

INJECTIVITY, MITTAG-LEFFLER SERIES AND MOTZKIN PRODUCTS

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RÉSUMÉ. Soit K un corps ultramétrique complet algébriquement clos, soit D un infraconnexe et soit $H(D)$ l'espace vectoriel des éléments analytiques sur D . Soit $f \in H(D)$ une fonction homographique et soit $\phi \in H(D)$ satisfaisant $|\phi(a)| = 1$ pour un certain a dans D . On montre que trois conditions α , β , γ concernant f et ϕ sont équivalentes. Quand ces conditions sont satisfaites, la fonction $g = f\phi$ vérifie les égalités $|g(x) - g(y)| = |x - y|\sqrt{|g'(x)g'(y)|} = |f(x) - f(y)|$ et $|g'(x)| = |f'(x)|$. La condition α concerne la série de Mittag-Leffler de ϕ , la condition β concerne le produit de Motzkin de ϕ et la condition γ utilise une factorisation de ϕ en fonctions homographiques et a été obtenue dans un précédent article. En particulier soit D non borné et soit $\varphi \in H(D)$ tel que ses termes de Mittag-Leffler φ_T satisfassent $\|\varphi_T\|_D \|f/f'\|_D < \text{diam}(T)$ pour chacun des trous singuliers T de φ et $|\varphi(a)| < 1$ pour un certain $a \in D$. Alors la fonction $g = f(1 + \varphi)$ satisfait toutes les égalités ci-dessus. Voir le résumé substantiel en français à la fin de l'article.

ABSTRACT. Let K be a complete ultrametric algebraically closed field, let D be an infraconnected set and let $H(D)$ be the space of the analytic elements in D . Let $f \in H(D)$ be a homographic function and let $\phi \in H(D)$ satisfy $|\phi(a)| = 1$. Three conditions α , β , γ involving ϕ and f are shown to be equivalent. When these conditions are satisfied the function $g = f\phi$ satisfies $|g(x) - g(y)| = |x - y|\sqrt{|g'(x)g'(y)|} = |f(x) - f(y)|$ and $|g'(x)| = |f'(x)|$. Condition α involves the Mittag-Leffler series of ϕ , Condition β involves the Motzkin product of ϕ , Condition γ involves a factorization of ϕ into homographic functions and was already given in a previous article. In particular, let D be unbounded and let $\varphi \in H(D)$ have Mittag-Leffler terms φ_T satisfying $\|\varphi_T\|_D \|f/f'\|_D < \text{diam}(T)$ for every singular hole T of φ and $|\varphi(a)| < 1$ for some $a \in D$. Then the function $g = f(1 + \varphi)$ satisfies all the equalities above.

1. Introduction and Theorems. Let K be an algebraically closed complete ultrametric field whose absolute value is denoted by $|\cdot|$.

Let D be a subset of K whose closure is denoted by $\bar{D}$. We denote by $R(D)$ the K -algebra of the rational functions $h(x) \in K(x)$ with no pole in D , and we denote by $H(D)$ the topological linear space which is the completion of $R(D)$ for the topology of uniform convergence on D . The elements of $H(D)$ are called the *analytic elements* on D [3, 13].

We denote by $H_b(D)$ the set of the bounded elements of $H(D)$ and by $\|\cdot\|_D$ the norm of uniform convergence defined on $H_b(D)$, and then $H_b(D)$ provided with this norm is a K -Banach algebra. Actually $H_b(D)$ is equal to $H(D)$ if and only if D is closed and bounded [3, 4].

Let us recall that a space $H(D)$ does not contain characteristic functions different from 0 and 1 if and only if D is infraconnected [3, 4]. A set D is said to be *infraconnected* if

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for every $a \in D$, the mapping from D to R_+ $x \rightarrow |x - a|$ has an image whose closure is an interval.

Notations. Let $a \in K$, let D be a set in K . We denote by $\delta(a, D)$ the distance from a to D i.e., $\inf\{|x - a| \mid x \in D\}$.

Let $a \in K$ and let $r \in R_+$. We will denote by $d(a, r)$ the disk $\{x \in K \mid |x - a| \leq r\}$ by $d(a, r^-)$ the disk $\{x \in K \mid |x - a| < r\}$ by $C(a, r)$ the circle $\{x \in K \mid |x - a| = r\}$ and by $|K|$ the set $\{|x| \mid x \in K\}$.

For every $\alpha \in C(a, r)$, $d(\alpha, r^-)$ is called a *class* of $C(a, r)$. Let $r' > r$. We will denote by $\Gamma(a, r, r')$ the set $\{x \in K \mid r < |x - a| < r'\}$, and by $\Delta(a, r, r')$ the set $\{x \in K \mid r \leq |x - a| \leq r'\}$.

We can now recall the definition of the holes.

Definitions. Let D be a closed subset of K . Let R be its diameter and let $a \in D$. We put $\tilde{D} = d(a, R)$ if $R < +\infty$ and $\tilde{D} = K$ if $R = \infty$ and we know that D is in the form

$$\tilde{D} \setminus \left(\bigcup_{i \in I} d(\alpha_i, r_i^-) \right)$$

with $d(\alpha_i, r_i^-) \cap d(\alpha_j, r_j^-) = \emptyset$ whenever $i \neq j$ and with $r_i = \delta(\alpha_i, D)$ whenever $i \in I$. The disks $d(\alpha_i, r_i^-)$, $(i \in I)$ are called the *holes* of D .

When D is open, an element $f \in H(D)$ is said to be *strictly injective* [15] or to be a *universal generator* [6] if it is injective in D and such that $f'(x) \neq 0$ whenever $x \in D$.

When K has characteristic zero, it is well known that every injective element of $H(D)$ is strictly injective.

In [8] we found a sufficient condition to obtain strictly injective analytic elements with help of the strongly copiercing sequences, a little specialization of the copiercing sequences introduced in [18].

Let D be a set of diameter $R \in]0, \infty]$. A couple $(\alpha, \beta) \in (K \setminus \bar{D})^2$ will be said to be a *strongly copiercing couple associated to D* if $|\alpha - \beta| < \delta(\alpha, D)$.

Hence it is seen that either both α, β belong to a same hole of D or both α, β belong to $K \setminus \tilde{D}$ and more precisely to a same class of a circle whose center lies in $\tilde{D}$.

A finite sequence $(a_n, b_n)_{1 \leq n \leq q}$ will be said to be a *strongly copiercing sequence associated to D* if for every $n = 1, \dots, q$ (a_n, b_n) is a strongly copiercing couple associated to D .

Finally, an infinite sequence $(a_n, b_n)_{n \in \mathbb{N}}$ will be said to be a *strongly copiercing sequence associated to D* if for every $n \in \mathbb{N}$, (a_n, b_n) is a strongly copiercing couple associated to D and if

$$\lim_{n \rightarrow \infty} \frac{|a_n - b_n|}{\delta(a_n, D)} = 0$$

As usual, by *homographic function* we mean a function in the form $(ax + b)/(cx + d)$ with $ad - bc \neq 0$ (in particular so is a function $ax + b$).

Here a homographic function $\phi \in R(D)$ will be said to be *strongly copiercing with respect to D* if ϕ is in the form $(x - a)/(x - b)$ with (a, b) a strongly copiercing couple associated to D .

In the same way, a sequence $(\phi_n)_{n \in I}$ of homographic functions will be said to be *strongly copiercing with respect to D* if each ϕ_n is in the form $(x - a_n)/(x - b_n)$ with $(a_n, b_n)_{n \in I}$ a strongly copiercing sequence associated to D .

Given a homographic function f , both f/f' and $1/f'$ are easily seen to be polynomials of degree at most 2.

Proposition 1 below will help us apply Condition γ) in the theorem which follows.

Proposition 1. *Let D be an infraconnected set, let $f \in R(D)$ be a homographic function and let $\phi \in R(D)$ be a strongly copiercing homographic function with respect to D .*

If the pole of ϕ belongs to $K \setminus \tilde{D}$ then we have

$$\left\| \frac{f}{f'} \phi' \right\|_D = \left\| \frac{f}{f'} \right\|_D \|\phi'\|_D.$$

If the pole of ϕ belongs to a hole T of D then we have

$$\left\| \frac{f}{f'} \phi' \right\|_D = \left\| \frac{f}{f'} \right\|_T \left(\left\| \frac{1}{\phi'} \right\|_T \right)^{-1} = \left\| \frac{f}{f'} \right\|_T \|\phi'\|_D.$$

In [8] we have obtained this Theorem (Relation (4) is due to L. Haddad):

Theorem. *Let D be an open set, let $h \in R(D)$ be a homographic function, let I be a subset of N , let $(\phi_n)_{n \in I}$ be a strongly copiercing sequence with respect to D such that for every $i \in I$ the homographic function $\phi_i = (x - \alpha_i)/(x - \beta_i)$ satisfies $|h(x)\phi'_i(x)| < |h'(x)|$ whenever $x \in D$. Let $g = h \prod_{i \in I} \phi_i$.*

Then g is strictly injective in D and satisfies

- (1) $|g(x)| = |h(x)|$ whenever $x \in D$
- (2) $|g'(x)| = |h'(x)|$ whenever $x \in D$
- (3) $|g(y) - g(x)| = |h(y) - h(x)|$ whenever $x, y \in D$
- (4) $|g(y) - g(x)|^2 = |y - x|^2 |g'(x)| |g'(y)|$ whenever $x, y \in D$.

In particular we notice that the condition “ $|h(x)\phi'_i(x)| < |h'(x)|$ whenever $x \in D$ ” is obviously satisfied when $\|(h/h')\phi'_i\|_D < 1$.

Here we give two other sufficient conditions on the elements $\phi \in H(D)$ to assure that $h\phi$ is injective in D , provided D is infraconnected. The first condition involves the Mittag-Leffler series and the second one involves the Motzkin Factors of ϕ . These two conditions will be shown to be equivalent to the condition “ $\phi = \prod_{n \in I} \phi_n$ with $\|h/h'\phi'_n\|_D < 1 \forall n \in I$ ”.

First we have to remember the Mittag-Leffler series of an analytic element on an infraconnected closed set.

When D is not bounded, we denote by $H_0(D)$ the subspace of the $f \in H(D)$ such that

$$\lim_{\substack{|x| \rightarrow +\infty \\ x \in D}} |f(x)| = 0.$$

Let D be an infraconnected closed set and let $f \in H(D)$. We know that f admits a unique series $\sum_{n=0}^{\infty} h_n$ named the *Mittag-Leffler series of f on D* the sum of which is equal to f , where $h_0 \in H(\tilde{D})$ and where for each $n \geq 1$, there exists a hole T_n of D

such that $h_n \in H_0(K \setminus T_n)$. (The sequences (T_n) , (h_n) are obviously injective [12, 13, 16]).

Moreover, for each $n \in N^*$ we have $\|h_n\|_D = \|h_n\|_{K \setminus T_n}$ and $\lim_{n \rightarrow \infty} h_n = 0$.

The holes T_n are called the *f-holes* (assuming $h_n \not\equiv 0$) and h_n is called the *Mittag-Leffler term of f associated to the hole T_n* .

Moreover, if $f \in H_b(D)$ then $\|h_0\|_D = \|h_0\|_{\tilde{D}}$ and we have $\|f\|_D = \sup_{n \in N} \|h_n\|_D$.

Besides, if D is not bounded and if $f \in H_b(D)$ then h_0 is a constant.

Now given a *f-hole* T , the Mittag-Leffler term of f associated to T will be denoted by $\bar{f}_T$. Thus in the Mittag-Leffler series of f on D above, we have $h_n = \bar{f}_{T_n}$ whenever $n \in N^*$.

In the same way we will always denote by $\bar{f}_0$ the element h_0 of $H(\tilde{D})$ defined in the Mittag-Leffler series of f on D above.

Now we have to recall the definition of the Motzkin factors of f on D [14].

An element $f \in H(D)$ is said to be *invertible* in $H(D)$ if f has no zero in D and if the function $1/f$ belongs to $H(D)$.

Besides f is said to be *semi-invertible* in $H(D)$ if it factorizes in the form Pg where P is a polynomial whose zeros belong to D and g is an element of $H(D)$ invertible in $H(D)$. (In particular a quasi-invertible element [5] is semi-invertible while the two notions coincide when D is open).

In [14] E. Motzkin showed a semi-invertible element $f \in H(D)$ to have a unique factorization in the form $f^0 \left(\prod_{n=1}^{\infty} f^{T_n} \right)$ where $(T_n)_{n \in N^*}$ is the sequence of the *f-holes*, f^0 belongs to $H(\tilde{D})$ and has all its zeros in D , for each $n \in N^*$, f^{T_n} belongs to $H(K \setminus T_n)$ and is invertible in $H(K \setminus T_n)$, and there is a (unique) integer q_n such that for every $\alpha \in T_n$, $(x - \alpha)^{q_n} f^{T_n}(x)$ has limit 1 when $|x|$ goes to $+\infty$ ($x \in D$).

Besides q_n is equal to zero for all but finitely many $n \in N^*$. Given a *f-hole* T_n we put $m(f, T_n) = -q_n$. We are now able to state the Theorem.

Theorem. Let D be an infraconnected open set, let $a \in D$. Let $f \in R(D)$ be a homographic function and let $\phi \in H_b(D)$ be semi-invertible and satisfy $|\phi(a)| = 1$. Statements $\alpha)$, $\beta)$, $\gamma)$ below are equivalent.

$\alpha)$ For every ϕ -hole T we have $\|\bar{\phi}_T\|_D \|f/f'\|_T < \text{diam}(T)$ and if D is bounded $\bar{\phi}_0$ satisfies

$$\|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D \left\| \frac{f}{f'} \right\|_D < \text{diam}(D).$$

$\beta)$ ϕ is invertible in $H(D)$ and for every ϕ -hole T we have $\|\phi^T - 1\|_D \|f/f'\|_T < \text{diam}(T)$ and if D is bounded ϕ^0 satisfies

$$\|\phi^0 - \phi^0(a)\|_D \left\| \frac{f}{f'} \right\|_D < \text{diam}(D).$$

$\gamma)$ ϕ is in the form $B \prod_{n \in I} \phi_n$ with $B \in C(0, 1)$ and with $(\phi_n)_{(n \in I)}$ a strongly copiercing sequence of homographic functions with respect to D satisfying

$$\left\| \frac{f}{f'} \phi'_n \right\|_D < 1 \text{ whenever } n \in I.$$

Besides if Conditions α), β), γ) are satisfied, then ϕ satisfies $\|\phi - \phi(a)\|_D < 1$ and the product $g = f\phi$ is strictly injective in D and satisfies

- (1) $|g(x)| = |f(x)|$
- (2) $|g'(x)| = |f'(x)|$
- (3) $|g(y) - g(x)| = |f(y) - f(x)|$
- (4) $|g(y) - g(x)|^2 = |g'(y)g'(x)||y - x|^2$ whenever $x, y \in D$.

Corollary. Let D be a bounded (resp. unbounded) open infraconnected set. Let $f \in R(D)$ be a homographic function, let $a \in D$ and let $\varphi \in H_b(D)$ satisfy $\|\bar{\varphi}_T\|_D \|f/f'\|_T < \text{diam}(T)$ for every φ -hole T and $\|\bar{\varphi}_0\|_D \|f/f'\|_D < \text{diam}(D)$ (resp. and $|\varphi(a)| < 1$).

Then we have $\|\varphi\|_D < 1$ and the function $g = f(1 + \varphi)$ is strictly injective in D and satisfies Conditions (1), (2), (3), (4) of the Theorem.

Example 1. Let $D = d(0, 1) \setminus d(b, \rho^-)$ with $|b| = r \in]0, 1[$ and $\rho \in]0, r[$. Let $q \in \mathbb{N}$, let $\varepsilon \in K$ satisfy $|\varepsilon| < \rho^{q+1}/r$, let $f(x) = x$ and let $g(x) = x(1 + \varepsilon/(x - b)^q)$. Then g is strictly injective in D and satisfies Conditions (1), (2), (3), (4).

Indeed, we may put $T = d(b, \rho^-)$. Then T is a hole of D and we have $\varphi = \bar{\varphi}_T$, $\|\bar{\varphi}_T\|_D = |\varepsilon|/\rho^q < \rho/r$ while $\|f/f'\|_T = r$ hence it is seen that $\|f/f'\|_T \|\bar{\varphi}_T\|_D < \text{diam}(T)$.

Example 2. Let $E = d(0, 1)$, let $(r_n)_{n \in \mathbb{N}}$ be a sequence in $]0, 1[$, let $(\beta_n)_{n \in \mathbb{N}}$ be a sequence in $d(0, 1)$ such that $|\beta_n - \beta_m| \geq \max(r_n, r_m)$ whenever $n \neq m$, let $T_n = d(\beta_n, r_n^-)$ ($n \in \mathbb{N}$) and let $D = d(0, 1) \setminus \left(\bigcup_{n \in \mathbb{N}} T_n \right)$

Let $\alpha \in K \setminus \left(\bigcup_{n \in \mathbb{N}} T_n \right)$, let $\beta \in (K \setminus d(0, 1)) \cup T_0$ with $\alpha \neq \beta$, and let $f(x) = (x - \alpha)/(x - \beta)$.

For each $n \in \mathbb{N}$, let $\varphi_n(x) = \sum_{j=1}^{\infty} \epsilon_{n,j}/(x - \beta_n)^j$ satisfy

$$|\epsilon_{n,j}| < \frac{|\alpha - \beta| r_n^{j+1}}{|\alpha - \beta_n| |\beta - \beta_n|} \quad (31)$$

$$\lim_{n+j \rightarrow +\infty} \frac{|\epsilon_{n,j}|}{r_n^j} = 0. \quad (32)$$

Hence φ_n is seen to belong to $H(D)$ and satisfy

$$\|\varphi_n\|_D = \sup_{j \in \mathbb{N}^*} \frac{|\epsilon_{n,j}|}{(r_n)^j} \quad (33)$$

so that the sequence (φ_n) goes to zero.

Now let $\varphi_0 = \sum_{j=1}^{\infty} \epsilon_{0,j} x^j$ satisfy

$$\lim_{j \rightarrow \infty} \epsilon_{0,j} = 0 \quad (34)$$

$$|\epsilon_{0,j}| < \frac{|\alpha - \beta|}{\max(|\alpha|, 1) \max(|\beta|, 1)} \quad (35)$$

whenever $j \in \mathbb{N}^*$.

Let $\phi = \prod_{n=0}^{\infty} (1 + \varphi_n)$ and let $\psi = 1 + \sum_{n=0}^{\infty} \varphi_n$

We will show that $f\phi$ and $f\psi$ are strictly injective in D and satisfy Conditions (1), (2), (3), (4) of the Theorem in place of g .

Actually we just check that ϕ satisfies Condition β) in the Theorem while ψ satisfies Condition α).

For every $n \geq 1$ we see that α and β do not belong to T_n hence we have

$$\left\| \frac{f}{f'} \right\|_{T_n} = \frac{|\alpha - \beta_n| |\beta - \beta_n|}{|\alpha - \beta|}. \quad (36)$$

In the same way f satisfies

$$\left\| \frac{f}{f'} \right\|_{\bar{D}} = \frac{\max(|\alpha|, 1) \max(|\beta|, 1)}{|\alpha - \beta|}. \quad (37)$$

Now by Proposition 3 we have $\|f/f'\|_{T_n} \geq r_n$. Hence by (31) and (36) we see that $\|\varphi_n\| < 1$ whenever $n \in N^*$. Besides by Proposition 3 we have $\|f/f'\|_{\bar{D}} \geq 1$ hence (as D is infraconnected) $\|f/f'\|_D = \|f/f'\|_{\bar{D}} \geq 1$. Now for each $n \in N^*$, by (31) and (33) we see that $\|\varphi_n\|_D = \|\varphi_n\|_{T_n} \leq r_n |\alpha - \beta| / |(\alpha - \beta_n)(\beta - \beta_n)|$, hence by (36) we have

$$\|\varphi_n\|_D < r_n \left(\left\| \frac{f}{f'} \right\|_{T_n} \right)^{-1} \leq 1. \quad (38)$$

In the same way by (35) and (37) we see that

$$\|\varphi_0\|_D < \left(\left\| \frac{f}{f'} \right\|_D \right)^{-1} \leq 1. \quad (39)$$

Hence by (38) and (39) ϕ and ψ satisfy

$$\|\phi - 1\|_D = \|\psi - 1\|_D < 1. \quad (40)$$

Now for each $n \in N^*$ we have $\phi^{T_n} = 1 + \varphi_n$ and $\phi^0 = 1 + \varphi_0$ while $\bar{\psi}_{T_n} = \varphi_n$ and $\bar{\psi}_0 = \varphi_0$.

Besides by (38) we see that we have

$$\|\phi^{T_n} - 1\|_D \left\| \frac{f}{f'} \right\|_{T_n} < r_n \quad (41)$$

and by (39) we have $\|\phi^0 - 1\|_D \|f/f'\|_D < 1$ hence

$$\|\phi^0 - \phi^0(0)\|_D \left\| \frac{f}{f'} \right\|_D < 1.180 \quad (42)$$

Thus ϕ clearly satisfies Condition β in the Theorem.

In the same way ψ satisfies

$$\bar{\psi}_{T_n} = \varphi_n = \phi^{T_n} - 1 \quad (43)$$

whenever $n \in N^*$ and

$$\bar{\psi}_0 = \varphi_0 = \phi^0 - 1. \quad (44)$$

Hence by (40) we have $|\psi(0) - 1| < 1$ and by (41), (42), (43), (44) ψ satisfies Condition α) in place of ϕ .

In [8] a conjecture was given in order to suggest a characterization of the injective analytic elements when D has certain properties. Let us remember the definition of the hyper-infraconnected sets.

Definition. A set D will be said to be *hyper-infraconnected* if for every $a \in \tilde{D}$, for every $r \in [\delta(a, D), \delta(a, K \setminus \tilde{D})] \cap |K|$ all the classes of $C(a, r)$ are included in D but finitely many ones.

Example. A disk $d(a, r)$ is hyper-infraconnected. But if $r \in |K|$ then $d(a, r^-)$ is not. Every affine infraconnected set is hyper-infraconnected.

Here is the first conjecture given in [8].

Conjecture 1. Let D be an open hyper-infraconnected bounded (resp. unbounded) set and let $g \in H(D)$. Then g is strictly injective if and only if it is in the form $f \prod_{i \in I} \phi_i$ with f a homographic function in $R(D)$ and $(\phi_i)_{i \in I}$ a strongly copiercing sequence of homographic functions with respect to D , satisfying $\|(f/f')\phi'_i\|_D < 1$ whenever $i \in I$.

Thus when D is hyper-infraconnected, this Conjecture suggests that an element $g \in H(D)$ is injective if and only if it has a factorization $f\phi$ satisfying Condition γ). Since α), β), γ) are actually equivalent, this conjecture seems to be somewhat strenghtened by this equivalence.

Besides, as it was noticed in [8], when D is hyper-infraconnected, the condition " $\|(f/f')\phi'_i\|_D < 1$ " is equivalent to the apparently weaker condition " $|f(x)\phi'_i(x)| < |f'(x)|$ for all $x \in D$ " because in a strongly infraconnected set and in particular in a hyper-infraconnected set, given an element $g \in H(D)$ such that $\|g\|_D \in |K|$, g does reach its maximum at a point of D [7].

Now we notice that the injective analytic elements we know in an infraconnected set (hyper-infraconnected or not) do satisfy the finite increasings Relation (4) again. For example so does $(1+x)^q$ (with q prime to p), in $d(0, 1^-)$.

Reciprocally when a non constant analytic element satisfies Relation (4) in an infraconnected set D it is clearly injective in D . Indeed let $f \in H(D)$ satisfy Relation (4). If f' admits any zero a in D , by Relation (4) we have $f(x) = f(a)$ for every x in D . Hence if f is not a constant, f' has no zero and therefore f is injective by Relation (4). Finally if Conjecture 1 is true we see that an analytic element f which is not constant in the union of a chained family of hyper-infraconnected open sets, is injective if and only if it satisfies Relation (4).

However, this equivalence has no chance to hold any more when D is not infraconnected. Indeed, let D be not infraconnected and let a, b belong to D so that certain annulus $\Gamma(a, r', r'')$ satisfy $\Gamma(a, r', r'') \cap D = \emptyset$ with $0 < r' < r'' < |a - b|$. Let $f(x) = \alpha(x - a)$ with $0 < |\alpha| < 1$ for $|x - a| \leq r'$ and let $f(x) = x - a$ for $|x - a| \geq r''$. We know that $f \in H(D)$ [4]. Besides f is seen to be injective in D . However we have $|f(b) - f(a)| = |b - a|$ while $|f'(a)f'(b)| = |\alpha| < 1$ hence Relation (4) is not satisfied.

Besides we may construct certain infraconnected open sets D with injective elements $f \in H(D)$ which do not satisfy Relation (4). Indeed, for example let us assume the residue characteristic p different from 2. Let $\rho \in]0, 1[$. For every $r \in [\rho, 1] \cap |K|$ we may choose $a_r \in d(1, 1^-)$, $b_r \in d(-1, 1^-)$ satisfying $|a_r - 1| = |b_r + 1| = r$ and $|a_r + b_r| = r$. Now, we put $E = \bigcup_{\rho \leq r \leq 1} (d(a_r, r^-) \cup d(b_r, r^-))$. Let $f(x) = x^2$. We check that f is injective in E and satisfies $|f(a_r) - f(b_r)| = r$ while $|a_r - b_r| = 1$. As $|f'(x)| = 1$ whenever $x \in E$, f does not satisfy Relation (4).

Let us remember that an analytic element f of $H(D)$ is said to be a *bianalytic element* in D if f is injective in D and such that the reciprocal function f^{-1} defined in $f(D)$

belongs to $H(f(D))$. Here the counter-example $f(x) = x^2$ we have just constructed is not bianalytic in E . By [6] and [15] we know that if D is an affine set and if an element f of $H(D)$ is strictly injective in D then f is bianalytic. Finally, the Theorem above suggests Conjecture 2 and Conjecture 3.

Conjecture 2. Let D be an infraconnected set, let $a \in D$, let f be a homographic function and let $\phi \in H_b(D)$ satisfy $\phi(a) \neq 0$ and Conditions α , β , γ of the Theorem. Then $f\phi$ is a bianalytic element of $H(D)$.

Conjecture 3. Let D be an infraconnected open set. A non constant element of $H(D)$ is bianalytic if and only if it satisfies Relation (4).

2. Circular filters, Propositions 2 and 3. Let us recall the valuation defined on K and the valuation function associated to an analytic element.

Let $\log$ be a real logarithm function of base $\theta > 1$. We define a valuation v in K^* by $v(x) = -\log |x|$.

Let $a \in K$ and let $r \in R_+$. We call *circular filter of center a of diameter r* the filter $\mathcal{F}$ that admits as a generating system the annuli $\Gamma(\alpha, r', r'')$ with $\alpha \in d(a, r^-)$, $r' < r < r''$ [5, 11].

Now let D be a set such that $\mathcal{F}$ is secant with D . The intersection $\mathcal{F}_D$ of $\mathcal{F}$ with D is called *circular filter of center a , of diameter r , on D* [5, 11].

Then we know that for every $f \in H(D)$, $|f(x)|$ has a limit $|f|_D$ along $\mathcal{F}_D$ and the mapping $|\cdot|_{\mathcal{F}_D}$ is a multiplicative semi-norm in $H(D)$ [5, 11].

In particular, when $D \cap d(a, r^-) \neq \emptyset$ or when $D \cap (K \setminus d(a, r)) \neq \emptyset$ we know that

$$|f|_{\mathcal{F}_D} = \lim_{\substack{|x-a| \rightarrow r \\ |x-a| \neq r \\ x \in D}} |f(x)|.$$

Besides, given a polynomial, or more generally an element f of $H(D \cup d(a, r))$, we have $|f|_{\mathcal{F}_D} = \|f\|_{d(a, r)} = \|f\|_{d(a, r^-)}$. As a consequence, if r belongs to $|K|$ so does $\|f\|_{d(a, r^-)}$.

Now let $\mu \in R$ be such that the circular filter $\mathcal{F}$ of center a , of diameter $r = \theta^{-\mu}$ is secant with D . We put $v_a(f, \mu) = -\log |f|_{\mathcal{F}_D}$.

In particular when $a = 0$ we just put $v(f, \mu) = v_a(f, \mu)$.

When $D \cap d(a, r^-) \neq \emptyset$ or when $D \cap (K \setminus d(a, r)) \neq \emptyset$ we see that

$$v_a(f, -\log(r)) = \lim_{\substack{|x-a| \rightarrow r \\ |x-a| \neq r \\ x \in D}} v(f(x)).$$

Now let D be an infraconnected set and let $T = d(a, r^-)$ be a hole of T . The circular filter $\mathcal{F}$ of center a , of diameter r is certainly secant with D . Let $\mathcal{F}_D$ be its intersection with D . For convenience we denote by $|\cdot|_T$ the multiplicative semi-norm $|\cdot|_{\mathcal{F}_D}$. Hence, in particular, if D has diameter $R > r$ then we have

$$|f|_T = \lim_{\substack{|x-a| \rightarrow r \\ |x-a| > r \\ x \in D}} |f(x)|.$$

In the same way when D is bounded and such that $\tilde{D} = d(\alpha, R)$ we will denote by $|f|_D$ the multiplicative semi-norm defined by the circular filter on D of center α , of diameter R .

Proposition 2 below is a little bit more accurate than Proposition 1 by using the notations $|\cdot|_T$ and $|\cdot|_D$.

Proposition 2. *Let D be an infraconnected set, let $f \in R(D)$ be a homographic function and let $\phi \in R(D)$ be a strongly copiercing homographic function with respect to D . If the pole of ϕ belongs to $K \setminus \tilde{D}$ then we have*

$$\left\| \frac{f}{f'} \phi' \right\|_D = \left\| \frac{f}{f'} \right\|_D \|\phi'\|_D = \left| \frac{f}{f'} \right|_D |\phi'|_D$$

If the pole of ϕ belongs to a hole T of D then we have

$$\left\| \frac{f}{f'} \phi' \right\|_D = \left| \frac{f}{f'} \phi' \right|_T = \left\| \frac{f}{f'} \right\|_T \left(\left\| \frac{1}{\phi'} \right\|_T \right)^{-1} = \left\| \frac{f}{f'} \right\|_T \|\phi'\|_D.$$

Proof. We first suppose that the pole b of ϕ belongs to $K \setminus \tilde{D}$, hence its zero a also belongs to $K \setminus \tilde{D}$, and we have $\|\phi'\|_D = \|\phi'\|_{\tilde{D}} = |\phi'(x)| = |a - b|/(\delta(b, D))^2$ whenever $x \in D$.

Hence we see that $\|(f/f')\phi'\|_D = \|f/f'\|_D |a - b|/(\delta(b, D))^2 = \|f/f'\|_D \|\phi'\|_D$.

We now suppose that b belongs to a hole $T = d(b, r^-)$ of D , hence so does a .

We know that f/f' is a polynomial P of degree ≤ 2 . Since $\phi' = (a - b)/(x - b)^2$ we see that for every $x \in D$ we have $v(P(x)) \geq v_b(P, v(x - b))$ while $v(\phi'(x)) = -2v(x - b) + v(a - b)$ hence $v(\phi'(x)P(x)) \geq v_b(P, v(x - b)) - 2v(x - b) + v(a - b)$.

But as $\deg P \leq 2$ it is clear that $v_b(P, \mu) \geq v_b(P, -\log(r)) + 2(\mu + \log(r))$.

Hence we have $v(\phi'(x)P(x)) \geq v_b(P, -\log(r)) + v(a - b) + 2(v(x - b) - (-\log(r)) - v(x - b))$ and then we see that $v(\phi'(x)P(x)) \geq v_b(\phi'P, -\log(r))$.

But $v_b(\phi'P, -\log(r)) = -\log |\phi'P|_T$ hence we have $|\phi'(x)P(x)| \leq |\phi'P|_T$ whenever $x \in D$ and therefore $\|\phi'P\|_D = |\phi'P|_T = |\phi'|_T |P|_T = |\phi'|_T \|P\|_T$.

Now $|\phi'|_T = |b - a|/r^2 = \|\phi'\|_D$ hence we have obtained $\|\phi'P\|_D = \|P\|_T \|\phi'\|_D$.

Since $(\phi')^{-1}$ is a polynomial we also have $|(\phi')^{-1}|_T = \|(\phi')^{-1}\|_T$ hence $|\phi'|_T = (\|1/\phi'\|_T)^{-1}$ and finally $|\phi'|_T \|P\|_T = (\|1/\phi'\|_T)^{-1} \|P\|_T$.

This ends the Proof of Proposition 2. $\square$

Proposition 3. *Let h be a homographic function and let $\mathcal{F}$ be a circular filter on K , of center α , of diameter $r > 0$. Then we have $|h/h'|_{\mathcal{F}} \geq r$.*

Proof. P still denotes h/h' . Let $\mu = -\log(r)$. It is clearly equivalent to show

$$v_\alpha(P, \mu) \leq \mu \quad (a)$$

When h is in the form $A(x - \beta)$ we have $P = x - \beta$ hence

$$v_\alpha(P, \mu) = \inf(\mu, v(\beta)) \leq \mu.$$

In the same way when h is in the form $A/(x - \gamma)$ we have $P = -(x - \gamma)$ hence

$$v_\alpha(P, \mu) = \inf(\mu, v(\gamma)) \leq \mu.$$

We now suppose h in the form $A((x - \beta)/(x - \gamma))$. Hence we have $P = (x - \beta) \times (x - \gamma)/(\beta - \gamma)$ and then we obtain

$$v_\alpha(P, \mu) = v_\alpha(x - \beta, \mu) + v_\alpha(x - \gamma, \mu) - v(\beta - \gamma). \quad (b)$$

It is seen that

$$v_\alpha(x - \beta, \mu) \leq \mu \text{ and } v_\alpha(x - \gamma, \mu) \leq \mu$$

hence if $v(\beta - \gamma) \geq \mu$, Relation (a) is clearly satisfied.

If $v(\beta - \gamma) < \mu$ then we have at least $v(\beta - \alpha) < \mu$ or $v(\gamma - \alpha) < \mu$.

For example, if $v(\beta - \alpha) < \mu$, we have $v_\alpha(x - \beta, \mu) = v(\beta - \alpha)$ hence by (b) we obtain

$$v_\alpha(P, \mu) = v(x - \gamma, \mu) \leq \mu.$$

In the same way when $v(\gamma - \alpha) < \mu$ we have

$$v_\alpha(P, \mu) = v(x - \beta, \mu) \leq \mu.$$

Relation (a) is then proven in all the cases and this ends the proof of Proposition 3. $\square$

3. Proposition 4 and 5. As a consequence of Theorem 1 in [18] we may easily deduce this Proposition :

Proposition 4. Let T be a disk $d(\alpha, r^-)$ and let $D = K \setminus d(\alpha, r^-)$. Let $f \in H(D)$ satisfy $\|f - 1\|_D < 1$ and $\lim_{|x| \rightarrow \infty} f(x) = 1$.

Let $\epsilon > 0$ be such that $\|f - 1\|_D + \epsilon < 1$. There exists a sequence $(a_n)_{n \in \mathbb{N}}$ in T such that the sequence $(a_n, b_n)_{n \in \mathbb{N}}$ is a strongly copiercing sequence associated to D satisfying $\sup_{n \in \mathbb{N}} |a_n - b_n| \leq r(1 + \epsilon)\|f - 1\|_D$ and such that $f(x) = \prod_{n=0}^{\infty} (x - a_n)/(x - b_n)$.

Proof. Let (b_n) be a stigmacyclical sequence of T (i.e. a sequence such that $|b_{n+1} - \alpha| > |b_n - \alpha|$, $\lim_{n \rightarrow \infty} |b_n - \alpha| = r$, and such that there exists a power series f in $(x - \alpha)$ convergent in $d(\alpha, r^-)$ satisfying $f(b_n) = 0 \forall n \in \mathbb{N}$). Let $a \in T$ be such that $|b_0 - a|/r > \|f - 1\|_D$. $\square$

Now it is seen that $f = f^T$ and $m(f, T) = 0$, hence f^T satisfies the hypothesis of Theorem 1 in [18]:

$$\|(x - a)^{-m(f, T)} f^T - 1\|_D < \frac{|b_0 - a|}{r}.$$

Consequently there exists a sequence (a_n) in T such that

$$\sup_{n \in \mathbb{N}} |a_n - b_n| < r(1 + \epsilon)\|f^T - 1\|_D$$

$$\lim_{n \rightarrow \infty} (a_n - b_n) = 0$$

and $f^T = \prod_{n \in \mathbb{N}} (x - a_n)/(x - b_n)$. (In Theorem 1 of [18] D is supposed to be bounded but it is seen that this hypothesis is useless when dealing with f^T).

Proposition 5. Let D be a closed infraconnected set and let $\phi \in H(D)$ satisfy $\|\phi - 1\|_D < 1$.

Let $(T_m)_{m \in N^*}$ be the sequence of the ϕ -holes of D . Let $(c_m)_{m \in N^*}$ be a sequence of strictly positive numbers such that $\lim_{n \rightarrow \infty} c_m = 0$. Let $a \in D$.

If D is bounded there exists a strongly copiercing sequence $(a_{n,m}, b_{n,m})_{(n,m) \in N \times N}$ associated to D such that

$$\lambda) \phi(x) = A \prod_{(n,m) \in N \times N} (x - a_{n,m}) / (x - b_{n,m}) \text{ with}$$

$$A = \phi(a) \prod_{n=0}^{\infty} \frac{b_{n,0} - a}{a_{n,0} - a} \text{ and } |A - \phi(a)| < \|\phi - \phi(a)\|_D + c_0$$

$\mu)$ For each $m \in N^*$, the sequence $(a_{n,m}, b_{n,m})_{n \in N}$ is a strongly copiercing sequence associated to $K \setminus T_m$ satisfying

$$|a_{n,m} - b_{n,m}| \leq r_m (\|\phi_{T_m}\|_D + c_m)$$

$$\text{while } \phi^{T_m} = \prod_{n=0}^{\infty} (x - a_{n,m}) / (x - b_{n,m})$$

$\nu)$ The sequence $(a_{n,0}, b_{n,0})_{n \in N}$ is a strongly copiercing sequence associated to $\tilde{D}$ satisfying

$$|a_{n,0} - b_{n,0}| \leq r_0 (\|\phi_0 - \phi(a)\|_D + c_0)$$

$$\text{while } \phi^0 = A \prod_{n=0}^{\infty} (x - a_{n,0}) / (x - b_{n,0}).$$

If D is not bounded there exists a strongly copiercing sequence $(a_{n,m}, b_{n,m})_{(m,n) \in N \times N^*}$ associated to D such that

$$\lambda') \phi(x) = A \prod_{(n,m) \in N \times N^*} (x - a_{n,m}) / (x - b_{n,m}) \text{ with } A = \lim_{\substack{|x| \rightarrow \infty \\ x \in D}} \phi(x)$$

$\mu')$ For each $m \in N^*$ the sequence $(a_{n,m}, b_{n,m})_{n \in N}$ is a strongly copiercing sequence associated to $K \setminus T_m$ satisfying $|a_{n,m} - b_{n,m}| \leq r_m (\|\phi_m\|_D + c_m)$

$$\text{while } \phi^{T_m} = \prod_{n=0}^{\infty} (x - a_{n,m}) / (x - b_{n,m}).$$

In [9] we gave this Theorem:

Theorem. Let D be a closed infraconnected set and let $a \in D$. Let $\phi \in H_b(D)$ be such that $\phi(a) \neq 0$. Then Conditions i), ii), iii) are equivalent

$$i) \|\phi - \phi(a)\|_D < |\phi(a)|.$$

$$ii) \text{ For every } \phi\text{-hole } T \text{ we have } \|\bar{\phi}_T\|_D < |\phi(a)| \text{ and } \|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D < |\phi(a)|$$

$$iii) \phi \text{ is invertible and for every } \phi\text{-hole } T, \phi^T \text{ satisfies } \|\phi^T - 1\|_D < 1 \text{ and } \phi^0 \text{ satisfies } \|\phi^0 - \phi^0(a)\|_D < |\phi(a)|.$$

Besides if Conditions i), ii), iii) are satisfied then we have

$$(u) m(\phi, T) = 0 \text{ for every } \phi\text{-hole } T$$

$$(v) \|\bar{\phi}_T\|_D = \|\phi^T - 1\|_D |\phi(a)| \text{ for every } \phi\text{-hole } T$$

$$(w) \|\phi^0 - \phi^0(a)\|_D = \|\phi_0 - \phi_0(a)\|_D$$

Remark. When D is not bounded, since $\phi \in H_b(D)$, both $\bar{\phi}_0$ and ϕ^0 are constant and therefore the relations $\|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D < |\phi(a)|$ and $\|\phi^0 - \phi^0(a)\|_D < |\phi(a)|$ are useless. Condition ii) is then equivalent to:

$$ii') \text{ For every } \phi\text{-hole } T \text{ we have } \|\bar{\phi}_T\|_D < |\phi(a)|.$$

In the same way, Condition iii) is equivalent to:

iii') ϕ is invertible in $H_b(D)$ and for every ϕ -hole T , ϕ satisfies $\|\phi^T - 1\|_D < 1$.

The theorem was proven with the help of this proposition renamed Proposition A [9]:

Proposition A. *Let D be a closed infraconnected set and let $\phi \in H(D)$ satisfy $\|\phi - 1\|_D < 1$. Then ϕ is invertible in $H(D)$ and for every ϕ -hole T we have $m(\phi, T) = 0$ and $\|\phi^T - 1\|_D = \|\phi_T\|_D$. Besides ϕ^0 satisfies $\|\phi^0 - 1\|_D = \|\phi_0 - 1\|_D$ and if D is not bounded then $\phi^0 \in K$.*

Proof of Proposition 5. Since $\|\phi - 1\|_D < 1$, by Theorem of [9] each ϕ^{T_m} is in the form $1 + \varphi_m$ with $\varphi_m \in H_0(K \setminus T_m)$ and $\|\varphi_m\|_D < 1$ while ϕ^0 is in the form $1 + \varphi_0$ with $\varphi_0 \in H(\tilde{D})$ and $\|\varphi_0\|_D < 1$. With no loss of generality we may obviously assume $a = 0$ and $\phi(0) = 1$.

Let $m \in N^*$. By Proposition 4 there exists a strongly copiercing sequence $(a_{n,m}, b_{n,m})_{n \in N}$ associated to T_m such that $|a_{n,m} - b_{n,m}|/r_m \leq \|\varphi_m\|_D + c_m$ and $\phi^{T_m} = \prod_{n=0}^{\infty} (x - a_{n,m})/(x - b_{n,m})$.

If D is not bounded, the sequence $(a_{n,m}, b_{n,m})_{(n,m) \in N \times N^*}$ clearly satisfies

$$\lim_{n+m \rightarrow \infty} \frac{|a_{n,m} - b_{n,m}|}{r_m} = 0$$

because the convergence of the sequence to 0 is uniform with respect to m . Besides, for each couple $(n, m) \in N \times N^*$, $a_{n,m}$ and $b_{n,m}$ belong to the hole T_m hence the sequence is strongly copiercing.

Now let us suppose D to be bounded, of diameter R . Let $c'_0 \in]0, c_0[$ satisfy

$$\|\phi^0 - 1\|_D + c'_0 < 1. \quad (51)$$

Let $t = 1/x$, let $r = 1/R$, let $T = d(0, r^-)$, let $\psi(t) = \phi^0(1/t)$ and let $E = K \setminus T$. Hence ψ is invertible in $H_b(E)$. Let $B = \lim_{|t| \rightarrow \infty} \psi(t)$. Since $\phi(0) = 1$ we have $B = 1$ and therefore $\psi = \psi^T$. Hence we have

$$|B - 1| \leq \|\psi - 1\|_E = \|\phi^0 - 1\|_{\tilde{D}} = \|\phi^0 - 1\|_D. \quad (52)$$

Let $\rho \in]r\|\phi^0 - 1\|_D, r[$ satisfy

$$\left(\frac{r}{\rho}\right)^2 \|\phi^0 - 1\|_D + c'_0 < \|\phi^0 - 1\|_D + c_0. \quad (53)$$

By Proposition 4 there exists a strongly copiercing sequence $(\alpha_n, \beta_n)_{n \in N}$ associated to E satisfying

$$|\alpha_n - \beta_n| < \rho < |\beta_n| \quad (54)$$

whenever $n \in N$

$$\frac{|\alpha_n - \beta_n|}{r} < \|\psi^T - 1\|_E = c'_0 \left(\frac{\rho}{r}\right)^2 \quad (55)$$

whenever $n \in N$ and such that $\psi(t) = \psi^T(t) = \prod_{n=0}^{\infty} (t - \alpha_n)/(t - \beta_n)$.

Now let $a_{n,0} = 1/\alpha_n$, $b_{n,0} = 1/\beta_n$, $A = \prod_{n=0}^{\infty} \alpha_n/\beta_n$.

It is easily seen that we have $\phi(x) = A \prod_{n=0}^{\infty} (x - a_{n,0}) / (x - b_{n,0})$.

Besides by (54) we have $|\alpha_n - \beta_n| < |\beta_n|$ hence

$$|a_{n,0} - b_{n,0}| = \frac{|\alpha_n - \beta_n|}{|\alpha_n \beta_n|} = \frac{|\alpha_n - \beta_n|}{|\beta_n|^2} < \frac{1}{|\beta_n|} = |b_{n,0}|. \quad (56)$$

Thus the sequence $(a_{n,m}, b_{n,m})_{(n,m) \in N \times N^*}$ is clearly a strongly copiercing sequence associated to D because when $m > 0$ both $a_{n,m}, b_{n,m}$ belong to T_m and when $m = 0$ both $a_{n,m}, b_{n,m}$ belong to $K \setminus \tilde{D}$ and they satisfy $|a_{n,m} - b_{n,m}| < \delta(b_{n,m}, D)$.

Finally, we have constructed the strongly copiercing sequence satisfying $\lambda), \mu)$ and then we have just to check the inequalities given in $\nu)$.

By (55) we have $|a_{n,0} - b_{n,0}| \leq r|b_{n,0}|^2(\|\phi^0 - 1\|_D + c'_0(\rho/r)^2)$, hence by (4) we see that $|a_{n,0} - b_{n,0}|/R \leq r^2/\rho^2(\|\phi^0 - 1\|_D + c'_0(\rho/r)^2)$. and finally by (53) we have $|a_{n,0} - b_{n,0}|/R \leq \|\phi^0 - 1\|_D + c_0$. Hence with greater reason we have $|a_{n,0} - b_{n,0}|/|b_{n,0}| < \|\phi^0 - 1\|_D + c_0$ and therefore we see that the sequence $A_n = \prod_{j=1}^n (b_j/a_j)$ satisfies $|A_n - 1| < \|\phi^0 - 1\|_D + c_0$ hence $|A - 1| < \|\phi^0 - 1\|_D = c_0$.

This finishes showing $\nu)$ and it ends the proof of Proposition 5. $\square$

4. Proof of the Theorem. Without loss of generality we may obviously assume $\phi(a) =$

1. If Condition $\gamma)$ is satisfied, by Theorem 5 of [8] it is clearly seen that g is strictly injective in D and satisfies (1), (2), (3), (4). Besides, as the sequence (ϕ_n) is strongly copiercing with respect to D , each ϕ_n satisfies $\|\phi_n - 1\|_D < 1$ hence we have $\|\phi - B\|_D < 1$ hence $\|\phi - \phi(a)\|_D < 1$.

Now we have just to show that $\alpha), \beta), \gamma)$ are equivalent. Let P be the polynomial f/f' .

First we will check that $\alpha)$ and $\beta)$ are equivalent. By Proposition 3 it is seen that for every hole T of D we have

$$\|P\|_T \geq \text{diam}(T) \quad (5)$$

and if D is bounded we have

$$\|P\|_D \geq \text{diam}(D). \quad (6)$$

Hence by the equivalence between ii) and iii) in Theorem of [9] and by the Remark which follows it in the case when D is unbounded Conditions $\alpha)$ and $\beta)$ are easily seen to be equivalent.

Indeed, let us assume $\alpha)$ to be satisfied. By (5) and (6) we see that $\|\bar{\phi}_T\|_D < 1$. Besides if D is bounded we see that $\|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D < 1$ while this inequality is trivial if D is not bounded because $\bar{\phi}_0$ is then a constant. Hence ϕ satisfies ii) in the Theorem of [9] and therefore we have $\|\phi^T - 1\|_D = \|\bar{\phi}_T\|_D$ for every ϕ -hole T and $\|\phi^0 - \phi^0(a)\|_D = \|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D$. Hence $\beta)$ clearly comes from $\alpha)$.

In the same way, let us suppose $\beta)$ satisfied. By (5) and (6) we have $\|\phi^T - 1\|_D < 1$ and we have $\|\phi^0 - \phi^0(a)\|_D < 1$ if D is bounded, while this last inequality is trivial if D is unbounded because ϕ^0 is a constant. Hence ϕ satisfies iii) in Theorem of [9] and then $\alpha)$ comes from $\beta)$.

Besides, when $\alpha), \beta)$ are satisfied, the set of conditions $\|\bar{\phi}_T\|_D < 1, \|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D < 1$ shows that $\|\phi - \phi(a)\|_D < 1$ hence $\|\phi - 1\|_D < 1$ and then by Proposition A of [9] we have

$$\|\phi^T - 1\|_D = \|\bar{\phi}_T\|_D \quad (7)$$

for every ϕ -hole T and

$$\|\phi^0 - 1\|_D = \|\bar{\phi}_0 - 1\|_D. \quad (8)$$

Now we will show that β) and γ) are equivalent.

We suppose that β) is satisfied and will show that so is γ). Since $\phi(a) = 1$, by (5) and (6) we have $\|\phi - 1\|_D < 1$.

Let $(T_m)_{m \in J}$ be the sequence of the ϕ -holes with $J \subset N^*$ and for each $m \in J$ let $T_m = d(\beta_m, r_m^-)$.

Let $(c_m)_{m \in N^*}$ be a sequence in $]0, +\infty[$ such that

$$\left(\|\phi^{T_m} - 1\|_D + c_m \right) \|P\|_{T_m} < r_m \quad (9)$$

while $\lim_{m \rightarrow \infty} c_m = 0$ if J is infinite. It is seen that we may assume $J = N^*$ without loss of generality.

By Proposition 5, for each $m \in N^*$ there exists a strongly copiercing sequence associated to $K \setminus T_m$, $(a_{n,m}, b_{n,m})_{n \in N}$ such that $|a_{n,m} - b_{n,m}|/r_m \leq \|\phi^{T_m} - 1\|_D + c_m$ hence by (9) we have $|a_{n,m} - b_{n,m}|/r_m^2 \|P\|_{T_m} < 1$, whenever $(n, m) \in N \times N^*$.

We put $\phi'_{n,m}(x) = (x - a_{n,m})/(x - b_{n,m})$, $(n, m) \in N \times N^*$. It is seen that $|\phi'_{n,m}|_{T_m} = |a_{n,m} - b_{n,m}|/r_m^2$ hence $|\phi'_{n,m}P|_{T_m} < 1$.

But by Proposition 2 we know that $|\phi'_{n,m}P|_{T_m} = \|\phi'_{n,m}P\|_D$ hence we obtain

$$\|\phi'_{n,m}P\|_D < 1 \quad (10)$$

whenever $(n, m) \in N \times N^*$.

Thus if D is not bounded, by Proposition 5 we know that ϕ is in the form

$$A \prod_{(n,m) \in N \times N^*} \phi_{n,m}$$

with $A \in d(1, 1^-)$ and therefore by (10) we see that γ) is clearly satisfied.

Now let us suppose D to be bounded, of diameter r_0 . By Condition β) and as $\phi(a) = 1$ we have $\|\phi^0 - 1\|_D \|P\|_D < r_0$.

Let $c_0 > 0$ be such that

$$\left(\|\phi^0 - 1\|_D + c_0 \right) \|P\|_D < r_0. \quad (11)$$

By Proposition 5 there exists a strongly copiercing sequence associated to $\tilde{D}$, $(a_{n,0}, b_{n,0})_{n \in N}$ satisfying

$$\frac{|a_{n,0} - b_{n,0}|}{r_0} < \|\phi^0 - 1\|_D + c_0 \quad (12)$$

and such that $\phi^0 = B \prod_{n=0}^{\infty} (x - a_{n,0})/(x - b_{n,0})$ with $B = \prod_{n=0}^{\infty} (b_{n,0}/a_{n,0})$

Since the sequence $(a_{n,0}, b_{n,0})$ is a strongly copiercing sequence associated to D it is seen that we have

$$|B - 1| < 1. \quad (13)$$

Now by (11) and (12) we have

$$\frac{|a_{n,0} - b_{n,0}|}{r_0^2} \|P\|_D < 1. \quad (14)$$

We put $\phi_{n,0}(x) = (x - a_{n,0})/(x - b_{n,0})$ ($n \in N$) and see that both $\phi'_{n,0}$ and P belong to $H(\tilde{D})$. Then as D is infraconnected, we have

$$\|\phi'_{n,0}P\|_D = \|\phi'_{n,0}P\|_{\tilde{D}} = \|\phi'_{n,0}\|_{\tilde{D}}\|P\|_{\tilde{D}} = \|\phi'_{n,0}\|_D\|P\|_D$$

while

$$\|\phi'_{n,0}\|_D = \frac{|a_{n,0} - b_{n,0}|}{\delta(b_{n,0}, D)^2} \leq \frac{|a_{n,0} - b_{n,0}|}{r_0^2}$$

hence by (14) we obtain

$$\|\phi'_{n,0}P\|_D = \|\phi'_{n,0}\|_D\|P\|_D < 1. \quad (15)$$

Obviously the sequence $(a_{n,m}, b_{n,m})_{(n,m) \in N \times N}$ is a strongly copiercing sequence associated to D and by (10), (13), (15) the family $(\phi_{n,m})_{(n,m) \in N \times N}$ satisfies γ). This finishes proving that β) implies γ).

Now we will show that γ) implies β). For every $n \in N$, let $\phi_n(x) = (x - a_n)/(x - b_n)$. Let T be a ϕ -hole of D of diameter r , and let I_T be the set of the $n \in N$ such that $a_n, b_n \in T$. By Theorem 5 of [18] we have $\phi^T(x) = \prod_{n \in I_T} \phi_n$ hence it is seen that we have

$$\|\phi^T - 1\|_D \leq \sup_{n \in I_T} \|\phi_n - 1\|_D = \sup_{n \in I_T} \frac{|a_n - b_n|}{r} \quad (16)$$

But we have

$$\frac{|a_n - b_n|}{r} = r|\phi'_n|_T \quad (17)$$

hence we see that $\|\phi^T - 1\|_D \leq r \sup_{n \in I_T} |\phi'_n|_T$ and therefore we have

$$\|\phi^T - 1\|_D \|P\|_T \leq r \left(\sup_{n \in I_T} |\phi'_n|_T \right) \|P\|_T \quad (18)$$

Now $|\phi'_n|_T = (1/|\phi'_n|_T)^{-1}$ and as $1/\phi'_n$ is a polynomial we have $|1/\phi'_n|_T = \|1/\phi'_n\|_T$ hence by Proposition 2, we see that $|\phi'_n|_T \|P\|_T = \|\phi'_n P\|_D$.

Consequently, by Hypothesis γ) we have

$$|\phi'_n|_T \|P\|_T < 1 \quad (19)$$

whenever $n \in I_T$. But by (17) the sequence $(|\phi'_n|_T)_{n \in I_T}$ either is finite or has limit zero and therefore it reaches its maximum for some $n \in I_T$. Hence by (19) we have $\sup_{n \in I_T} |\phi'_n|_T \|P\|_T < 1$. Finally, by (18) we obtain $\|\phi^T - 1\|_D \|P\|_T < r$ so that β) is satisfied when D is not bounded.

We now suppose D to be bounded, and then we have to show that $\|\phi^0 - \phi^0(a)\|_D < r_0$. Actually it is clear that $\|\phi^0 - \phi^0(x)\|_D$ is not depending on x whenever $x \in D$, hence we have just to show that $\|\phi^0 - \phi^0(a)\|_D < r_0$. By Theorem 5 of [18] ϕ^0 is in the form $A \prod_{n \in I_0} (x - a_n)/(x - b_n)$ where I_0 is the set of the $n \in N$ such that $a_n, b_n \in K \setminus \tilde{D}$.

For every $n \in I_0$ let $\psi_n = (b_n/a_n)(x - a_n)/(x - b_n)$. Obviously we have $\lim_{n \rightarrow \infty} b_n/a_n = 1$.

Let $\psi(x) = \prod_{n \in I_0} \psi_n(x)$. We see that $\psi(a) = 1$ hence $\phi^0 = \phi^0(a)\psi$ and therefore

$$\|\phi^0 - \phi^0(a)\|_D = |\phi^0(a)| \quad \|\psi - 1\|_D = \|\psi - 1\|_D$$

Hence we have just to show

$$\|\psi - 1\|_D \|P\|_D < r_0. \quad (20)$$

We see that $|\psi_n(x) - 1| = |x(b_n - a_n)|/|a_n(x - b_n)| = |x||b_n - a_n|/|b_n|^2$ whenever $x \in D$, whenever $n \in I$. Hence we obtain

$$\|\psi_n - 1\|_D = r_0 \frac{|b_n - a_n|}{|b_n|^2} \quad (21)$$

Now since $|\phi'_n(x)| = |a_n - b_n|/|(x - b_n)^2| = |a_n - b_n|/|b_n|^2$, we have $|\phi'_n(x)P(x)| = |a_n - b_n|/|b_n|^2 |P(x)|$ whenever $x \in D$, $n \in I_0$ and $\|\phi'_n P\|_D = |a_n - b_n|/|b_n|^2 \|P\|_D$ whenever $n \in I_0$. Hence by (12) we see that $\|\psi_n - 1\|_D \|P\|_D \leq r_0 |a_n - b_n|/|b_n|^2 \|P\|_D = r_0 \|\phi'_n P\|_D$. But by Hypothesis γ) we have $\|\phi'_n P\|_D < 1$. Hence we obtain

$$\|\psi_n - 1\|_D \|P\|_D < r_0 \quad (22)$$

whenever $n \in I_0$.

It is clear that $\lim_{n \in I_0} \psi_n = \lim_{n \in I_0} \phi_n = 1$ in $H(D)$ hence by (22) we have $\sup_{n \in I_0} \|\psi_n - 1\|_D < r_0/\|P\|_D$. But we see that $\|\psi - 1\|_D < r_0/\|P\|_D$ and therefore (20) is satisfied. This finishes proving that γ) implies β) and it ends the proof of the Theorem. $\square$

Résumé substantiel en français. Soit K , $|\cdot|$ un corps ultramétrique complet algébriquement clos. Soit D une partie de K . Soit $R(D)$ la K -algèbre des fractions rationnelles $h(x) \in K(x)$ sans pôle dans D , et soit $H(D)$ l'espace vectoriel complété de $R(D)$ pour la topologie de la convergence uniforme sur D . Les éléments de $H(D)$ sont appelés *éléments analytiques sur D* [3,13]. Soit $H_b(D)$ l'ensemble des éléments de $H(D)$ bornés dans D et soit $\|\cdot\|_D$ la norme de la convergence uniforme sur D . Maintenant, on note $\tilde{D}$ le plus petit disque circonferencié qui contient D si D est borné et on pose $\tilde{D} = K$ si D n'est pas borné. Soit $f \in H(D)$. Tout trou T de D tel que $f \notin H(D \cup T)$ est appelé *f -trou*. Alors $f(x)$ est égal à sa série de Mittag-Leffler $\tilde{f}_0 + \sum_{n=1}^{\infty} \tilde{f}_{T_n}$ avec $\tilde{f}_0 \in H(\tilde{D})$ et $\tilde{f}_{T_n} \in H(K \setminus T_n)$, avec $\lim_{|x| \rightarrow +\infty} \tilde{f}_{T_n}(x) = 0$ (les T_n sont les f -trous). En outre quand f est semi-inversible, en particulier quand il est inversible, E. Motzkin a introduit une factorisation $f^0 \left(\prod_{n=1}^{\infty} f^{T_n} \right)$ avec $f^0 \in H(\tilde{D})$ et $f^{T_n} \in H(K \setminus T_n)$, avec $\lim_{|x| \rightarrow \infty} f^{T_n}(x) = 1$. Dans [8] on a introduit des suites de fonctions homographiques de la forme $(x - a_n)/(x - b_n)$. Une telle suite est appelée *suite fortement copercante associée à D* si pour tout n , $|a_n - b_n|$ est strictement inférieur à la distance $\delta(a_n, D)$ de a_n à D et si $\lim_{n \rightarrow \infty} |a_n - b_n|/\delta(a_n, D) = 0$ (quand la suite est infinie). Rappelons également qu'une fonction dérivable est dite *strictement injective* si elle est injective et si sa dérivée ne s'annule pas. Voici le théorème principal.

Théorème. Soit D un ouvert infraconnexe, soit $a \in D$. Soit $f \in R(D)$ une fonction homographique et soit $\phi \in H_b(D)$ un élément semi-inversible satisfaisant $|\phi(a)| = 1$. Les affirmations $\alpha)$, $\beta)$, $\gamma)$ ci-dessous sont équivalentes.

$\alpha)$ Pour tout ϕ -trou T on a $\|\bar{\phi}_T\|_D \|f/f'\|_T < \text{diam}(T)$ et si D est borné, $\bar{\phi}_0$ satisfait $\|\bar{\phi}_0 - \bar{\phi}_0(a)\|_D \|f/f'\|_D < \text{diam}(D)$.

$\beta)$ ϕ est inversible dans $H(D)$ et pour tout ϕ -trou T on a $\|\phi^T - 1\|_D \left\| \frac{f}{f'} \right\|_T < \text{diam}(T)$ et si D est borné ϕ^0 satisfait

$$\|\phi^0 - \phi^0(a)\|_D \left\| \frac{f}{f'} \right\|_D < \text{diam}(D).$$

$\gamma)$ ϕ est de la forme $B \prod_{n \in I} \phi_n$ avec $B \in C(0, 1)$ et avec $(\phi_n)_{(n \in I)}$ une suite fortement coperçante de fonctions homographiques associées à D satisfaisant

$$\left\| \frac{f}{f'} \phi'_n \right\|_D < 1$$

pour tout $n \in I$.

En outre si les Conditions $\alpha)$, $\beta)$, $\gamma)$ sont satisfaites, alors ϕ satisfait $\|\phi - \phi(a)\|_D < 1$ et le produit $g = f\phi$ est strictement injectif dans D et satisfait

- (1) $|g(x)| = |f(x)|$
- (2) $|g'(x)| = |f'(x)|$
- (3) $|g(y) - g(x)| = |f(y) - f(x)|$
- (4) $|g(y) - g(x)|^2 = |g'(y)g'(x)||y - x|^2$ pour tous $x, y \in D$.

Exemple. Soit $D = d(0, 1) \setminus d(b, \rho^-)$ avec $|b| = r \in]0, 1[$ et $\rho \in]0, r[$. Soit $q \in \mathbb{Z}$, soit $\varepsilon \in K$ satisfaisant $|\varepsilon| < \rho^{q+1}/r$, soit $f(x) = x$ et soit $g(x) = x(1 + \varepsilon/(x - b)^q)$. Alors g est strictement injective dans D et satisfait les conditions (1), (2), (3), (4).

Plusieurs conjectures sont suggérées par le théorème.

Un élément $f \in H(D)$ est dit *bianalytique* s'il est injectif et tel que $f^{-1} \in H(f(D))$.

Conjecture. Soit D un infraconnexe, soit $a \in D$, soit f une fonction homographique et soit $\phi \in H_b(D)$ satisfaisant $\phi(a) \neq 0$ ainsi que les conditions $\alpha)$, $\beta)$, $\gamma)$ du théorème. Alors $f\phi$ est un élément bianalytique de $H(D)$.

Conjecture. Soit D un infraconnexe ouvert. Un élément non constant de $H(D)$ est bianalytique si et seulement s'il satisfait la relation (4).

Une autre conjecture déjà donnée dans [8] semble être renforcée par le théorème et s'appliquerait à des ensembles «hyper-infraconnexes», (en particulier les ensembles affines). Suivant cette conjecture, en gros, les éléments analytiques injectifs seraient ceux qui se factorisent sous la forme $f\phi$ satisfaisant les conditions $\alpha)$, $\beta)$, $\gamma)$.

Dans la démonstration du théorème on utilise la proposition suivante qui peut être utile par ailleurs.

Proposition. Soit D un infraconnexe, soit $f \in R(D)$ une fonction homographique et soit $\phi \in R(D)$ une suite de fonctions homographiques fortement copercantes associées à D .

Si le pôle de ϕ appartient à $K \setminus \tilde{D}$ alors on a $\|(f/f')\phi'\|_D = \|f/f'\|_D \|\phi'\|_D$. Si le pôle de ϕ appartient à un trou T de D alors on a

$$\left\| \frac{f}{f'} \phi' \right\|_D = \left\| \frac{f}{f'} \right\|_T \left(\left\| \frac{1}{\phi'} \right\|_T \right)^{-1} = \left\| \frac{f}{f'} \right\|_T \|\phi'\|_D$$

En outre la connection entre la Condition γ) et les relations (1), (2), (3), (4) dans le théorème est obtenue grâce au théorème de [8].

Théorème. Soit D un ouvert, soit $h \in R(D)$ une fonction homographique, soit I une partie de $\mathbb{Z}$, soit $(\phi_n)_{n \in I}$ une suite de fonctions homographiques fortement copercantes associées à D telles que pour tout $i \in I$ la fonction homographique $\phi_i = (x - \alpha_i)/(x - \beta_i)$ satisfait $|h(x)\phi'_i(x)| < |h'(x)|$ quel que soit $x \in D$. Soit $g = h \prod_{i \in I} \phi_i$.

Alors g est strictement injective dans D et satisfait

- (1) $|g(x)| = |h(x)|$ pour tout $x \in D$
- (2) $|g'(x)| = |h'(x)|$ pour tout $x \in D$
- (3) $|g(y) - g(x)| = |h(y) - h(x)|$ pour tous $x, y \in D$
- (4) $|g(y) - g(x)|^2 = |y - x|^2 |g'(x)| |g'(y)|$ pour tous $x, y \in D$.

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