

UNIQUENESS PROPERTIES OF ISOSCELES ORTHOGONALITY IN NORMED LINEAR SPACES

JAVIER ALONSO

RÉSUMÉ. On donne quelques estimations de l'orthogonalité isocèle dans les espaces normés, introduite par R. C. James. On obtient ainsi quelques propriétés d'unicité et quelques caractérisations de la convexité stricte.

ABSTRACT. Some estimations of isosceles orthogonality in normed linear spaces, introduced by R. C. James, are given. This provides uniqueness properties and strict convexity characterizations.

1. Introduction. An orthogonality in a normed linear space is a binary relation that coincides with the usual notion of orthogonality when the norm is induced by an inner product. These generalizations of the concept of orthogonality provide a good frame for developing properties about the geometric structure of the space. For example, the role they play in the characterization of inner product spaces among normed linear spaces is well-known [2]. This paper deals with isosceles orthogonality, introduced by R. C. James [3]. In a few words, the position that isosceles orthogonal vectors can adopt, which allows us to get uniqueness properties and strict convexity characterizations, is studied.

Let $(E, \|\cdot\|)$ be a real normed linear space of dimension at least two. If $x, y \in E$, then x is said to be *isosceles orthogonal* to y , ($x \perp y$), if $\|x + y\| = \|x - y\|$. Trivial properties of isosceles orthogonality, which will be used throughout the paper, are the following:

- a) It is symmetric: $x \perp y$ implies $y \perp x$.
- b) If $x \perp y$, then $x \perp -y$.
- c) If $x \perp \alpha x$, $\alpha \in \mathbb{R}$, then $\alpha x = 0$.
- d) If $x \perp y$, then $\alpha x \perp \alpha y$, for every $\alpha \in \mathbb{R}$.
- e) If $x \perp y$, then $\|x + y\| \geq \max\{\|x\|, \|y\|\}$.

On the other hand, isosceles orthogonality is, in general, *non-homogeneous* ($x \perp y$ does not imply $x \perp \alpha y$, $\alpha \in \mathbb{R}$) and *non-additive* ($x \perp y$, $x \perp z$ do not imply $x \perp y + z$). In fact, homogeneity and additivity are characteristic properties of inner product spaces [3].

Reçu le 21 décembre 1992 et, sous forme définitive, le 3 décembre 1993.

Isosceles orthogonality has the property of α -existence, that is, for every $x, y \in E$ there exists $\alpha \in \mathbb{R}$ such that $x \perp \alpha x + y$ [3]. O. P. Kapoor and J. Prasad [4] proved that the property of α -uniqueness (i.e. the above α is unique when $\|x\|\|y\| \neq 0$) is characteristic of strictly convex spaces. Corollary 2 shows that any normed linear space has the α -uniqueness property if we assume that $\|y\| \leq \|x\|$.

Another existence property, which is useful to bear in mind, is stated in the following proposition.

Proposition 1. *Let $x \in L \subset E$, where $\dim L = 2$. Then, for every $\gamma > 0$ there exists $y \in L$ such that $\|y\| = \gamma$ and $x \perp y$.*

Proof. Let $x \in L$ and let $z \in L$ be linearly independent with x . For every $\gamma > 0$ the continuous function

$$f(\lambda) = \left\| \frac{\gamma z + \gamma \lambda x}{\|z + \lambda x\|} + x \right\| - \left\| \frac{\gamma z + \gamma \lambda x}{\|z + \lambda x\|} - x \right\|, \quad \lambda \in \mathbb{R},$$

satisfies $\lim_{\lambda \rightarrow +\infty} f(\lambda) = 2 \min\{\gamma, \|x\|\} = -\lim_{\lambda \rightarrow -\infty} f(\lambda)$. Therefore, there exists $\lambda_0 \in \mathbb{R}$ such that $f(\lambda_0) = 0$ and taking $y = (\gamma z + \gamma \lambda_0 x) / \|z + \lambda_0 x\|$ we get $\|y\| = \gamma$ and $x \perp y$. $\square$

We shall name the existence property in the terms of Proposition 1 simply as *existence*. In this case, the corresponding uniqueness refers to the uniqueness (except in the sign) of y . Geometrically, the α -existence property means that in each straight line parallel to the vector x there exist orthogonal vectors to x , whereas the existence property means that in each sphere of each two-dimensional subspace to which x belongs there exist orthogonal vectors to x . It is easy to see that if an orthogonality is homogeneous then both properties, existence and α -existence, are equivalent; as well as uniqueness and α -uniqueness. Although, in general, isosceles orthogonality is not homogeneous, uniqueness and α -uniqueness are equivalent properties for this orthogonality, because from Proposition 5 uniqueness is also equivalent to strict convexity. On the other hand, it is known [1, 5] that in any normed linear space isosceles orthogonality has the uniqueness property if we fix $\gamma = \|x\|$. Corollary 4 shows that also in every normed linear space this orthogonality has the uniqueness property if we assume $\gamma \leq \|x\|$.

Along the paper we denote by $S = \{x \in E : \|x\| = 1\}$, the unit sphere of E .

2. Results. The next proposition shows where the isosceles orthogonal vectors to a given one are located in each two-dimensional subspace.

Proposition 2. *Let $x, y \in S$ and $\alpha, \beta \in \mathbb{R}$ be such that $x \perp y$ and $x \perp \alpha x + \beta y$. Then,*

$$|\alpha| \leq \begin{cases} \min \left\{ |\beta| (\|x + y\| - 1), \frac{(1 - |\beta|) (2 - \|x + y\|)}{\|x + y\|} \right\} & \text{if } |\beta| \leq 1 \\ \max \left\{ |\beta| (2 - \|x + y\|) - 1, \frac{(|\beta| - 1) (2 - \|x + y\|)}{\|x + y\|} \right\} & \text{if } |\beta| \geq 1, \end{cases}$$

and the estimations are sharp.

Proof. There is no loss of generality in taking $\alpha \geq 0$ and $\beta \geq 0$. The hypothesis says that $\|x + y\| = \|x - y\|$ and $\|(\alpha + 1)x + \beta y\| = \|(\alpha - 1)x + \beta y\|$.

Firstly, we shall see that $\alpha \leq |\beta - 1|$. The function $f(\lambda) = \|\lambda x + \beta y\|$, $\lambda \in \mathbb{R}$, is a convex function that satisfies $f(0) = \beta$, $f(-\beta) = f(\beta)$ and $f(\alpha - 1) = f(\alpha + 1)$. If $\alpha > |\beta - 1|$, then $-\beta < \alpha - 1$ and $\beta < \alpha + 1$. Therefore, $f(\lambda) = \beta$ for any $\lambda \in [-\beta, \alpha + 1]$. Then,

$$\alpha + \beta + 1 = \|(\alpha + 1)x + \beta y + \beta(x - y)\| \leq f(\alpha + 1) + f(-\beta) = 2\beta,$$

and we get $\beta \geq \alpha + 1$, against the assumption.

Let us now assume that $\beta \leq 1$. Then,

$$\|(\alpha - 1)x + \beta y\| = \|\beta(x - y) + (1 - \alpha - \beta)x\| \leq \beta\|x - y\| + 1 - \alpha - \beta, \quad (1)$$

and, taking into account that $2 = \|(\alpha - 1)x + \beta y - (\alpha + 1)x - \beta y\| \leq 2\|(\alpha - 1)x + \beta y\|$, we get

$$\alpha \leq \beta(\|x + y\| - 1).$$

On the other hand,

$$(\alpha + 1)\|x + y\| = \|(\alpha + 1)x + \beta y + (\alpha + 1 - \beta)y\| \leq \|(\alpha + 1)x + \beta y\| + \alpha + 1 - \beta,$$

and then, from (1),

$$(\alpha + 1)\|x + y\| - \|(\alpha + 1)x + \beta y\| + \beta - 1 \leq \alpha \leq \beta(\|x + y\| - 1) + 1 - \|(\alpha + 1)x + \beta y\|.$$

Therefore,

$$\alpha \leq \frac{(1 - \beta)(2 - \|x + y\|)}{\|x + y\|}.$$

Now, assume that $\beta \geq 1$. Then,

$$\beta\|x + y\| = \|(\alpha + 1)x + \beta y + (\beta - \alpha - 1)x\| \leq \|(\alpha + 1)x + \beta y\| + \beta - \alpha - 1,$$

and, therefore,

$$\alpha \leq \|(\alpha + 1)x + \beta y\| - 1 + \beta(1 - \|x + y\|). \quad (2)$$

If $\alpha \leq 1$ we have

$$\|(\alpha + 1)x + \beta y\| = \|(1 - \alpha)(x + y) + (\beta - 1 + \alpha)y\| \leq (1 - \alpha)\|x + y\| + \beta - 1 + \alpha,$$

and, from (2),

$$\alpha \leq \frac{(\beta - 1)(2 - \|x + y\|)}{\|x + y\|}.$$

On the other hand, if $\alpha \geq 1$,

$$(\alpha + 1)\|(\alpha + 1)x + \beta y\| = \|(\alpha - 1)[(\alpha + 1)x + \beta y] + 2\beta y\| \leq (\alpha - 1)\|(\alpha + 1)x + \beta y\| + 2\beta$$

and then,

$$\|(\alpha + 1)x + \beta y\| \leq \beta,$$

from which follows by (2)

$$\alpha \leq \beta(2 - \|x + y\|) - 1. \quad \square$$

The next example shows that the estimations are sharp.

Example 1. Let $1/2 \leq t \leq 1$ and let E be $\mathbb{R}^2$ endowed with a norm whose unit ball is the convex envelope of the points

$$u_1 = \left(\frac{1-2t}{t}, 1 \right), \quad u_2 = (0, 1), \quad u_3 = \left(\frac{t}{(1-t)^2 + t^2}, \frac{t(2t-1)}{(1-t)^2 + t^2} \right)$$

and their opposites. Then, the points $x = (1, 0)$ and $y = (0, 1)$ are in the unit sphere and $\|x + y\| = \|x - y\| = 1/t$, that is, $x \perp y$. If $x \perp \alpha x + \beta y$, then, by Proposition 2,

$$|\alpha| \leq \begin{cases} \alpha_1 = |\beta| \left(\frac{1}{t} - 1 \right) & \text{if } 0 \leq |\beta| \leq \frac{t(2t-1)}{(1-t)^2 + t^2} \\ \alpha_2 = (1 - |\beta|)(2t-1) & \text{if } \frac{t(2t-1)}{(1-t)^2 + t^2} \leq |\beta| \leq 1 \\ \alpha_3 = (|\beta| - 1)(2t-1) & \text{if } 1 \leq |\beta| \leq \frac{2t}{2t-1} \quad (1 \leq |\beta| \text{ if } t = \frac{1}{2}) \\ \alpha_4 = |\beta| \left(2 - \frac{1}{t} \right) - 1 & \text{if } |\beta| \geq \frac{2t}{2t-1} \quad (\text{and } t > \frac{1}{2}). \end{cases}$$

Let us assume that $\beta \geq 0$. Then it can be seen that for every β in the corresponding interval we have $x \perp \alpha_i x + \beta y$, $i = 1, 2$, and $x \perp -\alpha_j x + \beta y$, $j = 3, 4$. For example, if $0 \leq \beta \leq t(2t-1)/[(1-t)^2 + t^2]$, then,

$$0 \leq \lambda_1 := \frac{[(1-t)^2 + t^2](\beta + 1)}{4t^2 - 3t + 1} \leq 1, \quad 0 \leq \lambda_2 := \frac{[(1-t)^2 + t^2](1 - \beta)}{4t^2 - 3t + 1} \leq 1$$

and

$$(\alpha_1 + 1)x + \beta y = \lambda_1 u_3 + (1 - \lambda_1)(-u_1), \quad (\alpha_1 - 1)x + \beta y = \lambda_2(-u_3) + (1 - \lambda_2)u_1.$$

The remaining cases follows in a similar way.

Corollary 1. Let $x, y \in S$ be linearly independent.

- i) If $x \perp y$ and $x \perp \alpha x + y$, then $\alpha = 0$.
- ii) If $x \perp \alpha_1 x + \beta y$ and $x \perp \alpha_2 x + \beta y$, with $|\beta| \leq 1$, then $\alpha_1 = \alpha_2$.

Proof. Property i) is obvious from Proposition 2. To prove ii) we can assume without loss of generality that $\alpha_2 \leq \alpha_1$, $\alpha_1 \geq 0$ and $\beta > 0$. First we shall see that $\alpha_1 \leq 1$. Let us suppose that $\alpha_1 > 1$. Then,

$$\begin{aligned} (\alpha_1 + 1) \|(\alpha_1 - 1)x + \beta y\| &= \|(\alpha_1 - 1)[(\alpha_1 + 1)x + \beta y] + 2\beta y\| \\ &\leq (\alpha_1 - 1) \|(\alpha_1 + 1)x + \beta y\| + 2\beta, \end{aligned}$$

and, therefore, $\|(\alpha_1 + 1)x + \beta y\| \leq \beta$. On the other hand,

$$\alpha_1 + 1 = \|(\alpha_1 + 1)x + \beta y - \beta y\| \leq \|(\alpha_1 + 1)x + \beta y\| + \beta \leq 2\beta \leq 2,$$

and then, $\alpha_1 \leq 1$, against the assumption.

Let us suppose now that $\alpha_2 < \alpha_1$. Then, $0, \alpha_2 + 1$ and $\alpha_1 - 1$ are in the interior of the interval $[\alpha_2 - 1, \alpha_1 + 1]$ and we get from the convexity of the function $f(\lambda) = \|\lambda x + \beta y\|$ that $f(\lambda) = \beta$ for every $\lambda \in [\alpha_2 - 1, \alpha_1 + 1]$. Therefore, $\|\alpha_i x + \beta y\| = f(\alpha_i) = \beta$

and $\beta x \perp \alpha_i x + \beta y$ for $i = 1, 2$, because $\alpha_i - \beta$ and $\alpha_i + \beta$ are in $[\alpha_2 - 1, \alpha_1 + 1]$. Then,

$$x \perp \frac{\alpha_1 x + \beta y}{\|\alpha_1 x + \beta y\|}, \quad x \perp \left(\frac{\alpha_2 - \alpha_1}{\beta} \right) x + \frac{\alpha_1 x + \beta y}{\|\alpha_1 x + \beta y\|}.$$

From i) we get $\alpha_1 = \alpha_2$, contradicting the assumption. $\square$

Remark 1. Part ii) in Corollary 1 is not true when $|\beta| > 1$. For example, if E is $\mathbb{R}^2$ endowed with the maximum norm and $x = (1, 0)$ and $y = (0, 1)$, then $x \perp \alpha x + \beta y$ if $|\alpha| \leq |\beta| - 1$.

Corollary 2. For every $x, y \in E$ with $0 < \|y\| \leq \|x\|$ there exists a unique $\alpha \in \mathbb{R}$ such that $x \perp \alpha x + y$.

Proof. We only need to prove the uniqueness of α . There is no loss of generality in assuming that x and y are linearly independent. If $x \perp \alpha_i x + y$, $i = 1, 2$, then

$$\frac{x}{\|x\|} \perp \alpha_i \frac{x}{\|x\|} + \frac{\|y\|}{\|x\|} \frac{y}{\|y\|}, \quad i = 1, 2$$

and from Corollary 1 we get $\alpha_1 = \alpha_2$. $\square$

The interest of the next cumbersome proposition is to be found in its geometric interpretation. For each pair of isosceles orthogonal vectors x and y , with $\|x\| \geq \|y\|$, Proposition 3 determines the region in the two-dimensional subspace spanned by them where the corresponding restriction of the sphere of center 0 and radius $\|y\|$ can be.

Proposition 3. Let $x, y \in S$ linearly independent and $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $0 < |\beta| \leq 1$, $x \perp \beta y$ and $\|\alpha x + \gamma y\| = |\beta|$.

i) If $|\alpha\beta| \leq |\gamma|$, then

$$0 \leq |\alpha| \leq \frac{|\beta|}{\|x + \beta y\|},$$

$$\frac{\beta^2}{\|x + \beta y\|} \leq |\gamma| \leq \frac{\beta^2 (\|x + \beta y\| - |\beta| + 1)}{|\beta| + (\|x + \beta y\| - |\beta|) (\|x + \beta y\| - 1)}$$

and, more precisely,

$$(|\beta| - \|x + \beta y\|) |\alpha| + |\beta| \leq |\gamma| \leq \delta$$

where

$$\delta = \begin{cases} \min \left\{ (\|x + \beta y\| - |\beta|) |\alpha| + |\beta|, \frac{(|\beta| - |\alpha|) |\beta|}{\|x + \beta y\| - 1} \right\} & \text{if } \|x + \beta y\| \neq 1 \\ (\|x + \beta y\| - |\beta|) |\alpha| + |\beta| & \text{if } \|x + \beta y\| = 1. \end{cases}$$

ii) If $|\alpha\beta| \geq |\gamma|$, then

$$\frac{|\beta|}{\|x + \beta y\|} \leq |\alpha| \leq \frac{|\beta| (\|x + \beta y\| + |\beta| - 1)}{|\beta| + (\|x + \beta y\| - |\beta|) (\|x + \beta y\| - 1)}$$

$$0 \leq |\gamma| \leq \frac{\beta^2}{\|x + \beta y\|}$$

and

$$\nu \leq |\gamma| \leq (|\beta| - \|x + \beta y\|) |\alpha| + |\beta|,$$

where

$$\nu = \begin{cases} \frac{||\beta| - |\alpha|| \cdot |\beta|}{\|x + \beta y\| - 1} & \text{if } \|x + \beta y\| \neq 1 \\ 0 & \text{if } \|x + \beta y\| = 1. \end{cases}$$

In addition, all the above estimations are sharp.

Proof. There is no loss of generality in assuming that α, β and γ are positive. Let us suppose that $\alpha\beta \leq \gamma$. Then,

$$\gamma\|x + \beta y\| = \|(\gamma - \alpha\beta)x + \beta(\alpha x + \gamma y)\| \leq \gamma - \alpha\beta + \beta^2,$$

and, therefore,

$$\alpha\beta \leq \beta^2 + \gamma(1 - \|x + \beta y\|) \leq \beta^2 + \alpha\beta(1 - \|x + \beta y\|),$$

from which follows

$$\alpha \leq \frac{\beta}{\|x + \beta y\|}.$$

From

$$\beta = \|\alpha x + \gamma y\| = \|\alpha(x + \beta y) + (\gamma - \alpha\beta)y\| \leq \alpha\|x + \beta y\| + \gamma - \alpha\beta \quad (1)$$

it follows

$$\beta\gamma \geq \beta^2 + \alpha\beta(\beta - \|x + \beta y\|) \geq \beta^2 + \gamma(\beta - \|x + \beta y\|),$$

and then,

$$\gamma \geq \frac{\beta^2}{\|x + \beta y\|}.$$

From

$$\gamma\|x + \beta y\| = \|\beta(\alpha x + \gamma y) + (\gamma - \alpha\beta)x\| \leq \beta^2 + \gamma - \alpha\beta \quad (2)$$

we get

$$\alpha\beta \leq \gamma(1 - \|x + \beta y\|) + \beta^2,$$

and from

$$\alpha\beta + \gamma = \|\alpha(x - \beta y) - (\alpha x + \gamma y)\| \leq \alpha\|x + \beta y\| + \beta \quad (3)$$

we get

$$\gamma - \beta \leq \alpha(\|x + \beta y\| - \beta).$$

Therefore,

$$\beta(\gamma - \beta) \leq \alpha\beta (\|x + \beta y\| - \beta) \leq [\gamma(1 - \|x + \beta y\|) + \beta^2] (\|x + \beta y\| - \beta),$$

from which follows

$$\gamma \leq \frac{\beta^2 (\|x + \beta y\| - |\beta| + 1)}{|\beta| + (\|x + \beta y\| - |\beta|) (\|x + \beta y\| - 1)}.$$

Now, from (1) and (3) we get

$$(\beta - \|x + \beta y\|) \alpha + \beta \leq \gamma \leq (\|x + \beta y\| - \beta) \alpha + \beta,$$

and, if $\|x + \beta y\| \neq 1$, we get from (2)

$$\gamma \leq \frac{(\beta - \alpha)\beta}{\|x + \beta y\| - 1}.$$

Let us now assume that $\alpha\beta \geq \gamma$. Then,

$$\beta^2 = \|(\alpha\beta - \gamma)x + \gamma(x + \beta y)\| \leq \alpha\beta - \gamma + \gamma\|x + \beta y\|, \quad (4)$$

and, therefore,

$$\alpha\beta \geq \beta^2 + \gamma(1 - \|x + \beta y\|) \geq \beta^2 + \alpha\beta(1 - \|x + \beta y\|),$$

from which follows

$$\alpha \geq \frac{\beta}{\|x + \beta y\|}.$$

From

$$\alpha\|x + \beta y\| = \|(\alpha x + \gamma y) + (\alpha\beta - \gamma)y\| \leq \beta + \alpha\beta - \gamma \quad (5)$$

and

$$\alpha\beta + \gamma = \|\beta(\alpha x + \gamma y) + \gamma(x - \beta y)\| \leq \beta^2 + \gamma\|x + \beta y\| \quad (6)$$

we get

$$\alpha\beta - \beta^2 \leq \gamma(\|x + \beta y\| - 1) \leq (\beta + \alpha\beta - \alpha\|x + \beta y\|)(\|x + \beta y\| - 1),$$

from which follows

$$\alpha \leq \frac{\beta(\|x + \beta y\| + \beta - 1)}{\beta + (\|x + \beta y\| - \beta)(\|x + \beta y\| - 1)}.$$

From (5) we get

$$\gamma \leq (\beta - \|x + \beta y\|) \alpha + \beta,$$

and then,

$$\beta\gamma \leq (\beta - \|x + \beta y\|) \alpha\beta + \beta^2 \leq (\beta - \|x + \beta y\|) \gamma + \beta^2,$$

from which follows

$$\gamma \leq \frac{\beta^2}{\|x + \beta y\|}.$$

Now, if $\|x + \beta y\| \neq 1$, we get from (4)

$$\gamma \geq \frac{(\beta - \alpha)\beta}{\|x + \beta y\| - 1},$$

and from (6)

$$\gamma \geq \frac{(\alpha - \beta)\beta}{\|x + \beta y\| - 1},$$

which completes the proof. $\square$

The next example shows the sharpness of the above estimations.

Example 2. Let β and t be such that $0 < \beta \leq 1 \leq t \leq \beta + 1$ and let

$$x_1 = \left(\frac{\beta + 1 - t}{\beta + (t - \beta)(t - 1)}, \frac{\beta(t - \beta + 1)}{\beta + (t - \beta)(t - 1)} \right),$$

$$x_2 = \left(\frac{t + \beta - 1}{\beta + (t - \beta)(t - 1)}, \frac{\beta(t - \beta - 1)}{\beta + (t - \beta)(t - 1)} \right).$$

Let E be the linear space $\mathbb{R}^2$ endowed with a norm whose unit sphere S is the parallelepiped of vertex $\pm x_1, \pm x_2$. Then, we have that $x = (1, 0)$ and $y = (0, 1)$ belong to S because

$$x = \lambda(x_1 - x_2) + x_2, \quad y = \delta(x_1 + x_2) - x_2$$

with

$$0 \leq \lambda = \frac{\beta + 1 - t}{2} < 1, \quad 0 < \delta = \frac{\beta + t - 1}{2\beta} \leq 1.$$

We also have that

$$x + \beta y = t \left[\frac{\beta}{t} (x_1 - x_2) + x_2 \right], \quad x - \beta y = t \left[\frac{1}{t} (x_2 + x_1) - x_1 \right]$$

and, therefore, $\|x + \beta y\| = \|x - \beta y\| = t$.

Now, we have $\beta x_1 = \alpha_1 x + \gamma_1 y$ with $\alpha_1 \beta \leq \gamma_1$, where

$$\alpha_1 = \frac{\beta(\beta - t + 1)}{\beta + (t - \beta)(t - 1)}, \quad \gamma_1 = \frac{\beta^2(1 - \beta + t)}{\beta + (t - \beta)(t - 1)} = (t - \beta)\alpha_1 + \beta.$$

Moreover, if $t > 1$, then $\gamma_1 = (\beta - \alpha_1)\beta/(t - 1)$.

On the other hand, we have $\beta x_2 = \alpha_2 x + \gamma_2 y$, with $\alpha_2 \beta \geq |\gamma_2|$ and where

$$\alpha_2 = \frac{\beta(t + \beta - 1)}{\beta + (t - \beta)(t - 1)}, \quad \gamma_2 = \frac{\beta^2(t - \beta - 1)}{\beta + (t - \beta)(t - 1)}.$$

If $t > 1$, we also have $|\gamma_2| = |\beta - \alpha_2|\beta/(t - 1)$.

Finally, we have $\|\alpha_3 x + \gamma_3 y\| = \beta$, where

$$\alpha_3 = \frac{\beta}{t}, \quad \gamma_3 = \frac{\beta^2}{t} = (\beta - t)\alpha_3 + \beta, \quad \alpha_3 \beta = \gamma_3.$$

The next corollary removes the term $\|x + \beta y\|$ in the estimations of Proposition 3.

Corollary 3. *Let us consider the hypothesis of Proposition 3.*

i) *If $|\alpha\beta| \leq |\gamma|$, then*

$$0 \leq |\alpha| \leq |\beta|$$

and

$$\frac{\beta^2}{|\beta| + 1} \leq |\gamma| \leq \begin{cases} |\beta|(2 - |\beta|) & \text{if } 0 < |\beta| \leq 2 - \sqrt{2} \\ \frac{\beta^2}{|\beta| + 2\sqrt{2} - 3} & \text{if } 2 - \sqrt{2} \leq |\beta| \leq 1 \end{cases}$$

ii) *If $|\alpha\beta| \geq |\gamma|$, then*

$$\frac{|\beta|}{|\beta| + 1} \leq |\alpha| \leq \frac{|\beta|}{|\beta|(2\sqrt{2} - 3) + 1}$$

and

$$0 \leq |\gamma| \leq \beta^2.$$

All the above estimations are sharp.

Proof. We can assume that α, β and γ are positive.

The only non-trivial estimation in i) is the right one for γ . We know from Proposition 3 that $\gamma \leq f(\|x + \beta y\|)$, where

$$f(t) = \frac{\beta^2(t - \beta + 1)}{\beta + (t - \beta)(t - 1)}, \quad t \geq 1.$$

If $\beta \geq 2 - \sqrt{2}$, then $f(t)$ attains its maximum value at $t = \beta - 1 + \sqrt{2}$. Therefore,

$$f(\|x + \beta y\|) \leq f(\beta - 1 + \sqrt{2}) = \frac{\beta^2}{\beta + 2\sqrt{2} - 3}.$$

On the other hand, if $\beta \leq 2 - \sqrt{2}$, then $f(t)$ is monotonically decreasing and then, $f(\|x + \beta y\|) \leq f(1) = \beta(2 - \beta)$, because $\|x + \beta y\| \geq 1$.

In order to prove ii), let us consider the function

$$g(t) = \frac{\beta(t + \beta - 1)}{\beta + (t - \beta)(t - 1)}, \quad t \geq 1.$$

This function attains its maximum value in $t = (\sqrt{2} - 1)\beta + 1$. Therefore,

$$\alpha \leq g(\|x + \beta y\|) \leq g((\sqrt{2} - 1)\beta + 1) = \frac{\beta}{\beta(2\sqrt{2} - 3) + 1}.$$

The remaining estimations follow trivially from Proposition 3.

Finally, to show that the estimations are sharp we only need to replace t by 1, $\beta + 1$, $\beta(\sqrt{2} - 1) + 1$ and $\beta - 1 + \sqrt{2}$, respectively, in Example 2. $\square$

Remark 2. Bearing in mind that $x \perp \beta y$ if and only if $y \perp (1/\beta)x$, it is easy to get similar versions to Proposition 3 and Corollary 3 for the case $|\beta| > 1$.

Proposition 4. *Let $x, y \in S$ be linearly independent.*

- i) *If $x \perp \beta y$ and $x \perp \alpha x + \gamma y$, with $\|\alpha x + \gamma y\| = |\beta| \leq 1$, then $\alpha = 0$ and $\beta = \pm\gamma$.*
- ii) *If $x \perp \alpha_1 x + \beta_1 y$ and $x \perp \alpha_2 x + \beta_2 y$, with $\|\alpha_1 x + \beta_1 y\| = \|\alpha_2 x + \beta_2 y\| \leq 1$, then $(\alpha_1, \beta_1) = \pm(\alpha_2, \beta_2)$.*

Proof. i) We can assume without loss of generality that $\alpha \geq 0$, $\beta > 0$ and $\gamma > 0$. From the hypothesis it follows that the convex function $f(\lambda) = \|\lambda x + \gamma y\|$ satisfies

$$f\left(-\frac{\gamma}{\beta}\right) = f\left(\frac{\gamma}{\beta}\right), \quad f(\alpha - 1) = f(\alpha + 1)$$

$$f(0) = \gamma, \quad f(\alpha) = \beta.$$

Then, in order to prove i) we only need to see that the intervals $[-\gamma/\beta, \gamma/\beta]$ and $[\alpha - 1, \alpha + 1]$ are different, and that none of them is included in the interior of the other one, because in such case $f(\lambda)$ is constant in its union and, therefore, $\beta = \gamma$ and, from Corollary 1, $\alpha = 0$.

If $-\gamma/\beta = \alpha - 1$ and $\gamma/\beta = \alpha + 1$, then $\alpha = 0$ and $\beta = \gamma$.

Let us suppose that $-\gamma/\beta < \alpha - 1$ and $\alpha + 1 < \gamma/\beta$. Then, $\gamma > \alpha\beta$ and from Corollary 3 we get that

$$\gamma \leq \begin{cases} \beta(2 - \beta) & \text{if } 0 < \beta \leq 2 - \sqrt{2} \\ \frac{\beta^2}{\beta + 2\sqrt{2} - 3} & \text{if } 2 - \sqrt{2} \leq \beta \leq 1 \end{cases}$$

from which follows (in both cases) that $\gamma \leq 2\beta$ and, hence, $\alpha < 1$. Then,

$$\|(\alpha - 1)x + \gamma y\| = \|(\alpha - 1)(x - \beta y) + [(\alpha - 1)\beta + \gamma]y\| \leq (1 - \alpha)\|x - \beta y\| + (\alpha - 1)\beta + \gamma.$$

On the other hand,

$$\gamma\|x + \beta y\| = \|\beta[(\alpha + 1)x + \gamma y] + [\gamma - \beta(\alpha + 1)]x\| \leq \beta\|(\alpha + 1)x + \gamma y\| + \gamma - \beta(\alpha + 1),$$

and then,

$$\frac{\gamma(\|x + \beta y\| - 1) + \beta(\alpha + 1)}{\beta} \leq \|(\alpha + 1)x + \gamma y\| \leq (1 - \alpha)\|x - \beta y\| + (\alpha - 1)\beta + \gamma.$$

Now, taking into account that $x \perp \beta y$ (which implies $\|x - \beta y\| \geq \max\{1, \beta\}$) and that $\beta = \|\alpha x + \gamma y\| \geq \gamma - \alpha$, it follows that

$$\begin{aligned} (1 - \alpha)\|x - \beta y\| + (\alpha - 1)\beta + \gamma &= \|x - \beta y\| + \alpha(\beta - \|x - \beta y\|) + \gamma - \beta \\ &\leq \|x - \beta y\| + \alpha \leq \frac{\gamma(\|x - \beta y\| - 1) + \beta}{\beta} + \alpha. \end{aligned}$$

Therefore, all the above inequalities become identities. But, the last one is an identity only if $\|x - \beta y\| = 1$. Thus, we have

$$1 - \alpha + (\alpha - 1)\beta + \gamma = 1 + \alpha,$$

and then,

$$\beta(\alpha - 1) + \gamma = 2\alpha \geq 2(\gamma - \beta),$$

from which we get $\beta(\alpha + 1) \geq \gamma$, which contradicts the assumption.

Let us now suppose that $\alpha - 1 < -\gamma/\beta$ and $\gamma/\beta < \alpha + 1$. Then, $\gamma < \beta$. From the identity

$$(\alpha + 1)\|x + \beta y\| = \|(\alpha + 1)x + \gamma y + [\beta(\alpha + 1) - \gamma]y\|,$$

we get that

$$\|(\alpha + 1)x + \gamma y\| \geq (\alpha + 1)\|x + \beta y\| + \gamma - \beta(\alpha + 1),$$

and from

$$\beta\|(\alpha - 1)x + \gamma y\| = \|\gamma(x - \beta y) + [\beta(1 - \alpha) - \gamma]x\|,$$

we get

$$\|(\alpha - 1)x + \gamma y\| \leq \frac{\gamma(\|x - \beta y\| - 1) + \beta(1 - \alpha)}{\beta}.$$

Therefore,

$$\frac{\gamma(\|x - \beta y\| - 1) + \beta(1 - \alpha)}{\beta} \geq (\alpha + 1)\|x + \beta y\| + \gamma - \beta(\alpha + 1).$$

On the other hand, taking into account that $\beta = \|\alpha x + \gamma y\| \leq \alpha + \gamma$, we get that

$$\begin{aligned} (\alpha + 1)\|x + \beta y\| + \gamma - \beta(\alpha + 1) &= \|x + \beta y\| + \alpha(\|x + \beta y\| - \beta) + \gamma - \beta \\ &\geq \|x + \beta y\| - \alpha \geq \frac{\gamma(\|x + \beta y\| - 1) + \beta(1 - \alpha)}{\beta}, \end{aligned}$$

where the last inequality is an identity only if $\|x + \beta y\| = 1$, because $\gamma < \beta$. Then,

$$1 + \alpha(1 - \beta) + \gamma - \beta = 1 - \alpha,$$

and thus,

$$\gamma - \beta(\alpha + 1) = -2\alpha \leq 2(\gamma - \beta),$$

from which follows $\alpha - 1 \geq -\gamma/\beta$, contradicting the hypothesis.

ii) We can assume that $\beta_1 \neq 0$. Let

$$u = \frac{\alpha_1 x + \beta_1 y}{\|\alpha_1 x + \beta_1 y\|}, \quad \alpha = \frac{\alpha_2 \beta_1 - \alpha_1 \beta_2}{\beta_1}, \quad \beta = \|\alpha_1 x + \beta_1 y\|, \quad \gamma = \frac{\beta_2 \beta}{\beta_1}.$$

Then $x \perp \beta u$, $x \perp \alpha x + \gamma u$ and $\|\alpha x + \gamma u\| = \beta \leq 1$. From i) it follows $\alpha = 0$ and $\beta = \pm\gamma$. Therefore, $(\alpha_1, \beta_1) = \pm(\alpha_2, \beta_2)$. $\square$

Remark 3. The example on Remark 1 is also valid to see that Proposition 4 is not true if the norm of the vectors is greater than one.

Corollary 4. *Let $x \in L \subset E$, where $\dim L = 2$. Then, for every $0 \leq \gamma \leq \|x\|$ there exists a unique (except in the sign) $y \in L$, with $\|y\| = \gamma$, such that $x \perp y$.*

Proof. We only need to prove the uniqueness. Let us suppose that $x \perp y_i$, $y_i \in L$, $\|y_i\| = \gamma$, $i = 1, 2$. We can assume $\gamma > 0$ and thus x and y_1 are linearly independent. Let α and β be such that $y_2 = \alpha x + \beta y_1$. Then,

$$\frac{x}{\|x\|} \perp \frac{\|y_1\|}{\|x\|} \frac{y_1}{\|y_1\|}, \quad \frac{x}{\|x\|} \perp \alpha \frac{x}{\|x\|} + \frac{\beta \|y_1\|}{\|x\|} \frac{y_1}{\|y_1\|},$$

and

$$\left\| \alpha \frac{x}{\|x\|} + \frac{\beta \|y_1\|}{\|x\|} \frac{y_1}{\|y_1\|} \right\| = \frac{\|y_1\|}{\|x\|} \leq 1.$$

By Proposition 4 it follows $\alpha = 0$, $\beta = \pm 1$ and, thus, $y_2 = \pm y_1$. $\square$

Proposition 5. *For a normed linear space E the following are equivalent:*

- i) E is strictly convex.
- ii) *If L is a two-dimensional subspace of E , then, for every $x \in L \setminus \{0\}$ and $\gamma \geq \|x\|$ there exists a unique (except in the sign) $y \in L$ such that $\|y\| = \gamma$ and $x \perp y$.*

Proof. Let us assume that E is not strictly convex and let $u, v \in S$ be linearly independent and such that $\|u + v\| = 2$. Then, let $x = u - v$ and $y_\mu = 2(u + v) + \mu(u - v)$ with $-1 \leq \mu \leq 1$. The convex function $f(\lambda) = \|u + v + \lambda(u - v)\|$ satisfies $f(-1) = f(0) = f(1) = 2$ and, therefore, $f(\lambda) = 2$ for every $\lambda \in [-1, 1]$. Then,

$$\|y_\mu\| = 2f\left(\frac{\mu}{2}\right) = 4 > 2 \geq \|x\|$$

and

$$\|x + y_\mu\| = 2f\left(\frac{\mu + 1}{2}\right) = 4 = 2f\left(\frac{\mu - 1}{2}\right) = \|x - y_\mu\|,$$

for every $\mu \in [-1, 1]$, against ii).

Let us now assume that L is a two-dimensional subspace of E and that there exist $x, y, z \in L$ such that $y \neq \pm z$, $0 < \|x\| \leq \|y\| = \|z\|$, $x \perp y$ and $x \perp z$. Then, x and y are linearly independent and there exist $\alpha, \beta \in \mathbb{R}$, such that $z = \alpha x + \beta y$. We can assume without loss of generality that $\|x\| = 1$ and that α and β are positive. Thus,

$$\alpha = \|\alpha x + \beta y - \beta y\| \geq \|\alpha x + \beta y\| - \beta \|y\| = \|y\| |1 - \beta| \geq |1 - \beta|,$$

and, therefore, $-\beta \leq \alpha - 1$ and $\beta \leq \alpha + 1$. Moreover, at least one of the last inequalities is strict because $y \neq z$. Taking into account that the convex function $g(\lambda) = \|\lambda x + \beta y\|$ satisfies $g(0) = \beta \|y\|$, $g(-\beta) = g(\beta)$ and $g(\alpha - 1) = g(\alpha + 1)$, we get that $g(\lambda) = \beta \|y\|$ for every $\lambda \in [-\beta, \alpha + 1]$. Then,

$$\|(y + x) + (y - x)\| = 2\|y\| = \frac{g(-\beta) + g(\beta)}{\beta} = \|y + x\| + \|y - x\|,$$

and E is not strictly convex. $\square$

Acknowledgment. I am indebted to two referees and to the Chief Editor, Prof. Gilbert Labelle, for a number of helpful suggestions that led to an improvement of this paper.

Résumé substantiel en français. Une orthogonalité dans un espace normé est une relation binaire qui coïncide avec la notion usuelle d'orthogonalité quand la norme est induite par un produit scalaire. Cette généralisation du concept d'orthogonalité nous donne un bon outil pour étudier la géométrie de l'espace. Par exemple, le rôle joué par les orthogonalités généralisées pour la caractérisation des espaces préhilbertiens est bien connu [2]. Dans ce travail nous étudions l'orthogonalité isocèle, introduite par R. C. James [3]. Essentiellement on détermine la position des paires de vecteurs isocèles orthogonaux dans chaque sous-espace bidimensionnel, ce qui nous permet d'obtenir quelques propriétés d'unicité et caractériser la convexité stricte.

Soit $(E, \|\cdot\|)$ un espace normé réel de dimension ≥ 2 . Pour $x, y \in E$, on dit que x est *isocèle orthogonal* à y ($x \perp y$) si $\|x + y\| = \|x - y\|$. Cette orthogonalité possède, entre autres, la propriété de α -existence, c'est-à-dire, pour $x, y \in E$ il existe $\alpha \in \mathbb{R}$ tel que $x \perp \alpha x + y$. L'orthogonalité isocèle a aussi la propriété d'existence suivante: soit L un sous-espace bidimensionnel de E , alors, pour tout $x \in L$ et tout $\gamma > 0$, il existe $y \in L$ tel que $\|y\| = \gamma$ et $x \perp y$. Géométriquement, la propriété de α -existence exprime le fait qu'on peut trouver des vecteurs orthogonaux à x dans chaque droite parallèle à x . D'autre part, la propriété d'existence affirme que dans chaque sphère de chaque sous-espace bidimensionnel qui contient le vecteur x on peut trouver des vecteurs orthogonaux à x .

On notera par S la sphère unité dans E .

Le résultat suivant décrit l'endroit où l'on peut trouver, dans chaque sous-espace bidimensionnel, les vecteurs orthogonaux à un vecteur donné.

Proposition 2. Soient $x, y \in S$ et $\alpha, \beta \in \mathbb{R}$ tels que $x \perp y$ et $x \perp \alpha x + \beta y$. Alors,

$$|\alpha| \leq \begin{cases} \min \left\{ |\beta| (\|x + y\| - 1), \frac{(1 - |\beta|)(2 - \|x + y\|)}{\|x + y\|} \right\} & \text{si } |\beta| \leq 1 \\ \max \left\{ |\beta| (2 - \|x + y\|) - 1, \frac{(|\beta| - 1)(2 - \|x + y\|)}{\|x + y\|} \right\} & \text{si } |\beta| \geq 1, \end{cases}$$

et les bornes sont atteignables.

Corollaire 1. Soient $x, y \in S$ linéairement indépendants.

- i) Si $x \perp y$ et $x \perp \alpha x + y$, alors $\alpha = 0$.
- ii) Si $x \perp \alpha_1 x + \beta y$ et $x \perp \alpha_2 x + \beta y$, avec $|\beta| \leq 1$, alors $\alpha_1 = \alpha_2$.

O. P. Kapoor et J. Prasad [4] ont montré que la propriété de α -unicité (c'est-à-dire, unicité de α quand $\|x\|\|y\| \neq 0$) pour l'orthogonalité isocèle caractérise les espaces strictement convexes. Le résultat suivant montre que dans la sphère qui contient x , l'orthogonalité est α -unique, quel que soit l'espace normé.

Corollaire 2. Pour tout $x, y \in E$ tel que $0 < \|y\| \leq \|x\|$ il existe un seul $\alpha \in \mathbb{R}$ tel que $x \perp \alpha x + y$.

En ce qui concerne la propriété d'*unicité* associée à la propriété d'existence, c'est-à-dire, l'unicité (au signe près) de y , pour chaque $L \subset E$ et $\gamma > 0$, on montre dans [1, 5] que tout espace normé vérifie cette propriété si l'on ajoute l'hypothèse $\gamma = \|x\|$. Le Corollaire 4 montre qu'on obtient toujours l'unicité si l'on a $\gamma \leq \|x\|$.

Proposition 4. Soient $x, y \in S$ linéairement indépendants.

- i) Si $x \perp \beta y$ et $x \perp \alpha x + \gamma y$, avec $\|\alpha x + \gamma y\| = |\beta| \leq 1$, on a $\alpha = 0$ et $\beta = \pm \gamma$.
- ii) Si $x \perp \alpha_1 x + \beta_1 y$ et $x \perp \alpha_2 x + \beta_2 y$, avec $\|\alpha_1 x + \beta_1 y\| = \|\alpha_2 x + \beta_2 y\| \leq 1$, on a $(\alpha_1, \beta_1) = \pm (\alpha_2, \beta_2)$.

Corollaire 4. Soient $x \in L \subset E$, où $\dim L = 2$. Alors, pour chaque $0 \leq \gamma \leq \|x\|$, il existe un seul (au signe près) $y \in L$ tel que $\|y\| = \gamma$ et $x \perp y$.

Si une orthogonalité dans un espace normé est *homogène* (c'est-à-dire, $x \perp y$ entraîne $\alpha x \perp \beta y$, $\forall \alpha, \beta \in \mathbb{R}$) il est aisé de voir que les propriétés d'unicité et de α -unicité sont équivalentes. L'orthogonalité isocèle est homogène seulement dans les espaces préhilbertiens [3]; néanmoins les deux propriétés sont aussi équivalentes à cette orthogonalité puisque l'unicité de l'orthogonalité isocèle, comme dans le cas de la α -unicité, est une propriété équivalente à la convexité stricte.

Proposition 5. Pour tout espace normé E les propriétés suivantes sont équivalentes:

- i) E est strictement convexe.
- ii) Soit L un sous-espace bidimensionnel de E . Alors, pour chaque $x \in L \setminus \{0\}$ et chaque $\gamma \geq \|x\|$, il existe (au signe près) un unique $y \in L$ tel que $\|y\| = \gamma$ et $x \perp y$.

REFERENCES

1. J. Alonso, *Ortogonalidad en Espacios Normados*, Ph.D. Dissertation, Univ. of Extremadura, Badajoz, España, 1984.
2. D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser, Boston, 1986.
3. R. C. James, *Orthogonality in normed linear spaces*, Duke Math. J. **12** (1945) 291-302.
4. O. P. Kapoor and J. Prasad, *Orthogonality and characterizations of inner product spaces*, Bull. Austral. Math. Soc. **19** (1978) 403-416.
5. L. Zheng and Z. Yadong, *Singer orthogonality and characterizations of inner product spaces*, Arch. Math. **55** (1990) 588-594.

J. ALONSO

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE EXTREMADURA
06071 BADAJOZ, ESPAÑA