_i$ to K_i^{-1} . Indeed, we have for each β in the root lattice: $K_\beta E_i^{(N)} = v^{N(\beta, \alpha_i)} E_i^{(N)} K_\beta$, $K_\beta F_i^{(N)} = v^{-N(\beta, \alpha_i)} F_i^{(N)} K_\beta$. On the other hand, S^2 is the algebra automorphism of U defined by $S^2(E_i^{(N)}) = v^{N(\alpha_i, \alpha_i)} E_i^{(N)}$, $S^2(F_i^{(N)}) = v^{-N(\alpha_i, \alpha_i)} F_i^{(N)}$. For each simple positive root α_i , $(2\delta, \alpha_i) = (\alpha_i, \alpha_i)$ (see [Hu1, 13.3A]), so the conjugation by $K_{2\delta}$ and S^2 coincide on generators of U , hence are equal. $\square$

Theorem 2.3 implies that $\text{Coeff}(U_{B_l}, V) \otimes_{U'} B_{l'}$ is an integral domain, when l and l' are prime powers.

Let us again suppose l to be arbitrary, and let us specialize the coefficient algebra construction to the case where V is the simple module with highest weight $\mathbf{k} \in [1, l-1]^n$. It is denoted by L_k (cf. [L2] in the simply-laced case, [L3] in the general one). Let ξ be a highest weight vector of $V = L_k$. A “non-simply connected Lusztig-Frobenius” morphism can be built in the following way.

Let $\mathbf{L}_k$ denote the simple $\overline{U}_{\mathbf{Q}}$ -module of highest weight $\mathbf{k}$, let ξ' be a highest weight vector for it, and denote by $\text{Coeff}(\overline{U}_{\mathbf{Q}}, \mathbf{L}_k) \subset \overline{U}_{\mathbf{Q}}^*$ the corresponding (commutative) coefficient algebra.

Let $\mathbf{L}_k^F$ stand for that U_{B_l} -module obtained from $\mathbf{L}_k$ after composition by F . The algebra morphism $F^* : \overline{U}_{\mathbf{Q}} \otimes B_l \rightarrow U_{B_l}$ restricts to a morphism $\text{Coeff}(\overline{U}_{\mathbf{Q}}, \mathbf{L}_k) \otimes B_l \rightarrow \text{Coeff}(U_{B_l}, \mathbf{L}_k^F)$, which we still denote by F^* ; this morphism is injective, since F is surjective.

Recall that $\mathbf{L}_k^F$ is known (cf. [L2], [L3]) to be the simple U_{B_l} -module with highest weight $l\mathbf{k}$. Remark that $U_{B_l} \xi^{\otimes l}$ is a submodule of $\mathbf{L}_k^{\otimes l}$, with highest weight $l\mathbf{k}$ and

¹This fact was mentioned in a talk by H. A. Anderson; see also [Ro].

highest weight vector $\xi^{\otimes l}$. Hence, there is a surjective morphism of U_{B_l} -modules $\iota : U_{B_l} \xi^{\otimes l} \rightarrow \mathbf{L}_k^F$, such that $\iota(\xi^{\otimes l}) = \xi'$.

The following general lemma allows to finish the construction:

Lemma. *Let $\mathbf{A}$ be a Hopf algebra over a field k . Let P, Q be k -finite $\mathbf{A}$ -modules and $\iota : P \rightarrow Q$ be a surjective morphism of $\mathbf{A}$ -modules. Then the algebra $\text{Coeff}(\mathbf{A}, Q) \subset \mathbf{A}^*$ is a subalgebra of $\text{Coeff}(\mathbf{A}, P) \subset \mathbf{A}^*$.*

Proof. Let us first prove that the image $\phi_Q(Q^* \otimes Q)$ is included in $\text{Coeff}(\mathbf{A}, P)$. Fix $\mu \in Q^*, v \in Q$. Let v' be a preimage of v in P , and $\mu' = \iota^*(\mu)$. Then $\phi_Q(\mu' \otimes v') = \phi_P(\mu \otimes v)$. $\text{Coeff}(\mathbf{A}, Q)$ is then the ${}^tS$ -invariant subalgebra of $\mathbf{A}^*$ generated by $\phi_Q(Q^* \otimes Q)$. As $\text{Coeff}(\mathbf{A}, P)$ is also a ${}^tS$ -invariant subalgebra of $\mathbf{A}^*$, this proves the inclusion. $\square$

Composing the injective algebra morphism $F^* : \text{Coeff}(\overline{U}_{\mathbf{Q}}, L_k) \otimes B_l \rightarrow \text{Coeff}(U_{B_l}, \mathbf{L}_k^F)$, the injection $\iota : \text{Coeff}(U_{B_l}, \mathbf{L}_k^F) \rightarrow \text{Coeff}(U_{B_l}, L_k^{\otimes l})$ the lemma and the injection $\iota' : \text{Coeff}(U_{B_l}, L_k^{\otimes l}) \rightarrow \text{Coeff}(U_{B_l}, L_k)$ we obtain an injective algebra morphism $\iota' \circ \iota F^* : \text{Coeff}(\overline{U}_{\mathbf{Q}}, L_k) \otimes B_l \rightarrow \text{Coeff}(U_{B_l}, L_k)$ dual to the Lusztig-Frobenius morphism. This morphism appears in the several works (cf. e.g. Manin, Parshall-Wang).

It is not known if $\text{Coeff}(U_{B_l}, L_k)$ is finite, in general, over $F(\text{Coeff}(\overline{U}_{\mathbf{Q}}, L_k) \otimes B_l)$.

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Résumé substantiel en français.

1. Généralités sur les algèbres de Hopf. Soit $\mathbf{A}$ une algèbre de Hopf sur un anneau R ; son *dual restreint* est par définition, l'ensemble des formes sur $\mathbf{A}$ s'annulant sur un idéal bilatère cofini.

Si R est noethérien, $\mathbf{A}^\circ$ est l'ensemble des coefficients de $\mathbf{A}$ dans ses modules R -finis; c'est une sous-algèbre de $\mathbf{A}^*$. Pour R intègre, on peut supposer les modules sans torsion; enfin, si R est de Dedekind, on peut munir $\mathbf{A}^\circ$ d'une structure d'algèbre de Hopf.

Étant donné un morphisme d'anneaux $R \rightarrow R'$ et un R -module X , notons $X_{R'} = X \otimes_R R'$. Alors, si R est intègre et $R \rightarrow R'$ est surjectif, $(\mathbf{A}^\circ)_{R'}$ est une sous-algèbre de $(\mathbf{A}^*)_{R'}$. Si $R \rightarrow R'$ est plat, $(\mathbf{A}^\circ)_{R'}$ est une sous-algèbre de $(A_{R'})^\circ$. Si $R' = R_S$, S une partie multiplicative de R , si $\mathbf{A}$ est libre comme R -module et si tout $\mathbf{A}_{R'}$ -module R' -fini est la localisation d'un $\mathbf{A}$ -module R -fini, $(\mathbf{A}^\circ)_{R'} = (A_{R'})^\circ$. Ce résultat est encore valable si $R \rightarrow R'$ est une extension finie de corps.

2. Intégrité des algèbres de coordonnées des groupes quantiques. Rappelons les notations de [L1]. Soit $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$, $\mathcal{A}' = \mathbf{Q}(v)$ son corps de fractions. Soit $\mathbf{U}$ l'algèbre de Hopf sur $\mathcal{A}'$ engendrée par E_i, F_i et K_i , et les relations (a1) à (a5), (b), (c1) et (c2) de [L1]. Soient $E_i^{(N)} = \frac{E_i^N}{[N]_{d_i}!}$, $F_i^{(N)} = \frac{F_i^N}{[N]_{d_i}!}$ les puissances divisées quantiques. Soit U la sous- $\mathcal{A}$ -algèbre de $\mathbf{U}$ engendrée par $E_i^{(N)}, F_i^{(N)}, K_i$ et K_i^{-1} . C'est un $\mathcal{A}$ -module libre, et une algèbre de Hopf ([L1]).

Soit ϕ_l le l -ième polynôme cyclotomique, et $B_l = \mathcal{A}/(\phi_l(v))$. B_l est le l -ième anneau d'entiers cyclotomiques; on note $B_l = B_l \otimes \mathbf{Q}$ son corps de fractions. On a des

morphismes $\mathcal{A} \rightarrow \mathcal{B}_l \rightarrow \mathbf{F}_p$ (le second par $v \mapsto 1$). Alors $U_{\mathbf{F}_p}/(K_i - 1)$ s'identifie à $\overline{U}_{\mathbf{F}_p} = \overline{U}_{\mathbf{Z}} \otimes \mathbf{F}_p$; $\overline{U}_{\mathbf{Z}}$ est la forme entière de Kostant des algèbres enveloppantes.

Nous définissons alors les algèbres de coordonnées suivantes: pour l entier, $\mathcal{O}_l$ est l'espace des coefficients de $U_{\mathcal{B}_l}$ -modules, $\mathcal{B}_l$ -finis, où les éléments centraux K_i^l agissent par 1. Notons $\mathcal{O}_l = (\mathcal{O}_l)_{\mathcal{B}_l}$. On montre que c'est une algèbre de Hopf.

On a alors:

Théorème.

- i) Si $l = p^\nu, l' = p^{\nu'}, \nu \leq \nu'$ sont des entiers positifs, et si p est un nombre premier $\neq 2$, $\mathcal{O}_l \otimes_{\mathcal{U}} \mathcal{B}_{l'}$ est une algèbre intègre.
- ii) Sous les mêmes hypothèses, $\mathcal{O}_l \otimes_{\mathcal{U}} \mathcal{B}_{l'}$ est une algèbre intègre.

La preuve utilise que $\mathcal{O}_l$ est divisible dans $U_{\mathcal{B}_l}^*$, et la spécialisation à $\mathbf{F}_p$, où l'algèbre se plonge dans une algèbre de séries formelles.

Nous en déduisons l'intégrité des algèbres de coordonnées pour un paramètre générique. Ces algèbres ont été définies dans [L1] comme suit: on a sur $\mathcal{A}'$ l'algèbre $\mathbf{O} \subset \mathbf{U}^\circ$, constituée des coefficients de représentations de dimension finie, où l'action des K_i est diagonalisable, de valeurs propres puissances de v^{d_i} . Elle contient une $\mathcal{A}_0$ -sous-algèbre de Hopf, définie par $\mathcal{O} = \mathbf{O} \cap U_{\mathcal{A}_0}^*$ ($\mathcal{A}_0 = \mathbf{Q}[v, v^{-1}]$). On a $\mathcal{O} \otimes_{\mathcal{A}_0} \mathcal{A}' = \mathbf{O}$. On peut encore définir sur $\mathcal{A}$, $\mathcal{O} = U^* \cap \mathcal{O}$.

On a alors:

Corollaire. Les algèbres $\mathcal{O}, \mathcal{O}$ et $\mathbf{O}$ sont intègres.

Ce résultat se montre en spécialisant d'abord de $\mathcal{O}$ à $\mathcal{O}_l$, puis en localisant.

3. Clôture intégrale des algèbres de coordonnées aux racines de l'unité. Rappelons que d'après [dCKP], une algèbre intègre Ω de centre Z est dite intégralement close si toute algèbre X , telle que $\Omega \subset X \subset \frac{1}{z}\Omega (\subset Q(Z) \otimes_Z \Omega)$, pour $z \neq 0$ dans Z , est égale à Ω . Dans cette situation, le centre de Ω est une algèbre normale.

Soient $\hat{\mathcal{B}}_l$ et $\hat{B}_l$ les complétions p -adiques de $\mathcal{B}_l$ et B_l ; soit E une extension finie de $\hat{\mathcal{B}}_l$, $\mathbf{o}_E$ son anneau d'entiers. Soit π une uniformisante de E . Notons $\hat{\mathcal{O}}' = \hat{\mathcal{O}}_l \otimes \mathbf{o}_E = \hat{\mathcal{O}}_l \otimes \mathbf{o}_E$, $\hat{\mathcal{O}}' = \hat{\mathcal{O}}_l \otimes \mathbf{o}_E$. Alors

Théorème. $\hat{\mathcal{O}}'$ et $\hat{\mathcal{O}}'$ sont intégralement closes.

Ces résultats utilisent la spécialisation de π à 1, qui nous fait alors travailler sur $\mathbf{F}_q$ (le corps résiduel de E). On utilise alors la normalité en caractéristique p .

4. Finitude sur la partie de Frobenius de centre. Montrons que $\hat{\mathcal{O}}'_l$ contient une sous-algèbre dense finiment engendrée. Pour cela, nous montrons le résultat suivant:

Proposition. Supposons que la matrice de Cartan A n'ait pas de composantes de type B_k, C_k, F_4 si $p = 2$, et pas de composante de type G_2 si $p = 3$.

- i) Soit k une clôture algébrique de $\mathbf{F}_p$. L'injection de $\overline{U}_{\mathbf{Z}}^\circ \otimes k$ dans $(\overline{U}_k)^\circ$ est un isomorphisme.
- ii) L'injection de $\overline{U}_{\mathbf{Z}}^\circ \otimes \mathbf{F}_p$ dans $(\overline{U}_{\mathbf{F}_p})^\circ$ est un isomorphisme.

Ce résultat utilise en particulier la classification des isogénies entre groupes semi-simples en caractéristique p , due à Chevalley ([CC]): elles sont toutes centrales sous nos hypothèses.

Soit alors ρ un $\overline{U}_{\mathbf{Z}}$ -module, $\mathbf{Z}$ -fini, tel que $\overline{U}_{\mathbf{Z}}^{\circ}$ soit égal à $\text{Coeff}(\overline{U}_{\mathbf{Z}}, \rho)$ (l'algèbre engendrée par les coefficients de $\overline{U}_{\mathbf{Z}}$ dans la représentation ρ). Soit $\tilde{\rho}$ un U -module $\mathbf{A}$ -libre quantifiant ρ . Nous avons alors:

Proposition. *Sous les mêmes hypothèses sur la matrice de Cartan, $\hat{\mathcal{O}}_l$ contient $\text{Coeff}(U_{\mathcal{B}_l}, \tilde{\rho}_{\mathcal{B}_l})$ comme sous-algèbre dense.*

Rappelons ([L1]) la construction de la partie de Frobenius du centre de $\mathcal{O}_l$. Nous avons un morphisme d'algèbres de Hopf $F : U_{\mathcal{B}_l} \rightarrow \overline{U}_{\mathbf{Z}} \otimes \mathcal{B}_l$, défini par les formules $F(E_i^{(N)}) = e_i^{N/l}$, $F(F_i^{(N)}) = f_i^{N/l}$, $F(K_i) = 1$ (avec la convention $e_i^{(r)} = 0$, etc. si $r \in \mathbf{Q} - \mathbf{Z}$. F est surjectif; son dual $F^* : \overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l \rightarrow U_{\mathcal{B}_l}^*$ est donc un morphisme injectif de $\mathcal{B}_l$ -algèbres. F est le morphisme de Frobenius en caractéristique zéro (il donne le Frobenius en caractéristique p en posant $v = 1$). Il a les propriétés suivantes:

Proposition. $F^*(\overline{U}_{\mathbf{Z}}^{\circ} \otimes \mathcal{B}_l) \subset \mathcal{O}_l$; d'autre part, $F^*(\overline{U}_{\mathbf{Z}}^* \otimes \mathcal{B}_l)$ est contenue dans le centre de $U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$.

(Nous identifions $l\mathbf{Z}^n$ à son image dans la partie « de Cartan » $U_{\mathcal{B}_l}^0$ de $U_{\mathcal{B}_l}$, par $(k_1, \dots, k_n) \mapsto K_i^{k_1} \dots K_n^{k_n}$; $U_{\mathcal{B}_l, l\mathbf{Z}^n}^*$ est alors la sous-algèbre de $U_{\mathcal{B}_l}^*$ formée des éléments invariants à gauche (ou à droite) sous $l\mathbf{Z}^n$.)

Soit $\mathcal{O}_{\mathbf{Z}}$ l'algèbre $\overline{U}_{\mathbf{Z}}^{\circ}$, et $\hat{\mathcal{O}}_{\mathbf{Z}}$ sa complétée p -adique. Nous avons alors:

Proposition. *Supposons p premier aux coefficients de la matrice de Cartan. Le dual du morphisme du Lusztig-Frobenius induit un morphisme continu central de $\hat{\mathcal{O}} \otimes \mathcal{B}_l$ vers $\hat{\mathcal{O}}_l$, noté F . De plus, $\hat{\mathcal{O}}_l$ est un module fini sur $F(\hat{\mathcal{O}}_{\mathbf{Z}} \otimes \mathcal{B}_l)$. Mêmes résultats pour $\hat{\mathcal{O}}_l$ et $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes \mathcal{B}_l$. Enfin, les algèbres $Z(\mathcal{O}_l)$ et $\hat{\mathcal{O}}_{\mathbf{Q}} \otimes \mathcal{B}_l$ sont noethériennes et la première est type fini sur la seconde.*

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