

A LOCAL EXISTENCE RESULT FOR THE EQUATION

$$\frac{d}{dt}Bu(t) = Au(t) + f(t)$$

IN HILBERT SPACES

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RÉSUMÉ. Dans ce travail nous décrivons une extension (partielle) du “Théorème 1” page 23 de la monographie [5]. Il s’agit d’un passage de l’équation $u' = Au + f$ à l’équation plus générale (en forme implicite) $\frac{d}{dt}Bu(t) = Au(t) + f(t)$ (*) où les deux opérateurs A et B sont linéaires et non bornés dans l’espace de Hilbert H . Avec une condition de borne uniforme pour l’opérateur $(\lambda B^* - A^*)^{-1}$ sur une droite verticale dans le plan complexe, on établit que pour toute $f \in L^2(a, b; H)$ il existe au moins une solution ultrafaible de l’équation (*) considérée plus haut.

ABSTRACT. In this work we describe a (partial) extension of “Théorème 1” page 23 of the monograph [5]. It consists in the passage from the equation $u' = Au + f$ to the more general (implicit) equation $\frac{d}{dt}Bu(t) = Au(t) + f(t)$ (*), where both A and B are linear unbounded operators in a Hilbert space H . Under a uniform boundedness condition for the operator $(\lambda B^* - A^*)^{-1}$ on a vertical line in the complex plane we demonstrate that for any $f \in L^2(a, b; H)$ there exists at least one ultraweak solution $u(t) \in L^2(a, b; H)$ of the above differential equation (*).

0. Introduction. Local (and also global) existence theorems for ultraweak solutions of abstract differential equations in normal form, as, for instance, $u' = Au + f$, $u'' = Au + f$, $u^n + A_1u^{n-1} + \dots + A_nu = f$, $u'' + A_1u' + A_2u = f$, in the spaces $L^2(a, b; H)$ or $L^2_{\text{loc}}(a, b; H)$, where H is a Hilbert space can be found in the papers [1], [2], [4], [6], [7], [8] and also in the monographs [5], [9]. In the present paper we consider an extension of the first order equation case; instead of the equality equation $u'(t) = Au(t) + f(t)$ we now introduce a second, unbounded linear operator B and with it the (implicit) differential equation $\frac{d}{dt}Bu = Au + f$. Replacing the condition on the resolvent operator $(\lambda I - A^*)^{-1}$ which appeared in [5], [9] we now put a similar condition on the operator $(\lambda B^* - A^*)^{-1}$ where A^* and B^* are the Hilbert-adjoints to the operators A and B .

We define a natural class of vector-valued test-functions for this new situation, which permits introduction of ultraweak solutions. Then we prove the fundamental lemma (an *a priori* estimate) which extends (say) lemme 2.1 in [5] page 23, and conclude about the local existence theorem as a (simple) application of Hahn-Banach’s extension theorem and of Riesz’s representation theorem in Hilbert spaces.

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1. Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. Next, let us consider two linear closed operators in H , A and B , with dense domain $D(A)$ and $D(B)$ and with range in H . Let then A^* and B^* be the Hilbert adjoints of A and B , so that $\langle Ah, k \rangle = \langle h, A^*k \rangle$, $\forall h \in D(A)$ and $k \in D(A^*)$ and also $\langle Bh, k \rangle = \langle h, B^*k \rangle$, $\forall h \in D(B)$ and $k \in D(B^*)$. We know that both A^* and B^* are linear closed operators with dense domain in H .

Let us define now a class of vector-valued test-functions, here denoted by $K_{A^*, B^*}(a, b)$ where (a, b) is an arbitrary open interval on the real line.

This class will consist of functions $\varphi(t) \in C_0^1([a, b]; H)$, such that $\varphi(t) \in D(A^*) \cap D(B^*)$ for all $t \in (a, b)$, $A^*\varphi$ and $B^*\varphi$ belong to $C([a, b]; H)$, $\varphi'(t) \in D(B^*)$, $B^*\varphi' \in C([a, b]; H)$. We shall make the following

REMARK 1. If $D(A^*) \cap D(B^*) = \{\theta\}$, there is only one test-function in our class, namely $\varphi(t) \equiv \theta$.

Once the class of test-functions was defined, we are able to define the (ultra) weak solutions of the abstract differential equation

$$-Au(t) + \frac{d}{dt}Bu(t) = f(t) \quad (1.1)$$

where f is a given function in $L_{loc}^2(a, b; H)$. Precisely, a function $u(t)$ belonging to the same space $L_{loc}^2(a, b; H)$ is an ultraweak solution of the above equation if the integral identity

$$\int_a^b \langle u(t), A^*\varphi(t) + B^*\varphi'(t) \rangle dt = - \int_a^b \langle f(t), \varphi(t) \rangle dt \quad (1.2)$$

holds, for all $\varphi \in K_{A^*, B^*}(a, b)$.

REMARK 2. In the situation of Remark 1, every function u in $L_{loc}^2(a, b; H)$ will be an ultraweak solution of the above considered equation.

Some elementary properties related to the definition (1.2) are explained below (compare with Propositions 1.1), 1.2), 1.3) in our monograph [9]).

PROPOSITION 1. Let us consider a function $f(t) \in C([a, b]; H)$; then let $u(t) \in D(A) \cap D(B)$, $\forall t \in]a, b[$, $Bu \in C^1([a, b]; H)$, $Au \in C([a, b]; H)$ and the equality

$$\frac{d}{dt}Bu = Au + f \quad (1.3)$$

holds true on $]a, b[$. Then the relation (1.2) is also verified, $\forall \varphi \in K_{A^*, B^*}(a, b)$.

PROOF: Take in fact any $\varphi \in K_{A^*, B^*}(a, b)$; from (1.3) we derive, after taking scalar products, the relation

$$\langle \frac{d}{dt}Bu, \varphi \rangle - \langle Au, \varphi \rangle = \langle f, \varphi \rangle. \quad (1.4)$$

We note also the obvious equation

$$\frac{d}{dt}\langle Bu, \varphi \rangle = \langle \frac{d}{dt}Bu, \varphi \rangle + \langle Bu, \varphi' \rangle$$

and also the equalities

$$\begin{aligned}\langle Au(t), \varphi(t) \rangle &= \langle u(t), (A^* \varphi)(t) \rangle, \\ \langle Bu(t), \varphi'(t) \rangle &= \langle u(t), B^* \varphi'(t) \rangle\end{aligned}\quad (1.5)$$

which are true for all $t \in (a, b)$.

We now integrate the equality (1.4) on the interval (a, b) ; using vanishing of φ near a and b , we obtain the relations

$$\int_a^b \frac{d}{dt} \langle Bu, \varphi \rangle dt - \int_a^b \langle Bu, \varphi' \rangle dt - \int_a^b \langle u(t), (A^* \varphi)(t) \rangle dt = \int_a^b \langle f(t), \varphi(t) \rangle dt$$

and then also

$$- \int_a^b \langle u(t), B^* \varphi'(t) \rangle dt - \int_a^b \langle u(t), (A^* \varphi)(t) \rangle dt = \int_a^b \langle f, \varphi \rangle dt$$

which is in fact (1.2). $\square$

The following simple fact is also worth mentioning.

PROPOSITION 2. Let $\{f_n\}_1^\infty \subset L_{\text{loc}}^2(a, b; H)$ such that $f_n \rightarrow f$ in this space. Let also $\{u_n\}_1^\infty \subset L_{\text{loc}}^2(a, b; H)$ be a sequence of solutions to (1.2) such that $u_n \rightarrow u$ in the same sense (that is, in the square mean on every compact subinterval of (a, b)). Then the functions u and f are related through the same relation (1.2).

The proof is now quite obvious, it is therefore omitted. More significant is the following.

PROPOSITION 3. Let be given $f \in C([a, b]; H)$ and u such that $u(t) \in D(A) \cap D(B)$ for all $t \in]a, b[$, $Bu \in C^1([a, b]; H)$, $Au \in C([a, b]; H)$ and assume that (1.2) holds for all $\forall \varphi \in K_{A^*, B^*}(a, b)$. Assume also that the intersection $D(A^*) \cap D(B^*)$ is dense in H . Then the equality

$$\frac{d}{dt}(Bu) = Au + f$$

holds true, in the usual sense.

PROOF: We assume validity of (1.2); from it we obtain without difficulty, using our present hypotheses, the relation

$$\int_a^b \left\langle \frac{d}{dt}(Bu) - Au - f, \varphi \right\rangle dt = 0$$

for all $\varphi \in K_{A^*, B^*}([a, b])$. We then take test-functions φ of the special form: $\varphi(t) = \nu(t)h$ where $\nu(t) \in C_0^1([a, b])$ while $h \in D(A^*) \cap D(B^*)$. We deduce accordingly the equality

$$\int_a^b \left\langle \frac{d}{dt}(Bu) - Au - f, h \right\rangle \nu(t) dt = 0$$

$\forall h \in D(A^*) \cap D(B^*)$ and $\nu \in C_0^1([a, b])$. Next we note that the function

$$\frac{d}{dt}(Bu) - Au - f$$

belongs to $C([a, b]; H)$. It follows that

$$\left\langle \frac{d}{dt}(Bu) - Au - f, h \right\rangle = 0$$

for all $t \in]a, b[$ and also for all $h \in D(A^*) \cap D(B^*)$. From the density of the intersection $D(A^*) \cap D(B^*)$ we then infer, for all $t \in]a, b[$, the equality

$$\frac{d}{dt}(Bu)(t) - Au(t) - f(t) = \theta. \quad \square$$

2. In the following lines we are going to prove the following

THEOREM. *Let us assume that $-\infty < a < b < +\infty$; also, the inverse operator $(\lambda B^* - A^*)^{-1}$ should exist on the line $\{\lambda \in \mathbb{C} \mid \Re(\lambda) = \sigma\}$, should belong to $L(H)$ for any such λ and satisfy the uniform estimate*

$$\|(\lambda B^* - A^*)^{-1}\| \leq L, \quad \forall \lambda \text{ such that } \Re(\lambda) = \sigma. \quad (2.1)$$

Then, given any function $f \in L^2(a, b; H)$, there exists at least one function $u \in L^2(a, b; H)$ which is an ultraweak solution of the equation

$$\frac{d}{dt} B u(t) = A u(t) + f(t) \quad (2.2)$$

on (a, b) .

Our demonstration has as a crucial point the following “*a priori* estimate”.

LEMMA. *Under the above assumptions of the Theorem, the inequality*

$$\int_a^b \|\psi(t)\|^2 dt \leq C \int_a^b \|B^* \psi'(t) + A^* \psi(t)\|^2 dt, \quad \psi \in K_{A^*, B^*}(a, b) \quad (2.3)$$

holds true.

In order to obtain this estimate we introduce the function φ defined by

$$\varphi(t) = e^{\sigma t} \psi(t)$$

so that $\psi(t) = e^{-\sigma t} \varphi(t)$ and we put

$$\xi(t) = B^* \psi'(t) + A^* \psi(t).$$

Simple computations give then

$$\psi'(t) = e^{-\sigma t} \varphi'(t) - \sigma e^{-\sigma t} \varphi(t)$$

and

$$\xi(t) = B^* \psi'(t) + A^* \psi(t) = e^{-\sigma t} B^* \varphi'(t) - \sigma e^{-\sigma t} B^* \varphi(t) + A^* (e^{-\sigma t} \varphi(t)).$$

It follows that

$$\begin{aligned} B^* \varphi'(t) + (A^* - \sigma B^*) \varphi(t) &= e^{\sigma t} \xi(t) \\ &= g(t) \end{aligned} \quad (2.4)$$

where (remember), both φ and g have compact support in the open interval $]a, b[$. Consider now the (inverse) Fourier transforms

$$\begin{aligned} \check{\varphi}(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\tau t} \varphi(t) dt, \\ \check{g}(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\tau t} g(t) dt \end{aligned} \quad (2.5)$$

and remember also Parseval's identity (for vector-valued functions in Hilbert spaces):

$$\begin{aligned}\int_{\mathbf{R}} \|\check{\varphi}(\tau)\|^2 d\tau &= \int_{\mathbf{R}} \|\varphi(t)\|^2 dt, \\ \int_{\mathbf{R}} \|\check{g}(\tau)\|^2 d\tau &= \int_{\mathbf{R}} \|g(t)\|^2 dt.\end{aligned}\quad (2.6)$$

Now, let us multiply both sides in (2.4) by $e^{i\tau t}$ and then integrate over $\mathbf{R}$; we obtain the equality

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (B^* \varphi')(t) e^{i\tau t} dt + \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (A^* - \sigma B^*) \varphi(t) e^{i\tau t} dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\tau t} g(t) dt. \quad (2.7)$$

Using closedness of the adjoint operator B^* and the integrability over $\mathbf{R}$ of the functions $e^{i\tau t}(B^* \varphi')(t)$ and $e^{i\tau t} \varphi'(t)$ we obtain, by a classical result, the relations

$$\int_{\mathbf{R}} e^{i\tau t} \varphi'(t) dt \in D(B^*)$$

and

$$B^* \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\tau t} \varphi'(t) dt \right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (B^* \varphi')(t) e^{i\tau t} dt.$$

On the other hand, it is easy to see that

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{i\tau t} \varphi'(t) dt &= -i\tau \check{\varphi}(\tau) \in D(B^*), \\ B^* (-i\tau \check{\varphi}(\tau)) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (B^* \varphi')(t) e^{i\tau t} dt.\end{aligned}\quad (2.8)$$

Applying two more times the same classical result as above we obtain that

$$\begin{aligned}\check{\varphi}(\tau) &\in D(A^*), \\ A^* \check{\varphi}(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (A^* \varphi)(t) e^{i\tau t} dt\end{aligned}\quad (2.9)$$

and also the relations

$$\begin{aligned}\check{\varphi}(\tau) &\in D(B^*), \\ \sigma B^* \check{\varphi}(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} (\sigma B^* \varphi)(t) e^{i\tau t} dt.\end{aligned}\quad (2.10)$$

Therefore it follows that $\check{\varphi}(\tau) \in D(A^*) \cap D(B^*)$ and from the above we readily derive the equality

$$B^* (-i\tau \check{\varphi}(\tau)) + (A^* - \sigma B^*) \check{\varphi}(\tau) = \check{g}(\tau) \quad (2.11)$$

that is (in a more convenient form)

$$[A^* - (\sigma + i\tau) B^*] \check{\varphi}(\tau) = \check{g}(\tau). \quad (2.12)$$

Using the main assumptions one finds now the representation formula

$$\check{\varphi}(\tau) = -[(\sigma + i\tau)B^* - A^*]^{-1} \check{g}(\tau) \quad (2.13)$$

and accordingly the uniform estimate

$$\|\check{\varphi}(\tau)\| \leq L\|\check{g}(\tau)\|, \quad \tau \in \mathbb{R}. \quad (2.14)$$

Using the relation (2.6) we now see that the inequality

$$\int_a^b \|\varphi(t)\|^2 dt \leq L^2 \int_a^b \|g(t)\|^2 dt \quad (2.15)$$

holds true and it implies that

$$\int_a^b e^{2\sigma t} \|\psi(t)\|^2 dt \leq L^2 \int_a^b e^{2\sigma t} \|\xi(t)\|^2 dt. \quad (2.16)$$

Assume now, for example, that $\sigma < 0$. We infer that

$$e^{2\sigma b} \int_a^b \|\psi(t)\|^2 dt \leq L^2 e^{2\sigma a} \int_a^b \|B^*\psi' + A^*\psi\|^2 dt \quad (2.17)$$

which is exactly (2.3) with $C = L^2 e^{2\sigma(a-b)} = L^2 e^{2|\sigma|(b-a)}$. This proves the Lemma.

PROOF OF THE THEOREM: Consider the subset $\mathcal{M}$ of $L^2(a, b; H)$ which is the range of the operator

$$B^* \frac{d}{dt} + A^*$$

with domain $K_{A^*, B^*}(a, b)$; thus

$$\mathcal{M} = \{B^*\psi' + A^*\psi \mid \psi \in K_{A^*, B^*}(a, b)\}.$$

Next, consider a mapping $F: \mathcal{M} \rightarrow \mathbb{C}$ defined by the relation

$$F(B^*\varphi' + A^*\varphi) = - \int_a^b \langle \varphi(t), f(t) \rangle dt, \quad \varphi \in K_{A^*, B^*}(a, b). \quad (2.18)$$

F is well-defined (as a single-valued mapping): for, if $B^*\varphi'_1 + A^*\varphi_1 = B^*\varphi'_2 + A^*\varphi_2$ it results that $B^*(\varphi_1 - \varphi_2)' + A^*(\varphi_1 - \varphi_2) = 0$ and, using the Lemma, $\varphi_1 = \varphi_2$. Also, F is a linear mapping, as easily seen. Moreover, F is a continuous functional on $\mathcal{M}$ considered as a linear subspace of $L^2(a, b; H)$. In fact, we see the inequality

$$\begin{aligned} |F(B^*\varphi' + A^*\varphi)| &= \left| \int_a^b \langle \varphi, f \rangle dt \right| \\ &\leq \|\varphi\|_{L^2(a, b; H)} \cdot \|f\|_{L^2(a, b; H)}. \end{aligned} \quad (2.19)$$

Using again the estimate (2.3), we derive from (2.19) the relation

$$|F(B^*\varphi' + A^*\varphi)| \leq \sqrt{C} \|f\|_{L^2(a, b; H)} \cdot \|B^*\varphi' + A^*\varphi\|_{L^2(a, b; H)}. \quad (2.20)$$

At this stage we use the extension theorem and find $\tilde{F}$ which is a linear continuous mapping of $L^2(a, b; H)$ into $\mathbb{C}$ and is such that $\tilde{F} = F$ on the subspace $\mathcal{M}$. According to the main representation theorem (Riesz's theorem) there is an element $u(\cdot)$ in $L^2(a, b; H)$ such that the representation formula

$$\tilde{F}(v) = \int_a^b \langle v(t), u(t) \rangle dt, \quad v \in L^2(a, b; H) \quad (2.21)$$

holds true.

In particular, taking $v = B^* \varphi' + A^* \varphi$ with $\varphi \in K_{A^*, B^*}(a, b)$, we derive the relation

$$\begin{aligned} \tilde{F}(B^* \varphi' + A^* \varphi) &= F(B^* \varphi' + A^* \varphi) \\ &= \int_a^b \langle B^* \varphi' + A^* \varphi, u \rangle dt \\ &= - \int_a^b \langle \varphi, f \rangle dt, \quad \varphi \in K_{A^*, B^*}(a, b). \end{aligned} \quad (2.22)$$

Finally, taking the complex conjugate we obtain the equality

$$\int_a^b \langle u(t), B^* \varphi' + A^* \varphi \rangle dt = - \int_a^b \langle f(t), \varphi(t) \rangle dt, \quad \varphi \in K_{A^*, B^*}(a, b) \quad \square \quad (2.23)$$

FINAL REMARK. Let us assume that $u(t) \not\equiv \theta$ is a solution to the homogeneous equation

$$\frac{d}{dt} B u(t) - A u(t) = 0.$$

Then (proposition 1), u is a ultraweak solution to the equation $(Bu)' - Au = 0$; it can thus be summed with any solution of the inhomogeneous equation $(Bu)' - Au = f$ (in ultraweak sense) and it will give a new solution to the same equation. Therefore, in the existence theorem above one cannot expect also unicity of solutions; we have solved a certain ill-posed problem.

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