

LINEAR OPERATORS INDUCED BY POINT TRANSFORMATIONS

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RÉSUMÉ. Soit i) $(Y, \mathcal{F}, \lambda)$ un espace mesuré σ -fini, ii) $(X, \mathcal{B}, \mu)$ un espace de Baire σ -fini où μ est équivalente à une mesure finie μ_1 sur $\mathcal{B}$ satisfaisant la propriété : $\forall \varepsilon > 0, \exists K$, un sous-ensemble compact de X , tel que $\mu_1^*(K) > \mu_1(X) - \varepsilon$, iii) $\mathcal{V}$ (respectivement $\mathcal{W}$) un espace vectoriel de fonctions à valeurs dans le corps $\mathbb{R}$ ou $\mathbb{C}$ définies sur X (respectivement Y), et iv) $U : \mathcal{V} \rightarrow \mathcal{W}$ un opérateur essentiellement linéaire tel que $(Uf)(Ug) = 0$ p.p. lorsque $fg = 0$ p.p. Nous montrons que si $\mathcal{V}$ est de plus un treillis auto-adjoint contenant les fonctions caractéristiques des ensembles de mesure finie et si U satisfait les propriétés : a) $f(x) = 0$ p.p. $\Rightarrow (Uf)(x) = 0$ p.p. et b) $f_n \downarrow 0 \Rightarrow (Uf_n)(x) \rightarrow 0$ p.p., alors il existe une fonction mesurable $h : Y \rightarrow \mathbb{C}$ et une transformation $T : Y \rightarrow X$ mesurable par rapport à la complétion de $(Y, \mathcal{F}, \lambda)$, telles que $Uf = h(f \circ T)$ p.p. $\forall f \in \mathcal{V}$. Ce résultat très général nous permet d'obtenir des applications concernant les opérateurs sur $L^p(X, \mu)$ (où $1 \leq p \leq \infty$) et les homomorphismes de σ -algèbres.

ABSTRACT. Let i) $(Y, \mathcal{F}, \lambda)$ be any σ -finite measure space, ii) $(X, \mathcal{B}, \mu)$ be a tight σ -finite Baire measure space (where X is an arbitrary topological space), iii) $\mathcal{V}$ (respectively $\mathcal{W}$) be a vector space of complex or real-valued measurable functions defined on X (respectively Y), and iv) $U : \mathcal{V} \rightarrow \mathcal{W}$ be an (essentially) linear support separating operator. We show that if $\mathcal{V}$ is also a self-adjoint lattice which contains the characteristic function of every set of finite measure, and if U satisfies the properties: a) $f(x) = 0$ a.e. $\Rightarrow (Uf)(x) = 0$ a.e. and b) $f_n \downarrow 0 \Rightarrow (Uf_n)(x) \rightarrow 0$ a.e., then U is induced by a completion measurable point transformation $T : Y \rightarrow X$ in the sense that $Uf = h(f \circ T)$ a.e. for all $f \in \mathcal{V}$. This general result yields applications to operators on $L^p(X, \mu)$ (where $1 \leq p \leq \infty$) and also to set homomorphisms defined on measure algebras.

1. Introduction. A problem of interest in measure theory is whether a given bounded support separating linear operator U on $L^p(X, \mu)$ ($1 \leq p < \infty$) can be induced by a completion measurable point transformation $T : X \rightarrow X$. An even more interesting problem is whether T can be chosen to be invertible whenever 1) $p \neq 2$ and U is an invertible isometry, or 2) $p = 2$ and U is an invertible positive linear isometry. Most authors have worked on the second problem by using a classical result proved by von Neumann [11] which states that any set automorphism of $([0, 1], \mu)$ (where μ is any σ -finite Borel measure on $[0, 1]$) is induced by a point automorphism $T : [0, 1] \rightarrow [0, 1]$. For example, Choksi used von Neumann's result combined with transfinite induction,

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to prove that the second problem can be solved when μ is any σ -finite Baire measure on X , where X is 1) an arbitrary product of complete separable metric spaces, or 2) a compact group, or, more generally, 3) a locally compact homogeneous space under the action of a locally compact σ -compact group. (See Theorem 2 in [2], [3], and [4].) The methods employed to prove these results also use a theorem obtained by J. Lamperti [9] which states that every invertible (positive when $p = 2$) linear isometry can be induced by a set automorphism of $\mathcal{F}/\mathcal{N}$ (where $\mathcal{F}$ is the σ -algebra on X and $\mathcal{N} \subset \mathcal{F}$ is the class of sets of measure zero). The task following this is to prove that the set automorphism is induced by a point automorphism. To complete a history of our second problem, we also recall the result obtained by von Neumann in [12] for unitary operators on $L^2([0, 1])$ with Lebesgue measure.

On the other hand, not much work has been done on the solving of our first problem although it is obviously an important tool used for solving the second problem. We therefore develop methods which are completely different from those described in the preceding paragraph, in order to establish a very general theorem concerning essentially linear support separating operators on a (self-adjoint) vector lattice of complex or real-valued measurable functions defined on an arbitrary tight σ -finite Baire measure space $(X, \mathcal{B}, \mu)$. This theorem gives an important contribution to the solving of the first problem. It also yields a new result for operators on $L^\infty(X, \mu)$ as well as gives applications to set homomorphisms which can be induced by point mappings.

2. Notation. $\mathbb{F}$ will always denote either the field $\mathbb{R}$ or $\mathbb{C}$. When $(X, \mathcal{F}, \mu)$ is a measure space, we will add the symbol $\sim$ over any relation between functions or sets, whenever this relation is satisfied almost everywhere. For example $f \stackrel{\sim}{\geq} 0$ means $f \geq 0$ a.e. and $f \geq 0$ means $f(x) \geq 0$ for every $x \in X$. As usual, the symbols $L^p(X, \mu, \mathbb{R})$ and $L^p(X, \mu, \mathbb{C})$ will respectively designate the real and complex L^p -spaces associated with $(X, \mathcal{F}, \mu)$, and $(X, \mathcal{F}^*, \mu)$ will designate the completion of $(X, \mathcal{F}, \mu)$. $M(X, \mathcal{F}, \mathbb{F})$ will represent the set of all $\mathbb{F}$ -valued $\mathcal{F}$ -measurable functions on X and $M^\infty(X, \mathcal{F}, \mathbb{F})$, the subset consisting of all its bounded functions.

When X is a topological space, we will use the symbol $C_b(X, \mathbb{F})$ to denote the class of all continuous $\mathbb{F}$ -valued bounded functions on X . We shall omit the subscript symbol b in $C_b(X, \mathbb{F})$ whenever X is compact. A *zero set* in X is a set $Z \subset X$ for which there exists $f \in C_b(X, \mathbb{F})$ such that $Z = f^{-1}(0)$. A *cozero set* in X is the complement (with respect to X) of a zero set in X .

We define a vector space $\mathcal{C}$ of $\mathbb{F}$ -valued functions to be 1) a *lattice* if $|f| \in \mathcal{C}$ whenever $f \in \mathcal{C}$, and 2) *self-adjoint* if for every $f \in \mathcal{C}$, the complex conjugate function associated to f , also belongs to $\mathcal{C}$. (Note that when $\mathbb{F} = \mathbb{R}$, $\mathcal{C}$ is automatically self-adjoint.)

Finally, when $\mathcal{V}$ is a collection of $\mathbb{F}$ -valued functions defined on an arbitrary set, then $\mathcal{V}^+$ is the subcollection of $\mathcal{V}$ consisting of all positive functions in $\mathcal{V}$.

3. A result on multiplicative linear functionals. In order to prove the main result of this paper, we will need the following which concerns multiplicative linear functionals.

Proposition 3.1. *Let i) X be a compact (not necessarily Hausdorff) space, ii) $\mathcal{V}$ be a self-adjoint algebra of $C(X, \mathbb{F})$, containing the constant function 1, and iii) $L : \mathcal{V} \rightarrow \mathbb{F}$*

be a non-zero positive multiplicative linear functional.

Then there exists a $x \in X$ with $Lf = f(x)$ for all $f \in \mathcal{V}$.

Proof. Suppose that $S = \{f_i\}_{i=1}^n \subseteq \ker(L) = \{f \in \mathcal{V} : Lf = 0\}$. Then there exists $x_S \in X$ with $f_i(x_S) = 0$ for all i . Indeed, if this is not the case, then $\sum |f_i|^2 \geq \varepsilon$ for a certain $\varepsilon > 0$ and so, using the fact that L is a positive multiplicative linear functional, we obtain

$$\sum |Lf_i|^2 = L\left(\sum |f_i|^2\right) \geq \varepsilon \cdot L1 = \varepsilon.$$

Thus, there must be a j for which $Lf_j \neq 0$, contradicting the fact that $S = \{f_i\} \subseteq \ker(L)$. Hence, there exists $x_S \in X$ with $f_i(x_S) = 0$ for all i . Now, $\{S : S \subseteq \ker(L) \text{ is finite}\}$ is directed with $\subseteq$ and so, there is a subset $\{x_\alpha\}$ of $\{x_S\}$ converging to $x \in X$. For this x , we have $f(x) = 0$ for all $f \in \ker(L)$ and so, $Lf = f(x)$ for all $f \in \mathcal{V}$. $\square$

Remark 3.2. Proposition 3.1 remains true if, instead of being an algebra, $\mathcal{V}$ is a lattice and if, instead of being multiplicative, L satisfies the property that $(Lf)(Lg) = 0$ whenever $f, g \in \mathcal{V}^+$ are such that $fg = 0$. The result is that there exists $x \in X$ such that $Lf = (L1) \cdot f(x)$ for all $f \in \mathcal{V}$.

4. The main result. Let us set up the terminology which will be used in this section. First, $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ will denote arbitrary σ -finite measure spaces. Furthermore, $\mathcal{V}$ and $\mathcal{W}$ will respectively be vector subspaces of $M(X, \mathcal{B}, \mathbb{F})$ and $M(Y, \mathcal{F}, \mathbb{F})$. In addition, we will suppose that $\mathcal{V}$ is a self-adjoint lattice which contains the characteristic function of every set of finite measure. Finally, we will use the following definitions:

Definition 4.1. An operator $U : \mathcal{V} \rightarrow \mathcal{W}$ is called

- 1) *essentially linear* (abbreviated *e-linear*) if $U(\alpha f + \beta g) \cong \alpha Uf + \beta Ug$ for all $\alpha, \beta \in \mathbb{F}$ and all $f, g \in \mathcal{V}$.
- 2) *essentially positive* (abbreviated *e-positive*) if $f \geq 0$ implies $Uf \gtrsim 0$.
- 3) *support separating* if $(Uf)(Ug) \cong 0$ whenever $f, g \in \mathcal{V}^+$ are such that $fg = 0$.
- 4) *essentially multiplicative* (abbreviated *e-multiplicative*) if $U(fg) \cong (Uf)(Ug)$ for all $f, g \in \mathcal{V}^+$ such that $fg \in \mathcal{V}$.

The following preliminary result will be very useful in the proof of our main theorem.

Proposition 4.2. Let $U : \mathcal{V} \rightarrow \mathcal{W}$ be an *e-linear support separating operator* satisfying the following property:

(*) $f_n \downarrow 0$ implies that for every $\varepsilon > 0$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $U(f_{n_k})(x) \rightarrow 0$ except possibly on a set of measure less than ε .

Then, there exists $h \in M(Y, \mathcal{F}, \mathbb{F})$ such that if we denote by W the support of h , we have that 1) $Uf \cong 0$ on $Y \setminus W$ for all $f \in \mathcal{V}$, and 2) the operator $U_1 : \mathcal{V} \rightarrow M(Y, \mathcal{F}, \mathbb{F})$ defined by $U_1 f = (Uf/h)\chi_W$ is an *e-linear e-positive e-multiplicative operator* such that $U_1 f$ is essentially bounded whenever $f \in \mathcal{V} \cap M^\infty(X, \mathcal{B}, \mathbb{F})$.

Proof. Since $(X, \mathcal{B}, \mathbb{F})$ is σ -finite, let $\{E_n\}$ be a sequence of pairwise disjoint sets of finite measure whose union is X . Since U is support separating, then $\sum U(\chi_{E_n})$ exists a.e. Therefore, we can choose $h \in M(Y, \mathcal{F}, \mathbb{F})$ such that $h \cong \sum U(\chi_{E_n})$. If we define

W and U_1 as in the statement of this proposition, then, obviously, U_1 is e-linear and support separating.

Now, let $E \in \mathcal{B}$. Then $U(\chi_{E_n}) \cong U(\chi_{E \cap E_n}) + U(\chi_{E_n \setminus E \cap E_n})$ and so, using the fact that U is support separating, we conclude that

$$\frac{U(\chi_{E \cap E_n})}{U(\chi_{E_n})} \cong \chi_{S_n} \text{ on } W_n = \{y \in Y : U(\chi_{E_n})(y) \neq 0\}, \text{ where } S_n \subseteq W_n.$$

Hence,

$$\frac{\sum U(\chi_{E \cap E_n})}{h} \cong \chi_S \text{ on } W, \text{ where } S = \cup S_n \subseteq W.$$

Put $F_N = \bigcup_{n=1}^N E_n$. Then $\chi_{E \cap F_N} \uparrow \chi_E$. Since U satisfies property (*), we conclude that $U_1(\chi_E) \cong \chi_S$. Thus, U_1 sends characteristic functions to functions which are essentially characteristic functions. Hence, if $E, F \in \mathcal{B}$, then by using the equations $U_1(\chi_E) \cong U_1(\chi_{E \cap F}) + U_1(\chi_{E \setminus F \cap F})$ and $U_1(\chi_F) \cong U_1(\chi_{E \cap F}) + U_1(\chi_{F \setminus E \cap F})$ and the fact that U_1 is support separating, we deduce that $U_1(\chi_E \chi_F) \cong U_1(\chi_E) \cdot U_1(\chi_F)$. Consequently, we obtain $U_1(st) \cong U_1(s) \cdot U_1(t)$ for all s, t belonging to $\mathcal{S}_X$, the class of all μ -integrable simple functions. (Recall that a μ -integrable simple function is a finite linear combination of characteristic functions of sets of finite measure.) Clearly, for all $s \in \mathcal{S}_X$, we have that $U_1(s)$ is essentially bounded and $U_1(s) \cong 0$ on $Y \setminus W$. Furthermore, $s \in \mathcal{S}_X^+$ implies $U_1(s) \geq 0$.

Finally, let $f \in \mathcal{V}^+ \cap M^\infty(X, \mathcal{B}, \mathbb{F})$. Since μ is σ -finite, it is easy to show that there exists a sequence $\{s_n\} \subseteq \mathcal{S}_X^+$ such that $s_n \uparrow f$. Using the fact that U (and so U_1) satisfies property (*), we deduce that $U_1 f$ is an essentially bounded positive function which vanishes a.e. outside W . Now, if $g \in \mathcal{V}^+$ is such that $fg \in \mathcal{V}^+$, the relation $U_1(fg) \cong (U_1 f)(U_1 g)$ is easily verified and so, the proof of Proposition 4.2 is complete. $\square$

Corollary 4.3. *The following are equivalent for an e-linear support separating operator $U : \mathcal{V} \rightarrow \mathcal{W}$:*

- 1) $f_n \downarrow 0$ implies $(U f_n)(x) \rightarrow 0$ a.e.
- 2) $f_n \downarrow 0$ implies the existence of a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ with $U(f_{n_k})(x) \rightarrow 0$ a.e.
- 3) $f_n \downarrow 0$ implies that for all $\varepsilon > 0$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $U(f_{n_k})(x) \rightarrow 0$ except possibly on a set of measure less than ε .

Proof. It is sufficient to prove that 3) $\Rightarrow$ 1). Now, by Proposition 4.2, there exists $h \in M(Y, \mathcal{F}, \mathbb{F})$ such that the operator $U_1 : \mathcal{V} \rightarrow M(Y, \mathcal{F}, \mathbb{F})$ defined by $U_1 f = (U f / h) \chi_W$ (where W denotes the support of h) is e-positive, e-linear and e-multiplicative. Obviously, U_1 satisfies 3) and so, the e-positivity of U_1 implies that U_1 satisfies 1). Now, the fact that $U f = (U_1 f) h$ on W and $U f \cong 0$ on $Y \setminus W$ yields that U satisfies 1). $\square$

Remarks 4.4. a) Conditions 1), 2), 3) of Corollary 4.3 are clearly equivalent for e-linear e-positive (not necessarily support separating) operators $U : \mathcal{V} \rightarrow \mathcal{W}$ and this fact was used in the preceding proof. Now, the corollary asserts that the conditions are also equivalent for e-linear (not necessarily e-positive) support separating operators.

b) When $U : \mathcal{V} \rightarrow \mathcal{W}$ is an $\mathcal{E}$ -linear support separating operator satisfying: $f_n \downarrow 0 \Rightarrow \{Uf_n\}$ converges to zero in measure, then U satisfies condition 2) of our corollary and so, it satisfies conditions 1) and 3) as well. Of course, the converse also holds when λ is a finite measure and hence, in this case, the four conditions are equivalent.

We are now ready to prove the main result of this paper. But first, recall from [13, p. 33] that a finite Baire measure μ on a topological space X is called *tight* if for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ such that $\mu^*(K) > \mu(X) - \varepsilon$, where μ^* denotes the outer measure determined by μ . In [8], it is shown that when X is completely regular, then μ is tight iff μ is the restriction of a (unique) finite Radon measure on X . Now, a (non-finite) σ -finite Baire measure is *tight* if it is equivalent to a tight finite measure. Note that a space X for which every finite (and so every σ -finite) Baire measure is tight, is called *strongly measure-compact* (abbreviated SMC). SMC spaces include many interesting spaces such as, for example, a) σ -compact spaces, b) continuous images of complete separable metric spaces (i.e. Suslin spaces), and c) Lindelöf, Čech complete spaces (see [13, p. 52]).

Theorem 4.5. *Let i) X be an arbitrary topological space, ii) μ be a tight σ -finite measure on the class $\mathcal{B}$ of all Baire sets in X , and iii) $U : \mathcal{V} \rightarrow \mathcal{W}$ be an $\mathcal{E}$ -linear support separating operator satisfying the conditions:*

- a) $f \cong 0 \Rightarrow Uf \cong 0$
- b) $f_n \downarrow 0 \Rightarrow (Uf_n)(x) \rightarrow 0$ a.e. (or, equivalently, condition 2) or 3) of Corollary 4.3).

Then, there exists 1) a function $h \in M(Y, \mathcal{F}, \mathbb{F})$ which is uniquely determined a.e. and 2) a transformation $T : Y \rightarrow X$ which is $\mathcal{F}^*$ -measurable in the sense that $T^{-1}E \in \mathcal{F}^*$ for every $E \in \mathcal{B}$, such that

$$i) \mu(E) = 0 \Rightarrow \lambda(T^{-1}E \cap W) = 0 \text{ (where } W \text{ denotes the support of } h)$$

and

$$ii) Uf \cong h(f \circ T) \text{ for all } f \in \mathcal{V}.$$

Moreover, the above conclusions hold even when $\mathcal{V}$ contains functions in $M(X, \mathcal{B}^*, \mathbb{F})$ which are not in $M(X, \mathcal{B}, \mathbb{F})$.

Remarks 4.6. a) When i) X is such that its Baire σ -algebra $\mathcal{B}$ contains a separating sequence $\{E_n\}$, ii) $(Y, \mathcal{F}, \lambda) = (X, \mathcal{B}, \mu)$, iii) $\mathcal{V} = \mathcal{W}$, and iv) U is invertible, then T can be redefined on a null set so to become an invertible non-singular completion measurable transformation on X (see step 6 of the proof of Theorem 4.5). If, furthermore, $\mathcal{B}$ is countably generated, then T can be chosen to be measurable and obviously, T is uniquely determined on W up to a set of measure zero.

b) If the operator of Theorem 4.5 does not satisfy the condition: $f \cong 0 \Rightarrow Uf \cong 0$, it is still possible to construct h and T such that $Uf \cong h(f \circ T)$, for spaces X which are the countable union of compact zero subsets. (The most important example of such a space is an arbitrary locally compact σ -compact Hausdorff space.)

Proof of Theorem 4.5. Throughout this proof, we will use the symbol $\mathcal{S}_X$ to denote the class of all μ -integrable simple functions on X . First, apply Proposition 4.2 to get $h \in M(Y, \mathcal{F}, \mathbb{F})$ such that 1) for all $f \in \mathcal{V}$, $Uf \cong 0$ outside $W = \{y \in Y : h(y) \neq 0\}$

and 2) $U_1 : \mathcal{V} \rightarrow M(Y, \mathcal{F}, \mathbb{F})$ defined by $U_1 f = (Uf/h) \chi_W$ is an e-positive e-linear e-multiplicative operator satisfying: i) $f_n \downarrow 0 \Rightarrow (U_1 f_n)(x) \rightarrow 0$ a.e. and ii) $f \in \mathcal{V} \cap M^\infty(X, \mathcal{F}, \mathbb{F}) \Rightarrow U_1 f$ is essentially bounded. We will now divide the remainder of the proof into several steps.

STEP 1. There exists an e-positive e-linear e-multiplicative operator

$$\bar{U}_1 : M^\infty(X, \mathcal{B}, \mathbb{F}) \rightarrow M^\infty(Y, \mathcal{F}, \mathbb{F})$$

such that $\bar{U}_1 f \cong U_1 f$ for all $f \in \mathcal{V} \cap M^\infty(X, \mathcal{B}, \mathbb{F})$.

Proof. Let f be a positive function in $M^\infty(X, \mathcal{B}, \mathbb{F})$. Then there exists $\{s_n\} \subseteq \mathcal{S}_X^+$ such that $s_n \uparrow f$. Now, $\lim_n U_1(s_n)$ exists a.e. Indeed, put $F_N = \bigcup_{n=1}^N E_n$ (where E_n are the sets used to define h in Proposition 4.2). Then, $\lim_N U_1(\chi_{F_N}) \cong \chi_W$ and so for all $n \geq 1$, we have

$$U_1(s_n) \cong \lim_N U_1(s_n \cdot \chi_{F_N}) \lesssim M \cdot \chi_W \text{ (where } M \text{ is such that } f \leq M \text{)}.$$

Since U_1 is e-positive and $\{s_n\}$ is increasing, we conclude that $\lim_n U_1(s_n)$ exists a.e. Furthermore, let $\{s_n\}, \{t_n\} \subseteq \mathcal{S}_X^+$ both increase to f . Then, if $N \geq 1$, there exists $\{r_n\} \subseteq \mathcal{S}_X^+$ such that $r_n \downarrow f \cdot \chi_{F_N}$. Since $\{r_n - s_n \cdot \chi_{F_N}\}_n$ and $\{r_n - t_n \cdot \chi_{F_N}\}_n$ both decrease to zero, we obtain $U_1(\chi_{F_N}) \cdot \lim_n U_1(s_n) \cong U_1(\chi_{F_N}) \cdot \lim_n U_1(t_n)$. But $\lim_N U_1(\chi_{F_N}) \cong \chi_W$ and for each $f \in \mathcal{V}$, $U_1 f \cong 0$ outside W . Hence, we deduce that $\lim_n U_1(s_n) \cong \lim_n U_1(t_n)$. Consequently, if $\mathcal{N}_Y$ denotes the class of sets $E \in \mathcal{F}$ with $\lambda(E) = 0$, we can define $\bar{U}_1 f \in M^\infty(Y, \mathcal{F}, \mathbb{F})/\mathcal{N}_Y$ by $\bar{U}_1 f \cong \lim_n U_1(s_n)$ where $\{s_n\} \subseteq \mathcal{S}_X^+$ increases to f . We have therefore constructed a positive linear multiplicative map $\bar{U}_1 : M^\infty(X, \mathcal{B}, \mathbb{F}) \rightarrow M^\infty(Y, \mathcal{F}, \mathbb{F})/\mathcal{N}_Y$ such that $U_1 f$ belongs to $\bar{U}_1 f$ whenever $f \in \mathcal{V} \cap M^\infty(X, \mathcal{B}, \mathbb{F})$. Finally, for each $f \in M^\infty(X, \mathcal{B}, \mathbb{F})$, choose a representative in $\bar{U}_1 f$ and call it also $\bar{U}_1 f$. Then, clearly, $\bar{U}_1$ satisfies the requirements of step 1. It should also be noted that $\bar{U}_1 1 \cong \lim_N U_1(\chi_{F_N}) \cong \chi_W$. The proof of step 1 is complete. $\square$

STEP 2. There exists a transformation $T : Y \rightarrow X$ such that

$$\bar{U}_1 f \cong \chi_W(f \circ T) \text{ for all } f \in C_b(X, \mathbb{F}).$$

Proof. Let μ_1 be a tight finite Baire measure which is equivalent to μ , and let $\{K_n\}$ be a sequence of increasing compact subsets of X , such that $\mu_1^*(K_n) \geq \mu_1(X) - 1/n$. Then, there exists an increasing sequence of Baire sets $\{B_n\}$ with $K_n \subseteq B_n$ and $\mu_1(B_n) = \mu_1^*(K_n)$. In particular, $\mu(X \setminus \bigcup B_n) = \mu_1(X \setminus \bigcup B_n) = 0$. Now, clearly, $\bar{U}_1$ can be viewed as a map from $M^\infty(X, \mathcal{B}, \mathbb{F})$ to $M^\infty(Y, \mathcal{F}, \mathbb{F})/\mathcal{N}_Y$, where $\mathcal{N}_Y$ is the class of sets of measure zero in Y . Extend $\bar{U}_1$ on the algebra $\mathcal{A}$ generated by $M^\infty(X, \mathcal{B}, \mathbb{F})$ and $\{\chi_{K_n}\}$, by defining

$$\bar{U}_1 \left(g + \sum_{i=1}^r f_i \cdot \chi_{K_{n_i}} \right) = \bar{U}_1 \left(g + \sum_{i=1}^r f_i \cdot \chi_{B_{n_i}} \right).$$

Since the map $\bar{U}_1$ of step 1 satisfies $f \cong 0 \Rightarrow \bar{U}_1 f \cong 0$, then the extension of $\bar{U}_1$ is well defined. Indeed, if $g + \sum f_i \cdot \chi_{K_{n_i}} = 0$, then $B = \left\{ x : \left(g + \sum f_i \cdot \chi_{B_{n_i}} \right) (x) \neq 0 \right\}$ is a Baire set contained in $\bigcup (B_{n_i} \setminus K_{n_i})$. Suppose, without loss of generality, that $n_1 < n_2 < \dots < n_r$. Then, clearly, $K_{n_1} \subseteq B_{n_1} \setminus B$ and so

$$\mu_1(B_{n_1}) = \mu_1^*(K_{n_1}) \leq \mu_1(B_{n_1}) - \mu_1(B \cap B_{n_1}),$$

which implies $\mu(B \cap B_{n_1}) = \mu_1(B \cap B_{n_1}) = 0$. Replace B with $B_1 = B \setminus B_{n_1}$ and repeat the preceding argument by first observing that $B_1 \subseteq \bigcup \{B_{n_i} \setminus K_{n_i} : i = 2, \dots, r\}$. Then $\mu(B_1 \cap B_{n_2}) = \mu(B \cap \{B_{n_1} \cup B_{n_2}\}) = 0$. By continuing this process, we obtain $\mu(B \cap \bigcup B_{n_i}) = 0$ i.e. $\mu(B) = 0$. Hence, $h = g + \sum f_i \cdot \chi_{B_{n_i}} \cong 0$ and so, $\bar{U}_1(h) = 0$. The extension of $\bar{U}_1$ is therefore well defined. Furthermore, $\bar{U}_1 : \mathcal{A} \rightarrow M^\infty(Y, \mathcal{F}, \mathbb{F})/\mathcal{N}_Y$ is positive, linear, and multiplicative. As usual, choose a representative in $\bar{U}_1 f$ and call it also $\bar{U}_1 f$. Then $\bar{U}_1 : \mathcal{A} \rightarrow M^\infty(Y, \mathcal{F}, \mathbb{F})$ is an e-positive e-linear e-multiplicative operator.

By an important theorem proved by A. Ionescu Tulcea and C. Ionescu Tulcea in [6], there exists a *lifting* of $(Y, \mathcal{F}^*, \lambda)$ i.e. there exists a mapping $\rho : M^\infty(Y, \mathcal{F}^*, \mathbb{F}) \rightarrow M^\infty(Y, \mathcal{F}^*, \mathbb{F})$ which satisfies:

- 1) $\rho(f) \cong f$
- 2) $f \cong g$ implies $\rho(f) = \rho(g)$
- 3) $\rho(1) = 1$
- 4) $f \geq 0$ implies $\rho(f) \geq 0$
- 5) $\rho(\alpha f + \beta g) = \alpha \rho(f) + \beta \rho(g)$
- 6) $\rho(fg) = \rho(f) \cdot \rho(g)$.

Put $U_2 = \rho \circ \bar{U}_1$. Then $U_2 : \mathcal{A} \rightarrow M^\infty(Y, \mathcal{F}^*, \mathbb{F})$ is (exactly) positive, linear and multiplicative. Denote by S the union of the sets $S_n = \{x : U_2(\chi_{K_n})(x) \neq 0\}$. Then $\chi_S = \lim_n U_2(\chi_{K_n}) \cong \lim_n \bar{U}_1(\chi_{K_n}) \cong \lim_n \bar{U}_1(\chi_{B_n})$ and since $\bar{U}_1(\chi_{B_n})(x) \rightarrow \chi_W(x)$ a.e., we deduce that $S \cong W$.

Now, let $m \geq 1$. Then for each $x \in S_m$, define $L_x(f|_{K_m}) = U_2(f)(x)$. Then L_x is a well-defined non-zero positive multiplicative linear functional on $\{f|_{K_m} : f \in C_b(X, \mathbb{F})\}$. By Proposition 3.1, there exists a point $x_m \in X$ such that

$$(U_2 f)(x) = L_x(f|_{K_m}) = f(x_m) \quad (f \in C_b(X, \mathbb{F})).$$

For all $x \in Y \setminus S$, define Tx arbitrarily and for all $x \in S$, put $Tx = x_m$, where m is the first $n \geq 1$ with $x \in S_n$. Then, clearly, $U_2 f = \chi_S(f \circ T)$ on S , for all $f \in C_b(X, \mathbb{F})$. This shows that $\chi_S(f \circ T)$ is $\mathcal{F}^*$ -measurable and since $S \cong W$, the same is true for $\chi_W(f \circ T)$. Furthermore, we know that $U_2 f \cong 0$ on $Y \setminus W$ and hence,

$$U_2 f \cong \chi_W(f \circ T) \text{ on } Y \quad (f \in C_b(X, \mathbb{F})).$$

Finally, $\rho(\bar{U}_1 f) \cong \bar{U}_1 f$ and consequently, we obtain the relation

$$\bar{U}_1 f \cong \chi_W(f \circ T) \quad (f \in C_b(X, \mathbb{F})).$$

and the proof of step 2 is complete. $\square$

STEP 3. $Uf \cong h(f \circ T)$ for all $f \in \mathcal{V}$.

Proof. Let V, C be cozero sets in X which both have finite measure and which satisfy $V \subseteq C$. Then, we can choose zero sets $\{Z_n\}$ and $\{f_n\} \subseteq C_b^+(X, \mathbb{F})$ such that $\chi_{Z_n} \leq f_n \leq \chi_V$ and $\chi_{Z_n} \uparrow \chi_V$. Thus, we have $U_1(\chi_{Z_n})(x) \rightarrow U_1(\chi_V)(x)$ a.e. Furthermore, $U_1(\chi_{Z_n}) \cong \bar{U}_1(\chi_{Z_n}) \lesssim \bar{U}_1 f_n \lesssim \bar{U}_1(\chi_V) \cong U_1(\chi_V)$ and so

$$U_1(\chi_V) \cong \lim_n \bar{U}_1 f_n \cong \lim_n \chi_W(f_n \circ T) = \chi_W(\chi_V \circ T).$$

Now, put $\mathcal{H}_C = \{B \in \mathcal{B} : E \subseteq C \text{ and } U_1(\chi_E) \cong \chi_W(\chi_E \circ T)\}$. Then, by the above, $\mathcal{H}_C$ is an algebra of sets (in C) containing all cozero sets in X which are subsets of C . Since $f_n \downarrow 0$ implies $(U_1 f_n)(x) \rightarrow 0$ a.e., $\mathcal{H}_C$ is also a σ -algebra and consequently $\mathcal{H}_C = \{B \in \mathcal{B} : E \subseteq C\}$. But, clearly, each Baire set of finite measure is contained in at least one cozero set C (in X) with $\mu(C) < \infty$ and so, we deduce that

$$U_1 s \cong \chi_W(s \circ T) \text{ for all } s \in \mathcal{S}_X.$$

This relation is obviously extended to all $f \in \mathcal{V}^+$ (and so to all $f \in \mathcal{V}$) by using once again the property: $f_n \downarrow 0 \Rightarrow (U_1 f_n)(x) \rightarrow 0$ a.e. Finally, for each $f \in \mathcal{V}$ we have $Uf \cong h(f \circ T)$ on W and since $Uf \cong 0$ on $Y \setminus W$, we get

$$Uf \cong h(f \circ T) \quad (\text{on } Y).$$

This completes the proof of step 3. $\square$

STEP 4. The transformation T of step 2 can be chosen so as to be $\mathcal{F}^*$ -measurable and furthermore, $\mu(E) = 0 \Rightarrow \lambda(T^{-1}E \cap W) = 0$.

Proof. For all $E \in \mathcal{B}$, we have $U(\chi_E) \cong h \cdot \chi_{(T^{-1}E)}$ and so, $T^{-1}E \cap W \in \mathcal{F}^*$. Since the definition of T outside W does not affect the relation $Uf \cong \chi_W(f \circ T)$ ($f \in \mathcal{V}$), put $Ty = x_0$ on $Y \setminus W$ (where x_0 is any point in X). Then, clearly, $T^{-1}E \in \mathcal{F}^*$ for all $E \in \mathcal{B}$. Now, if $\mu(E) = 0$, then $\chi_E \cong 0$ and so $U(\chi_E) \cong 0$. But the fact that $U(\chi_E) \cong h \cdot \chi_{(T^{-1}E)}$ therefore implies that $\lambda(T^{-1}E \cap W) = 0$. Finally, if $E \subseteq F \in \mathcal{B}$ and $\mu(F) = 0$, then $\lambda(T^{-1}E \cap W) = 0$ and since $T^{-1}E \cap W \subseteq T^{-1}F \cap W$, we deduce that $T^{-1}E \cap W \in \mathcal{F}^*$. Consequently, the definition of T outside W gives $T^{-1}E \in \mathcal{F}^*$. Thus, we have shown that T is $\mathcal{F}^*$ -measurable and the proof of step 4 is therefore complete. $\square$

STEP 5. If $h_1 : Y \rightarrow \mathbb{F}$ is a $\mathcal{F}$ -measurable function and $T_1 : Y \rightarrow X$ is a $\mathcal{F}^*$ -measurable transformation such that

$$Uf \cong h_1(f \circ T_1) \text{ for all } f \in \mathcal{V},$$

then $h \cong h_1$ and $T_1^{-1}E \cap W \cong T^{-1}E \cap W$ for all $E \in \mathcal{B}$ (where $W = \{y \in Y : h(y) \neq 0\}$).

Proof. Let $\{F_N\}$ be defined as in the proof of step 1. Then $\chi_{F_N} \uparrow 1$ and $h \cong \lim_N U(\chi_{F_N})$. Hence

$$h \cong \lim_N h_1(\chi_{F_N} \circ T_1) \cong h_1 \cdot \lim_N (\chi_{F_N} \circ T_1) = h_1.$$

The fact that $T_1^{-1}E \cap W \cong T^{-1}E \cap W$ for all $E \in \mathcal{B}$, now follows easily. $\square$

STEP 6. Suppose that i) $\mathcal{B}$ has a separating sequence $\{E_n\}$, ii) $(Y, \mathcal{F}, \lambda) = (X, \mathcal{B}, \mu)$, iii) $\mathcal{V} = \mathcal{W}$, and iv) U is invertible (i.e. there exists $U^{-1} : \mathcal{V} \rightarrow \mathcal{V}$ possessing the same properties as U and satisfying $U(U^{-1}f) \cong f$ and $U^{-1}(Uf) \cong f$ for all $f \in \mathcal{V}$).

Then $T : X \rightarrow X$ can be redefined on a null set of X , so as to become an invertible non-singular completion measurable transformation on X .

Proof. Take a disjoint sequence $\{E_n\} \subseteq \mathcal{B}$ such that $\bigcup E_n = X$ and $\mu(E_n) < \infty$ for all n . If we put $f_n = U^{-1}(\chi_{X \setminus W} \cdot \chi_{E_n})$, we thus have $Uf_n \cong \chi_{X \setminus W} \cdot \chi_{E_n}$. Since $Uf_n \cong 0$ on W , then $\mu(X \setminus W \cap E_n) = 0$ and so $\mu(X \setminus W) = 0$ i.e. we can suppose that $W = X$. Now, let us apply Theorem 4.5 to U^{-1} , to get $h_1 : X \rightarrow \mathbb{F}$ and $T_1 : X \rightarrow X$ such that $U^{-1}f \cong h_1(f \circ T_1)$. By the same argument as above, we can assume that $W_1 = \{x : h_1(x) \neq 0\} = X$. Using the relations $U(U^{-1}f) \cong f$ and $U^{-1}(Uf) \cong f$ applied to $f = \chi_E$ (where $E \in \mathcal{B}$) together with the fact that $W = W_1 = X$, we obtain $E \cong (T \circ T_1)^{-1}E$ and $E \cong (T_1 \circ T)^{-1}E$. Now, let $\{S_n\}$ be a separating sequence in $\mathcal{B}$ and put $Z_1 = \bigcup (S_n \triangle (T \circ T_1)^{-1}(S_n))$ and $Z_2 = \bigcup (S_n \triangle (T_1 \circ T)^{-1}(S_n))$. Then, if we define $\Omega_1 = (X \setminus Z_1) \cup (X \setminus Z_2)$, we easily prove that Ω_1 is an element of $\mathcal{B}$ satisfying $\mu(X \setminus \Omega_1) = 0$ and $(T \circ T_1)(x) = (T_1 \circ T)(x) = x$ for all $x \in \Omega_1$. Finally, put

$$X \setminus \Omega = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^n \dots \bigcup_{r_i, s_i=0}^{\infty} (T_1^{-r_1} T^{-s_1} \dots T_1^{-r_n} T^{-s_n} (X \setminus \Omega_1)).$$

Then, using the fact that $\mu(E) = 0 \Rightarrow \mu(T^{-1}E) = 0$ and $\mu(T_1^{-1}E) = 0$, we conclude that $\mu(X \setminus \Omega) = 0$. Furthermore, $T(\Omega) \subseteq \Omega$ and $T_1(\Omega) \subseteq \Omega$. Hence, by redefining T on Ω by $Tx = x$, then, clearly, $T : X \rightarrow X$ is an invertible non-singular completion measurable transformation, satisfying $Uf \cong h(f \circ T)$. $\square$

The proof of Theorem 4.5 is now complete. $\square$

Corollary 4.7. Let $(X, \mathcal{B}, \mu)$ be as in Theorem 4.5 and let $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^q(Y, \lambda, \mathbb{F})$ (where $1 \leq p, q \leq \infty$) be an e -linear support separating operator such that $f_n \downarrow 0$ implies $(Uf_n)(x) \rightarrow 0$ a.e.

Then, there exists 1) a function $h \in M(Y, \mathcal{F}, \mathbb{F})$ which is uniquely determined a.e. and 2) a $\mathcal{F}^*$ -measurable transformation $T : Y \rightarrow X$, such that

i) $\mu(E) = 0 \Rightarrow \lambda(T^{-1}E \cap W) = 0$ (where $W = \{y \in Y : h(y) \neq 0\}$)

and

ii) $Uf \cong h(f \circ T)$ for all $f \in L^p(X, \mathcal{B}, \mathbb{F})$.

Furthermore, if $p = q = \infty$, then U is bounded iff $h \in L^\infty(Y, \lambda, \mathbb{F})$.

Finally, if $(Y, \mathcal{F}, \lambda) = (X, \mathcal{B}, \mu)$, then it is possible to choose $T : X \rightarrow X$ so that $\mu(E) = 0$ implies $\mu(T^{-1}E) = 0$ for all $E \in \mathcal{B}$, and we also have

$$U^n f \cong \prod_{i=0}^{n-1} (h \circ T^i) \cdot (f \circ T^n)$$

for all $f \in L^p(X, \mu, \mathbb{F})$ and all $n \geq 1$.

Proof. For each $f \in \mathcal{L}^p(X, \mu, \mathbb{F}) = \{f \in M(X, \mathcal{B}^*, \mathbb{F}) : \int_X |f|^p d\mu < \infty\}$, choose a representative in $Uf \cap M(Y, \mathcal{F}, \mathbb{F})$ and call it also Uf . Then, by Theorem 4.5, we get the

existence of $h : Y \rightarrow \mathbb{F}$ and $T : Y \rightarrow X$ satisfying i) and ii). In the case where $p = \infty$, it is obvious that U is bounded iff $h \in L^\infty(Y, \lambda, \mathbb{F})$. Finally, if $(Y, \mathcal{F}, \lambda) = (X, \mathcal{B}, \mu)$, redefine T on $X \setminus W$ by $Tx = x$. Then, clearly, $\mu(E) = 0$ implies $\mu(T^{-1}E) = 0$ and, by induction, U satisfies the following relation

$$U^n f \cong \prod_{i=0}^{n-1} (h \circ T^i) \cdot (f \circ T^n)$$

for all $f \in L^p(X, \mu, \mathbb{F})$ and all $n \geq 1$. $\square$

Remark 4.8. We could have stated Corollary 4.7 using condition 2) or 3) of Corollary 4.3 which are both equivalent to the condition: $f_n \downarrow 0 \Rightarrow (U f_n)(x) \rightarrow 0$ a.e.

5. Applications to bounded operators on L^p ($1 \leq p < \infty$). In this section, p will always denote a real number in $[1, \infty[$. We will also use the following definition.

Definition 5.1. Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be two arbitrary measure spaces. Then, a bounded linear support separating operator $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^p(Y, \lambda, \mathbb{F})$ is called a *Lamperti operator*.

Remark 5.2. Many operators are Lamperti. For example, Lamperti operators include arbitrary linear isometries for $p \neq 2$ (see [10, pp. 331–334]) and also positive linear isometries for $p = 2$. They also include bounded linear positive operators which have a positive inverse (see [7, p. 1208]) and bounded multiplicative linear operators. There are also other examples such as $U : L^p(\mathbb{R}, m, \mathbb{C}) \rightarrow L^p(\mathbb{R}, m, \mathbb{C})$ defined by $(Uf)(x) \cong e^{i\pi x} \cdot f(\alpha x)$ (where $\alpha \in \mathbb{R}$ and m denotes Lebesgue measure).

The following simple proposition gives a method for constructing Lamperti operators on L^p .

Proposition 5.3. Let 1) $h : Y \rightarrow \mathbb{F}$ be a $\mathcal{F}$ -measurable function and 2) $T : Y \rightarrow X$ be a $\mathcal{F}^*$ -measurable transformation, (i.e. $T^{-1}E \in \mathcal{F}^*$ for all $E \in \mathcal{B}$) such that there exists $M > 0$ with

$$\int_{T^{-1}E} |h|^p d\lambda \leq M \cdot \mu(E) \quad \text{for all } E \in \mathcal{B}. \quad (*)$$

Then the operator $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^p(Y, \lambda, \mathbb{F})$ defined by

$$Uf \cong h(f \circ T)$$

is a *Lamperti operator*.

Proof. First, U is well defined. Indeed, if $f \cong g$, then using (*), we conclude that $\mu(E) = 0 \Rightarrow \lambda(T^{-1}E \cap W) = 0$ (where $W = \{y \in Y : h(y) \neq 0\}$) and so, $h(f \circ T) \cong h(g \circ T)$. U is therefore well defined. Since it is clear that U is linear and support separating, it remains only to prove that U is bounded.

Now, (*) shows that

$$\int_X |h|^p (f \circ T) d\lambda \leq M \int_X f d\mu$$

for all positive μ -integrable simple functions f on X . Hence, the same relation holds for all $f \in M^+(X, \mathcal{B}, \mathbb{F})$. Consequently, if $f \in L^p(X, \mu, \mathbb{F})$, we have

$$\|Uf\|_p^p = \int_X |h|^p (|f|^p \circ T) d\lambda \leq M \int_X |f|^p d\mu = M \cdot \|f\|_p^p$$

which shows that U is bounded. Hence, U is a Lamperti operator and the proof is therefore complete. $\square$

The following theorem states that the converse of Proposition 5.3 holds for many interesting measure spaces $(X, \mathcal{B}, \mu)$.

Theorem 5.4. *Let 1) $(X, \mathcal{B}, \mu)$ be a tight σ -finit Baire measure space of topological space X , and 2) $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^p(Y, \lambda, \mathbb{F})$ be a Lamperti operator.*

Then, there exists 1) a function $h \in M(Y, \mathcal{F}, \mathbb{F})$ which is uniquely determined a.e. and 2) a $\mathcal{F}^*$ -measurable transformation $T : X \rightarrow X$, such that

i) there is $M > 0$ with

$$\int_{T^{-1}E} |h|^p d\lambda \leq M \cdot \mu(E) \quad \text{for all } E \in \mathcal{B}$$

and

ii) $Uf \cong h(f \circ T)$ for all $f \in L^p(X, \mu, \mathbb{F})$.

Proof. Since U is bounded, then, clearly, $f_n \downarrow 0$ implies the existence of a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $U(f_{n_k})(x) \rightarrow 0$ a.e. By Corollary 4.3, U satisfies the property: $f_n \downarrow 0 \Rightarrow (Uf_n)(x) \rightarrow 0$ a.e. and so, we get ii) by applying Corollary 4.7. To get i), note that $\|Uf\|_p^p \leq M \cdot \|f\|_p^p$ ($f \in L^p(X, \mu, \mathbb{F})$) for some $M > 0$ and hence, this inequality applied to $f = \chi_E$ together with ii), implies i). $\square$

Remarks 5.5. a) When X is a complete separable metric space, then the Borel and Baire sets are the same and so, Theorem 5.4 combined with Remarks 4.6 a and the fact that X is SMC, contains the classical result proved in [10, p. 333].

b) Theorem 5.4 is also valid for a linear support separating operator $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^q(Y, \lambda, \mathbb{F})$ (where $1 \leq q < \infty$) even if $p \neq q$.

One could ask whether it is possible to obtain a result similar to that of Theorem 5.4 but for σ -finite Radon measure spaces $(X, \mathcal{B}_o, \mu)$, where X is a topological space, $\mathcal{B}_o$ is the Borel σ -algebra on X , and μ is a σ -finite Radon measure on $(X, \mathcal{B}_o)$. In order to state such a result, recall that a Borel measure space $(X, \mathcal{B}_o, \mu)$ is said to be *completion regular* if $(X, \mathcal{B}_o^*, \mu)$ coincides with $(X, \mathcal{B}_a^*, \mu|_{\mathcal{B}_a})$, where $\mathcal{B}_a$ denotes the σ -algebra of all Baire sets in X . Of course, this is the case when X is metrisable or when X is a locally compact Hausdorff space such that every open set is σ -compact. (Note that, in the latter case, X need not be metrisable as shown, by an example, in [1, p. 229].) There are also other examples of completion regular Borel measure spaces $(X, \mathcal{B}_o, \mu)$ which are not metrisable. Such is the case for 1) any uncountable product of compact metric spaces with Borel measures of full support and 2) any locally compact σ -compact group with Haar measure. (These facts are due to Kakutani, and to Kakutani and Kodaira. See [5, pp. 83–85].) For all these examples, the following holds:

Corollary 5.6. *Let 1) $(X, \mathcal{B}_o, \mu)$ be a completion regular σ -finite Radon measure space of a topological space X , and 2) $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^p(Y, \lambda, \mathbb{F})$ be a Lamperti operator. Then the conclusions of Theorem 5.4 hold for U .*

Proof. Let $\{E_n\}$ be a sequence of pairwise disjoint Borel sets such that $0 < \mu(E_n) < \infty$ and $\mu(X \setminus \bigcup E_n) = 0$. Define μ_1 on $\mathcal{B}_o$ by

$$\mu_1(E) = \sum \frac{\mu(E \cap E_n)}{2^n \mu(E_n)}.$$

Then, we easily prove that $\mu_1|_{\mathcal{B}_a}$ is a finite tight Baire measure (where $\mathcal{B}_a$ is the σ -algebra of all Baire sets in X). Since $\mu_1|_{\mathcal{B}_a}$ is equivalent to $\mu|_{\mathcal{B}_a}$, $\mu|_{\mathcal{B}_a}$ is therefore a σ -finite tight Baire measure. The result now follows from Theorem 5.4 and the fact that $(X, \mathcal{B}_o^*, \mu) = (X, \mathcal{B}_a^*, \mu|_{\mathcal{B}_a})$, where $\mathcal{B}_a^*$ denotes the completion of $\mathcal{B}_a$ with respect to $\mu|_{\mathcal{B}_a}$. $\square$

Von Neumann's multiplication theorem (see [12, pp. 618–619]) is one of the first attempts to obtain results on bounded linear operators on L^p which are induced by measurable point transformation. His result states that every unitary operator U on $L^2[0, 1]$ satisfying $(Uf)(Ug) \in L^2[0, 1]$ and $U(fg) \cong (Uf)(Ug)$ whenever f, g and $fg \in L^2[0, 1]$, can be induced by an (invertible) measure preserving transformation. In order to show how our Theorem 5.4 contains von Neumann's result, we present the following Corollary for bounded multiplicative linear operators.

Corollary 5.7. *Let 1) $(X, \mathcal{B}, \mu)$ be as in Theorem 5.4, and 2) $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^p(Y, \lambda, \mathbb{F})$ be a bounded linear operator such that $(Uf)(Ug) \in L^p(X, \mu, \mathbb{F})$ and $U(fg) \cong (Uf)(Ug)$ whenever f, g and $fg \in L^p(X, \mu, \mathbb{F})$.*

Then, there exists 1) a a.e. unique set $W \in \mathcal{B}$ and 2) a $\mathcal{F}^*$ -measurable transformation $T : Y \rightarrow X$ such that

i) there is $M > 0$ with $\lambda(T^{-1}E \cap W) \leq M \cdot \mu(E)$ for all $E \in \mathcal{B}$ and

ii) $Uf \cong \chi_W(f \circ T)$ for all $f \in L^p(X, \mathcal{B}, \mu)$.

Furthermore, if μ, λ are finite measures with $\lambda(Y) = \mu(X)$ and if U is an isometry (i.e. $\|Uf\|_p = \|f\|_p$ for every $f \in L^p(X, \mu, \mathbb{F})$), then $\lambda(Y \setminus W) = 0$, T is measure preserving, and

$$Uf \cong f \circ T \text{ for all } f \in L^p(X, \mu, \mathbb{F}).$$

Proof. Clearly, U is a Lamperti operator and so, by Theorem 5.4, there exists a (a.e.) unique $\mathcal{F}$ -measurable function $h : Y \rightarrow \mathbb{F}$ and a $\mathcal{F}^*$ -measurable transformation $T : Y \rightarrow X$, such that

a) there is $M > 0$ with

$$\int_{T^{-1}E} |h|^p d\lambda \leq M \cdot \mu(E) \quad \text{for all } E \in \mathcal{B}$$

and

b) $Uf \cong h(f \circ T)$ for all $f \in L^p(X, \mu, \mathbb{F})$.

Since U is multiplicative, then for every $E \in \mathcal{B}$ with $\mu(E) < \infty$, we have $h^2(\chi_E \circ T) \cong$

$h(\chi_E \circ T)$ and so $h^2 \cong h$ on $T^{-1}E$. But X is σ -finite and therefore, $h^2 \cong h$ (on X). Hence, $h \cong \chi_W$ (where $W = \{y \in Y : h(y) \neq 0\}$). It follows, from a) and b), that conclusions i) and ii) of Corollary 5.7 are satisfied.

Now, suppose that μ, λ are finite measures with $\lambda(Y) = \mu(X)$ and that U is an isometry. Then, in particular, $\lambda(W) = \|U1\|_p^p = \|1\|_p^p = \mu(X)$ and so $\lambda(Y \setminus W) = 0$. Hence, $Uf \cong f \circ T$ for all $f \in L^p(X, \mu, \mathbb{F})$ and consequently, for all $E \in \mathcal{B}$, we have

$$\lambda(T^{-1}E) = \|\chi_E \circ T\|_p^p = \|U(\chi_E)\|_p^p = \|\chi_E\|_p^p = \mu(E),$$

which shows that T is measure preserving. $\square$

6. A result on set mappings induced by point mappings. As a consequence of Theorem 4.5, we obtain the following result:

Theorem 6.1. *Let 1) $(X, \mathcal{B}, \mu)$ be, as usual, a tight σ -finite Baire measure space of a topological space X , 2) $\mathcal{N}_X = \{E \in \mathcal{B} : \mu(E) = 0\}$ and $\mathcal{N}_Y = \{E \in \mathcal{F} : \lambda(E) = 0\}$, and 3) $\varphi : \mathcal{B}/\mathcal{N}_X \rightarrow \mathcal{F}/\mathcal{N}_Y$ be a set σ -homomorphism (i.e. φ preserves complements and disjoint countable unions).*

Then, there exists a $\mathcal{F}^*$ -measurable transformation $T : Y \rightarrow X$ such that

$$T^{-1}E \cap \varphi X \cong \varphi E \text{ for all } E \in \mathcal{B}.$$

Proof. For all $\{E_i : i = 1, \dots, n\} \subseteq \mathcal{B}$ and all $\{\alpha_i : i = 1, \dots, n\} \subseteq \mathbb{F}$, choose $U(\sum \alpha_i \cdot \chi_{E_i}) \in M(Y, \mathcal{F}, \mathbb{F})$ such that $U(\sum \alpha_i \cdot \chi_{E_i}) \cong \sum \alpha_i \cdot \chi_{\varphi E_i}$. Then, clearly, U is an e-linear e-positive e-multiplicative operator on the set $\mathcal{S}_X$ of all (measurable) simple functions on X . Furthermore, U also satisfies the property: $s_n \downarrow 0 \Rightarrow U(s_n)(x) \rightarrow 0$ a.e. (see, for example, [7, pp. 1209–1210]). By Theorem 4.5, there exists a $\mathcal{F}$ -measurable function $h : Y \rightarrow \mathbb{F}$ and a $\mathcal{F}^*$ -measurable transformation $T : Y \rightarrow X$ such that $Us \cong h(s \circ T)$ for all $s \in \mathcal{S}_X$. Obviously, $h \cong \chi_{\varphi X}$ and so, for all $E \in \mathcal{B}$, we have

$$\chi_{\varphi E} \cong U(\chi_E) \cong \chi_{\varphi X}(\chi_E \circ T) = \chi_{(\varphi X \cap T^{-1}E)},$$

which implies that $T^{-1}E \cap \varphi X \cong \varphi E$. $\square$

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Résumé substantiel en français. Un problème intéressant en théorie de la mesure est le suivant: sous quelles conditions un opérateur linéaire $U : L^p(X, \mu) \rightarrow L^p(X, \mu)$ tel que $(Uf)(Ug) = 0$ p.p. lorsque $fg = 0$ p.p. peut-il être induit par une transformation $T : X \rightarrow X$ mesurable par rapport à la complétion de (X, μ) ? Un problème encore plus intéressant est de trouver les conditions sous lesquelles T peut être choisi inversible lorsque 1) $p \neq 2$ et U est une isométrie inversible, ou 2) $p = 2$ et U est une isométrie positive inversible. Dans la littérature mathématique, la plupart des auteurs ont travaillé sur la résolution du deuxième problème et ce en utilisant le résultat classique démontré par von Neuman [11], lequel affirme que tout isomorphisme de σ -algèbres sur $([0, 1], \mu)$ (où μ est une mesure de Borel σ -finie quelconque sur $[0, 1]$)

est induit par une transformation mesurable inversible $T : [0, 1] \rightarrow [0, 1]$. Par exemple, Choksi a utilisé le résultat de von Neumann ainsi que l'induction transfinie de façon à prouver que notre deuxième problème est résolu lorsque μ est une mesure de Baire σ -finie sur X , X étant soit 1) un produit arbitraire d'espaces métriques séparables et complets, ou soit 2) un groupe compact, ou, d'une manière plus générale, 3) un espace localement compact et homogène sous l'action d'un groupe localement compact et σ -compact (voir [2, 3, 4]). Les méthodes employées pour démontrer ces résultats utilisent également un théorème obtenu par J. Lamperti [9] qui affirme que toute isométrie linéaire $U : L^p(X, \mu) \rightarrow L^p(X, \mu)$ (qui doit être positive lorsque $p = 2$) peut être induite par un isomorphisme de σ -algèbres sur (X, μ) . Par la suite, il s'agit de montrer que cet isomorphisme de σ -algèbres est induit par une transformation mesurable inversible. Pour compléter l'historique de notre deuxième problème, mentionnons le résultat obtenu par von Neumann [12] sur les opérateurs unitaires sur $L^2([0, 1])$, où $[0, 1]$ est muni de la mesure de Lebesgue.

D'autre part, très peu de travaux ont été réalisés sur la résolution de notre premier problème, laquelle peut certainement aider à la résolution du deuxième problème. Par conséquent, nous développons des méthodes complètement différentes de celles qui ont été exposés dans le paragraphe précédent, de façon à montrer le résultat général qui suit (voir théorème 4.5) :

Soit i) $(Y, \mathcal{F}, \lambda)$ un espace mesuré σ -fini, ii) X un espace topologique, iii) μ une mesure σ -finie sur la classe $\mathcal{B}$ des ensembles de Baire sur X telle que μ est équivalente à une mesure finie μ_1 satisfaisant la propriété : $\forall \varepsilon > 0, \exists K$, un sous-ensemble compact de X , tel que $\mu_1^*(K) > \mu_1(X) - \varepsilon$, iv) $\mathcal{W}$ un espace vectoriel de fonctions mesurables sur Y à valeurs dans le corps $\mathbb{F}$ (où $\mathbb{F} = \mathbb{R}$ ou $\mathbb{C}$), v) $\mathcal{V}$ un treillis vectoriel auto-adjoint de fonctions mesurables sur X à valeurs dans $\mathbb{F}$, contenant les fonctions caractéristiques des ensembles de mesure finie, et vi) $U : \mathcal{V} \rightarrow \mathcal{W}$ une transformation satisfaisant les propriétés :

- 1) $U(\alpha f + \beta g) = \alpha Uf + \beta Ug$ p.p. ;
- 2) $f = 0$ p.p. $\Rightarrow Uf = 0$ p.p. ;
- 3) $fg = 0$ p.p. $\Rightarrow (Uf)(Ug) = 0$ p.p. ;
- 4) $f_n \downarrow 0 \Rightarrow \forall \varepsilon > 0$, il existe une sous-suite $\{f_{n_k}\}$ de $\{f_n\}$ telle que $(Uf_{n_k})(x) \rightarrow 0$ sauf sur un ensemble de mesure $< \varepsilon$.

Alors il existe une fonction mesurable $h : Y \rightarrow \mathbb{F}$ et une transformation $T : Y \rightarrow X$ qui est mesurable par rapport à la complétion de $(Y, \mathcal{F}, \lambda)$, telles que :

- i) $\mu(E) = 0 \Rightarrow \lambda(T^{-1}E \cap W) = 0$ (où $W = \{y \in Y : h(y) \neq 0\}$), et
- ii) $Uf = h(f \circ T)$ p.p. $\forall f \in \mathcal{V}$.

Ce résultat apporte une importante contribution à la résolution de notre premier problème et il permet, entre autres, d'obtenir les corollaires suivants (voir théorèmes 5.4 et 6.1) concernant les opérateurs sur L^p et les homomorphismes de σ -algèbres :

I) Soit i) $(X, \mathcal{B}, \mu)$ et $(Y, \mathcal{F}, \lambda)$ tels que définis dans le théorème 4.5, et ii) $U : L^p(X, \mu, \mathbb{F}) \rightarrow L^p(Y, \lambda, \mathbb{F})$ un opérateur linéaire tel que $fg = 0$ p.p. $\Rightarrow (Uf)(Ug) = 0$ p.p. Alors il existe une fonction mesurable $h : Y \rightarrow \mathbb{F}$ et une transformation $T : Y \rightarrow X$ qui est mesurable par rapport à la complétion de $(Y, \mathcal{F}, \lambda)$, telles que :

- i) $\int_{T^{-1}E} |h|^p d\lambda \leq M \cdot \mu(E) \forall E \in \mathcal{B}$, pour un certain $M > 0$, et
- ii) $Uf = h(f \circ T) p.p. \forall f \in L^p(X, \mu, \mathbb{F})$.

II) Soit i) $(X, \mathcal{B}, \mu)$ et $(Y, \mathcal{F}, \lambda)$ tels que définis dans le théorème 4.5, ii) $\mathcal{N}_X = \{E \in \mathcal{B} : \mu(E) = 0\}$ et $\mathcal{N}_Y = \{E \in \mathcal{F} : \lambda(E) = 0\}$, et iii) $\varphi : \mathcal{B}/\mathcal{N}_X \rightarrow \mathcal{F}/\mathcal{N}_Y$ une fonction d'ensembles satisfaisant :

- 1) $\varphi(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} \varphi(E_i)$ lorsque les E_i sont deux à deux disjoints ;
- 2) $\varphi(X \setminus E) = Y \setminus \varphi(E)$.

Alors il existe une transformation $T : Y \rightarrow X$ qui est mesurable par rapport à la complétion de $(Y, \mathcal{F}, \lambda)$, telle que $T^{-1}E \cap \varphi X \cong \varphi E$ pour tout $E \in \mathcal{B}$.

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