

SOME REMARKS ON THE ABSTRACT CAUCHY PROBLEM IN THE ULTRAWEAK SENSE

SAMUEL ZAIDMAN

RÉSUMÉ. Ce travail est formé de quatre parties. Dans la première, nous avons une extension (partielle) du résultat d'unicité bien connu de Hille-Phillips [2, Th. 23.7.1] au cas des solutions continues et ultra-faibles, dans la deuxième partie, on considère cette même classe de solutions pour le problème de Cauchy: $u'(t) = Au(t)$, $u(0) = u_0$, $0 \leq t < \infty$, et on suppose l'existence, l'unicité et la dépendance continue ponctuelle des données initiales dans le domaine $D(A)$; la solution $u(t)$ est aussi dans $D(A)$. On trouve alors un demi-plan $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq \omega\}$ contenant toutes les valeurs propres de l'opérateur A . En ajoutant des conditions on obtient que la solution $u(t)$ est continûment différentiable dans l'ouvert $]0, \infty[$. Dans la troisième partie nous examinons la validité de la formule:

$$(\lambda I - A)^{-1}x = \int_0^\infty \exp(-\lambda t)U(t)x \, dt, \text{ pour } x \in D(A),$$

où $U(t)$ est le semi-groupe d'opérateurs associés au problème de Cauchy bien posé et ultra-faible, dans un espace de Banach réflexif, A étant un opérateur linéaire, fermé de domaine dense. Enfin, dans la quatrième partie nous indiquons une nouvelle démonstration du résultat principal de [7] (concernant l'unicité rétrograde) et nous montrons aussi une extension où l'on remplace les solutions continues par des solutions de p^e puissance intégrables.

ABSTRACT. This work consists of four sections; in the first one we (partially) extend the well-known uniqueness result of Hille-Phillips [2, th. 23.7.1] to the case of continuous ultraweak solutions; in the second part we consider the same class of solutions for the Cauchy problem: $u'(t) = Au(t)$, $u(0) = u_0$, $t \geq 0$, and assume existence, unicity and pointwise continuous dependence on the initial data in $D(A)$, where $u(t) \in D(A)$ for all $t \geq 0$. One finds a half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq \omega\}$ containing all eigenvalues of the operator A ; under some more assumptions we obtain that the solution is continuously differentiable in the open set $(0, \infty)$; in the third part we are concerned with the Laplace transform formula:

$$(\lambda I - A)^{-1}x = \int_0^\infty \exp(-\lambda t)U(t)x \, dt, \quad x \in D(A) \tag{0.1}$$

where $U(t)$ is the operator semigroup associated to the well-posed ultraweak Cauchy problem in a reflexive Banach space, A being a linear closed operator of dense domain. Finally, in the last part of the paper we give a shorter proof of the result in [7] (about backward uniqueness) and then extend it from continuous to L^p -solutions.

Introduction. This paper is a continuation of some previous work concerning the abstract Cauchy problem in the ultraweak sense ([7], [8], [9], [10]).

In the first section we examine a result about uniqueness of the Cauchy problem for a situation which is not necessarily well-posed (that is, existence and continuous dependence

Reçu le 25 mars 1988 et, sous forme révisée, le 20 septembre 1988.

This work was done while visiting (part-time) the Centre de recherches mathématiques at the Université de Montréal.

©Association mathématique du Québec

is not taken into account). For a class of regular solutions, the Theorem 23.7.1 in Hille-Phillips [2] gives a sufficient condition for unicity of solutions to the equation: $u'(t) = Au(t)$, $0 \leq t < \infty$, $u(0) = 0$, in terms of some (simple) spectral properties of the operator A (which could have empty resolvent set, however). This solution must be of normal type ω , which means that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|u(t)\| = \omega < \infty. \quad (0.2)$$

The operator A is a linear closed transformation with domain $D(A)$, acting in the complex Banach space X .

In the present work we establish a (partial) extension: the space X must be reflexive, the operator A must have dense domain, but the solution needs not be differentiable anywhere, and it need not belong to the domain $D(A)$. In fact, it will be a continuous ultraweak solution as defined by Lions [6] and Kato-Tanabe [3]. The method of proof is via a regularization process: The solution u is replaced by a sequence of convolutions $u * \alpha_n$, and in this way it becomes possible to obtain Hille-Phillips's formula (23.7.4) involving the Laplace transform

$$F(\lambda) = \int_0^\infty \exp(-\lambda t) u(t) dt, \quad \text{for } \operatorname{Re} \lambda > \omega \quad (0.3)$$

(formula (1.12) below). In the second part of the paper, which is essentially a continuation of our recent article [10] we again consider continuous ultraweak solutions of the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = u_0, \quad 0 \leq t < \infty \quad (0.4)$$

and we now assume existence, uniqueness and pointwise continuous dependence on the initial data in the domain $D(A)$. Here too, existence of the resolvent operator $(\lambda I - A)^{-1} \in L(X)$ is not assumed (for any complex number λ) but the ultraweak solution must belong to the domain $D(A)$ for all $t \geq 0$. We obtain two simple results, about location of eigenvalues of A and about regularity of the solution in $(0, +\infty)$. Next, in the third section we arrive at the main representation formula

$$(\lambda I - A)^{-1} x = \int_0^\infty \exp(-\lambda t) U(t) x dt, \quad \forall x \in D(A) \quad (0.5)$$

where $U(t)$ is the operator semi-group associated to the above well-posed Cauchy problem, under the same supplementary hypothesis as above, namely that A is linear and closed and the (complex) Banach space is reflexive. This formula appears in [4] as (1.8) but the proof there is given under slightly different assumptions (the space X is not necessarily reflexive but the solution must be regular). Again, the method rests essentially on the well-known regularization process, $u \rightarrow u * \alpha_n$ which permits (as established in [11]) to replace an ultraweak solution by a very regular one. In the last section of this work we elaborate again on Theorem 1.1 in Agmon-Nirenberg's paper [1], following our previous article [7]. We give now a new (and shorter) proof of the Theorem in [7] concerning backward unicity result for the ultraweak Cauchy problem, when continuous solutions are considered.

Using the resolvent regularization method (established in our papers [12], [13]) the proof of the result appears really shorter and needs less hypotheses on the operator A (the density of the domain $D(A^*)$ is no longer necessary).

Then, we extend the uniqueness result to L^p solutions, $p \geq 1$; in this case the resolvent regularization does not seem to give too much, so that we use exactly the same method as in [7] (mollification) with some small changes, and obtain unicity in the almost-everywhere sense.

1. As indicated above, we are concerned in this section with a proof for the following

THEOREM 1. *Let X be a reflexive complex B -space; X^* be its dual space and A be a linear closed operator in X with dense domain $D(A) \subset X$ and with range in X too. Denote by A^* the dual operator to A (hence acting on the dense set $D(A^*) \subset X^*$). Assume also that the set of eigenvalues of A is not dense in any half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \alpha \in \mathbb{R}\}$. Let now $u(t) \in \mathbb{C}([0, \infty); X)$, such that*

- a) $u(0) = \theta$;
- b) $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|u(t)\| = \omega < \infty$;
- c) $\int_0^\infty \left\langle \frac{d}{dt} \phi^*(t) + (A^* \phi^*)(t), u(t) \right\rangle dt = 0$ for all $\phi^* \in K_{A^*}(0, \infty)$.

Then $u(t) = \theta$, for all $t \geq 0$.

(One denotes here with $K_{A^*}(0, \infty)$ the set of "test-functions" $\phi^*(t) \in C_0^1([0, \infty[; X)$, $\phi^*(t) \in D(A^*) \forall t \geq 0$, $A^* \phi^* \in C([0, \infty[; X^*)$)

PROOF: First we note that from condition b) above the following holds true: for $\omega_1 > \omega$ there is $R_1 > 0$ such that

$$\|u(t)\| \leq \exp(t\omega_1), \quad \text{for } t \geq R_1. \quad (1.1)$$

Next, we consider a sequence $\{\alpha_n(t)\}_1^\infty$ of scalar-valued, non negative C^1 -functions, vanishing for $|t| \geq 1/n$ and such that $\int_{\mathbb{R}} \alpha_n(s) ds = 1$, $n = 1, 2, \dots$ and then consider the convolutions

$$(u * \alpha_n)(t) = \int_{\mathbb{R}} u(s) \alpha_n(t-s) ds = \int_{t-1/n}^{t+1/n} u(s) \alpha_n(t-s) ds \quad (1.2)$$

which, given that $u(t)$ is only defined on $[0, \infty)$, will be well-defined for $t \geq t_0 > 0$, as soon as $n \geq n_0$ (where $t_0 - 1/n_0 \geq 0$).

We shall now use the estimate (1.1) in order to get a similar estimate for $u_n(t) = (u * \alpha_n)(t)$. Precisely

$$(u * \alpha_n)(t) = \int_{-1/n}^{1/n} u(t-\sigma) \alpha_n(\sigma) d\sigma,$$

hence

$$\|(u * \alpha_n)(t)\| \leq \int_{-1/n}^{1/n} \|u(t-\sigma)\| \alpha_n(\sigma) d\sigma. \quad (1.3)$$

Now, if $t \geq R_1 + 1$ and $|\sigma| < 1/n$, it follows that $t - \sigma \geq t - 1/n \geq t - 1 \geq R_1$; hence we get the estimate

$$\|(u * \alpha_n)(t)\| \leq \int_{-1/n}^{1/n} \exp((t-\sigma)\omega_1) \cdot \alpha_n(\sigma) d\sigma \quad \text{for } t \geq R_1 + 1. \quad (1.4)$$

Accordingly we obtain the inequality

$$\|(u * \alpha_n)(t)\| \leq \exp(t\omega_1) \int_{-1/n}^{1/n} \exp(-\sigma\omega_1) \cdot \alpha_n(\sigma) d\sigma \leq C_{\omega_1} \exp(t\omega_1), \quad (1.5)$$

which is valid for $t \geq R_1 + 1$, $n \geq n_0$ when $C_{\omega_1} = \sup_{|\sigma| \leq 1} \exp(-\sigma\omega_1)$. It results that the improper integral

$$\int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n)(t) dt, \quad n \geq n_0 \quad (1.6)$$

will exist, for all complex numbers λ such that $\operatorname{Re} \lambda > \omega$ (in fact once such λ is given, we take ω_1 such that $\omega < \omega_1 < \operatorname{Re} \lambda$ and then apply (1.5))

On the other hand, as can be inferred from Theorem 2 in [11], we see that from c) above, combined with the reflexivity of the space X and with the closedness of the operator A , it follows

$$(u * \alpha_n)(t) \in D(A), \quad \text{for } t \geq t_0, \quad n \geq n_0, \quad \text{and } (u * \alpha_n)' = A(u * \alpha_n), \quad (1.7)$$

holds, again for $t \geq t_0$ and $n \geq n_0$. Taking any number $N > t_0$ and using integration by parts we immediately obtain the relation

$$\begin{aligned} \int_{t_0}^N \exp(-\lambda t)(u * \alpha_n)'(t) dt &= \int_{t_0}^N \exp(-\lambda t)A(u * \alpha_n)(t) dt \\ &= \exp(-\lambda N)(u * \alpha_n)(N) - \exp(-\lambda t_0)(u * \alpha_n)(t_0) + \lambda \int_{t_0}^N \exp(-\lambda t)(u * \alpha_n)(t) dt, \end{aligned} \quad (1.8)$$

this being true for any complex number λ . Next, we assume that $\operatorname{Re} \lambda > \omega_1$ and let $N \rightarrow \infty$. Using (1.5) and (1.6), we obtain that

$$\lim_{N \rightarrow \infty} \int_{t_0}^N \exp(-\lambda t)A(u * \alpha_n)(t) dt,$$

exists and also the equality

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{t_0}^N \exp(-\lambda t)A(u * \alpha_n)(t) dt \\ = \lambda \int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n) dt - \exp(-\lambda t_0)(u * \alpha_n)(t_0), \end{aligned} \quad (1.9)$$

$n \geq n_0$. We can now use Theorem 1.3.5 in [5] to obtain that

$$\int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n)(t) dt \in D(A)$$

and

$$A \int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n)(t) dt = \lambda \int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n) dt - \exp(-\lambda t_0)(u * \alpha_n)(t_0), \quad (1.10)$$

for $n \geq n_0$, $\operatorname{Re} \lambda > \omega_1$, which becomes

$$(\lambda I - A) \int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n)(t) dt = \exp(-\lambda t_0)(u * \alpha_n)(t_0), \quad \text{for } n \geq n_0, \operatorname{Re} \lambda > \omega_1.$$

As we let $n \rightarrow \infty$, we obtain: $(u * \alpha_n)(t_0) \rightarrow u(t_0)$ and also

$$\int_{t_0}^{\infty} \exp(-\lambda t)(u * \alpha_n) dt \rightarrow \int_{t_0}^{\infty} \exp(-\lambda t)u(t) dt,$$

(this last limit is found as a consequence of the dominated convergence theorem, for $\operatorname{Re} \lambda > \omega_1$). Using now closedness of the operator $(\lambda I - A)$ we see, again for $\operatorname{Re} \lambda > \omega_1$, that

$$\int_{t_0}^{\infty} \exp(-\lambda t)u(t) dt \in D(A) \text{ and } (\lambda I - A) \int_{t_0}^{\infty} \exp(-\lambda t)u(t) dt = \exp(-\lambda t_0)u(t_0). \quad (1.11)$$

Now, let us make $t_0 \downarrow 0$; again by closedness of the operator $(\lambda I - A)$ and by assumption a) above we derive that

$$\int_0^{\infty} \exp(-\lambda t)u(t) dt \in D(A) \text{ and } (\lambda I - A) \int_0^{\infty} \exp(-\lambda t)u(t) dt = \theta, \quad \operatorname{Re} \lambda > \omega_1. \quad (1.12)$$

As ω_1 is any number greater than ω , it follows that (1.12) actually holds in the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\}$. Let us assume now that $u(t) \not\equiv \theta$ on $[0, \infty]$. By the uniqueness properties of the Laplace transform it follows that the function $F(\lambda)$ given by $\int_0^{\infty} \exp(-\lambda t)u(t) dt$ is not zero identically in the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\}$. By a classical property of analytic functions, we infer that the set of zeros of the function $F(\lambda)$ is discrete. Hence, the set $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega, F(\lambda) \neq \theta\}$ is everywhere dense in the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\}$. On the other hand, we note that any complex number λ such that $F(\lambda) \neq \theta$ is an eigenvalue of the operator A (using (1.12)). Hence, we arrive at the conclusion that the point spectrum of A is dense in the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\}$ which contradicts the initial assumption. This demonstrates Theorem 1.

2. In this section we are again in a complex Banach space X but reflexivity is not assumed until later on.

Next, we consider a linear (unbounded) operator A with dense domain $D(A) \subset X$, but closedness of A is not yet assumed. In any case, the dual operator A^* , acting on the subset $D(A^*)$ of the dual space X^* is well-defined.

This permits definition of the class of test-functions $K_{A^*}(0, \infty)$ as in the previous section and accordingly the continuous ultraweak solutions on the open interval $]0, \infty[$ of the differential equation: $u'(t) - Au(t) = \theta$, meaning that the assumption c) of Theorem 1 is satisfied. If moreover the solution $u(t)$ belongs to the space $C([0, \infty); X)$ and $u(0) = u_0$ is given in advance we have the ultraweak abstract Cauchy problem (for continuous functions). Now, we shall state the following simple Corollary of the "Theorem" in the recent paper [10] (just take $M = D(A)$).

THEOREM 2. *Let us assume that for any element u_0 of the domain $D(A)$ there exists a unique solution $u(t)$ to the ultraweak Cauchy problem on $[0, \infty)$, such that $u(0) = u_0$ and $u(t) \in D(A)$ for all $t \geq 0$. Assume also correctness (i.e. pointwise continuous dependence) with respect to the initial data, on $[0, \infty)$. It follows that there exists an operator semigroup $U(t), (t \geq 0) \rightarrow L(X)$, $U(0) = I$, strongly continuous for $t > 0$, and such that $U(t)u_0 = u(t)$ for all $u_0 \in D(A)$.*

Note also — as in Remark 1 of [10] — that if $x \in X$ and $x \notin D(A)$, the continuous function on $]0, \infty[$ given by the equality: $v(t) = U(t)x$ is a solution to the relation c) of Th. 1 above, but it is not necessarily in the Domain $D(A)$ and can be discontinuous for $t = 0$). For strong solutions, a result similar to Theorem 2 is the Theorem 1.1 in [4, p. 26]. Under some more assumptions we obtain a solution which is in $C^1([0, \infty[; X)$. We have in fact

THEOREM 3. *Let us assume the hypotheses of Theorem 2 and furthermore the conditions*

- a) *The domain of the dual operator $D(A^*)$ is total in X^* (that is, $x^*(x) = 0, \forall x^* \in D(A^*)$, implies $x = \theta$).*
- b) *The commutativity relation: $AU(t)x = U(t)Ax, x \in D(A), t \geq 0$ holds.*

It follows that, for $x \in D(A)$, the ultraweak solution $u(t) = U(t)x$ has a strong continuous derivative on $]0, \infty[$.

PROOF OF THEOREM 3: First we note that from the condition b) one can infer that $A(U(t)x) \in C([0, \infty[; X)$ for all $x \in D(A)$. Thus, the (ultraweak) solution $u(t) = U(t)x$ is strongly continuous for $t \geq 0$, belongs to $D(A)$ on $[0, \infty)$ and is such that $t \rightarrow (Au)(t)$ is continuous for $t > 0$. Now, we can derive existence and continuity of the (strong) derivative $u'(t)$ on $]0, \infty[$ in the following (usual) way (compare with article [11, Prop. 3]).

Let us take in fact, in the identity c) of Theorem 1 above, some special test function $\varphi^*(t) = \nu(t)x^*$, where $\nu(t) \in C_0^1([0, \infty[)$ and $x^* \in D(A^*)$. It results, accordingly, that the identity

$$\int_0^\infty \langle \nu'(t)x^* + \nu(t)A^*x^*, u(t) \rangle dt = 0, \forall \nu(t) \in C_0^1([0, \infty[), x^* \in D(A^*) \quad (2.1)$$

holds.

Hence we obtain

$$\int_0^\infty \nu'(t) \langle x^*, u(t) \rangle dt = - \int_0^\infty \nu(t) \langle A^*x^*, u(t) \rangle dt = - \int_0^\infty \nu(t) \langle x^*, Au(t) \rangle dt \quad (2.2)$$

It follows readily that

$$\langle x^*, \int_0^\infty \nu'(t)u(t)dt \rangle = - \langle x^*, \int_0^\infty \nu(t)Au(t)dt \rangle, \forall x^* \in D(A^*) \quad (2.3)$$

Using the assumption a) we now infer that

$$\int_0^\infty \nu'(t)u(t)dt = - \int_0^\infty \nu(t)Au(t)dt, \nu(t) \in C_0^\infty([0, \infty[)$$

Thus, we can say that the vector-valued distribution T_u defined on $C_0^\infty([0, \infty[)$ by the relation $T_u(\varphi) = \int_0^\infty u(t)\varphi(t)dt$ has a continuous derivative on $]0, \infty[$ which is the function $(Au)(t)$. This is enough to get the result. $\square$

Turning back to the assumptions of Theorem 2 we may note that the semigroup $U(t)$ obtained there has the same properties as in ([4, p. 28, 29]). Therefore we see that

$$\text{There exists a number } \omega \geq -\infty \text{ such that } \lim_{t \rightarrow \infty} \frac{1}{t} \log \|U(t)\| = \omega < +\infty. \quad (2.4)$$

This number ω will be called the type of the semigroup $U(t)$ and also the type of the corresponding ultraweak Cauchy problem.

We may close this section with the following

PROPOSITION 1. Under the assumptions of Theorem 2, the eigenvalues-values of the operator A are contained in the complex half-plane $\{\lambda \in \mathbb{C}, \operatorname{Re} \lambda \leq \omega\}$.

PROOF: Let in fact be $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$ with $x \in D(A)$ and $x \neq \theta$. Put then $u(t) = \exp(\lambda t)x$; we see that $u'(t) = \lambda u(t) = Au(t)$, $t \geq 0$. This relation implies the condition c) of Theorem 1 in an obvious manner. It follows that $u(t) = U(t)x = \exp(\lambda t)x$, and accordingly we find the estimate

$$\|\exp(\lambda t)x\| = \exp(t \operatorname{Re} \lambda)\|x\| = \|U(t)x\| \leq \|U(t)\|\|x\|$$

so that we get

$$\exp(t \operatorname{Re} \lambda) \leq \|U(t)\|, t > 0,$$

and then

$$t \operatorname{Re} \lambda \leq \log \|U(t)\|, \operatorname{Re} \lambda \leq \frac{1}{t} \log \|U(t)\|, t \geq 0. \quad (2.5)$$

As $t \rightarrow \infty$ we get $\operatorname{Re} \lambda \leq \omega$. $\square$

3. In this (short) section we study the Laplace transformation of the above obtained semigroup $U(t)$, that is the integral

$$\int_0^\infty \exp(-\lambda t)U(t)x dt, \quad \text{when } x \in D(A) \quad (3.1)$$

This integral is well-defined (as $\lim_{R \rightarrow \infty} \int_0^R \exp(-\lambda t)U(t)x dt$) for all complex numbers λ such that $\operatorname{Re} \lambda > \omega$.

In fact, for $x \in D(A)$, the function $U(t)x$ is continuous $[0, \infty) \rightarrow X$; this accounts for existence of the integral

$$\int_0^R \exp(-\lambda t)U(t)x dt, \forall \lambda \in \mathbb{C}, \forall R \geq 0 \quad (3.2)$$

Next, using relation (2.4), we find that if $\omega_1 > \omega$, $\exists \bar{t}$ such that $\|U(t)\| \leq \exp(t\omega_1)$, for $t \geq \bar{t}$ and accordingly we get

$$\|\exp(-\lambda t)U(t)x\| = \exp((- \operatorname{Re} \lambda)t)\|U(t)x\| \leq \exp(t(\omega_1 - \operatorname{Re} \lambda))\|x\|, t \geq \bar{t}.$$

Now, if $\operatorname{Re} \lambda > \omega$, $\exists \omega_1$ such that $\omega < \omega_1 < \operatorname{Re} \lambda$. This implies existence of the (Bochner) integral

$$\int_0^\infty \exp(-\lambda t)U(t)x dt, \text{ for } \operatorname{Re} \lambda > \omega \quad (3.3)$$

In the sequel we shall try to obtain a relation between this (Laplace) integral and the operator $(\lambda I - A)^{-1}$, similar (but however different) with the usual formula in the theory of C_0 -semigroups, where the equality

$$(\lambda I - A)^{-1}x = \int_0^\infty \exp(-\lambda t)U(t)x dt \quad (3.4)$$

holds for all $x \in X$, A being there the infinitesimal generator of the C_0 -semigroup $U(t)$. We shall denote by $u(t)$ the ultraweak solution $U(t)x$, for $x \in D(A)$; we know that $u(t)$ belongs to the domain $D(A)$ for all $t > 0$, but we know nothing about the existence of

the (continuous) derivative $u'(t)$ for the same values of t . We shall consider the restricted situation (as in section 1) where *the space X is a reflexive, complex B -space and A is a linear closed operator in X with dense domain.*

We can also infer, from relations $\|u(t)\| = \|U(t)x\| \leq \|U(t)\|\|x\|$ that

$$\frac{1}{t} \log \|u(t)\| \leq \frac{1}{t} \log \|U(t)\| + \frac{1}{t} \log \|x\|,$$

hence it follows that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \|u(t)\| \leq \omega$$

(here ω is the type of the semigroup $U(t)$). This estimate indicates that we can use some ideas and techniques in previous section 1, and in fact we now follow essentially the proof of Theorem 1 above, with only some minor modifications.

Again, we note that, for any $\omega_1 > \omega$ there is $R_1 > 0$ such that

$$\|u(t)\| \leq \exp(t\omega_1), t \geq R_1 \quad (3.5)$$

Next, we shall consider the convolutions $(u * \alpha_n)$ as in (1.2), where $t \geq t_0 > 0$, $n \geq n_0$. As in that proof we arrive at the situation (1.10) and later on at the equality

$$(\lambda I - A) \int_{t_0}^{\infty} \exp(-\lambda t) u(t) dt = \exp(-\lambda t_0) u(t_0) \quad (3.6)$$

Now, we let $t_0 \downarrow 0$; using closedness of the operator $(\lambda I - A)$ we obtain that

$$\int_0^{\infty} \exp(-\lambda t) u(t) dt \in D(A) \text{ and } (\lambda I - A) \int_0^{\infty} \exp(-\lambda t) u(t) dt = u(0) = x \quad (3.7)$$

Accordingly, for $x \in D(A)$ we obtain that $\int_0^{\infty} \exp(-\lambda t) U(t)x dt \in D(A)$ and the equality

$$x = (\lambda I - A) \left(\int_0^{\infty} \exp(-\lambda t) U(t)x dt \right), \quad \text{for } \operatorname{Re} \lambda > \omega \quad (3.8)$$

Therefore, the domain $D(A)$ is contained in the range of the operator $(\lambda I - A)$.

On the other hand, we can use Proposition 1 to infer the existence of the inverse operator $(\lambda I - A)^{-1}$ for $\operatorname{Re} \lambda > \omega$ (which is defined on the image of $(\lambda I - A)$ hence (*a fortiori*) on $D(A)$.) It follows that the fundamental formula

$$(\lambda I - A)^{-1} x = \int_0^{\infty} \exp(-\lambda t) U(t)x dt, x \in D(A), \operatorname{Re} \lambda > \omega \quad (3.9)$$

is satisfied.

4.

We start this section with an improvement of the "Theorem" in [7]. Precisely, we have the following.

THEOREM 4. Let X be a complex Banach space, A a linear closed operator with dense domain $D(A) \subset X$ and with the following supplementary property: there exists a simple Jordan arc Γ going to ∞ , lying in a closed angle in the open upper half λ -plane, on which the operator $R(\lambda) = (\lambda I - A)^{-1}$ belongs to $L(X)$ and satisfies an estimate

$$\|R(\lambda)\| = O(\exp(\alpha \operatorname{Im} \lambda)) \quad \text{for some } \alpha \geq 0. \quad (4.1)$$

Let $u(t) \in C([0, T]; X)$, such that $u(T) = \theta$ and satisfying the equality

$$\int_0^\infty \left\langle \frac{1}{i} \frac{d\Phi^*}{dt} + A^* \Phi^*, u \right\rangle dt = 0, \quad \forall \Phi^* \in K_{A^*}(0, T) \quad (4.2)$$

(class of test-functions as in [7]).

Then, if $T > \alpha$, it follows that

$$u(t) = \theta \quad \text{for } \alpha \leq t \leq T. \quad (4.3)$$

(Note that we do not assume here the density of the domain $D(A^*)$)

In order to prove this result we shall use the resolvent regularization method, following our articles [12] and [13] (precisely, the Proposition 1 in [13])

First, if we multiply by $(-1)^{1/2}$ in (4.2) we obtain the relation

$$\int_0^T \left\langle \frac{d\Phi^*}{dt} + (iA)^* \Phi^*, u \right\rangle dt = 0, \quad \text{for all } \Phi^* \in K_{A^*}(0, T) \quad (4.4)$$

Let us take in the complex plane a number λ such that $\lambda/i \in \rho(A)$ (due to the inclusion $\Gamma \subset \rho(A)$ it is easy to find many such λ 's). It follows that $\lambda \in \rho(iA)$ (in fact, the bounded operator $1/i (\frac{\lambda}{i} I - A)^{-1}$ is the inverse of the operator $\lambda I - iA$, as easily seen). Consider now the function v which is defined by: $v(t) = (\lambda I - iA)^{-1} u(t)$, $0 \leq t \leq T$. It follows that $v'(t) = (iA)v(t)$ on $[0, T]$. Applying now Theorem 1.1 in [1] we find that $v(t) = \theta$ for $\alpha \leq t \leq T$. Then we get: $u(t) = (\lambda I - iA)v(t) = \theta$ for $\alpha \leq t \leq T$.

REMARK. The above method of proof (using resolvent regularization) is very short indeed. Unfortunately, it does not seem to work when instead of continuous ultraweak solutions we consider solutions belonging to the (Bochner) space $L^p([0, T]; X)$ with any $p \geq 1$. We shall therefore assume now that $u \in L^p([0, T]; X)$, $p \geq 1$ and the equality

$$\int_0^T \left\langle \frac{1}{i} \frac{d\Phi^*}{dt} + A^* \Phi^*, u(t) \right\rangle dt = 0, \quad \forall \Phi^* \in K_{A^*}(0, T] \quad (4.5)$$

(The class of vector-valued test-functions here denoted by $K_{A^*}(0, T]$ is somewhat different from previous $K_{A^*}(0, T)$: it consists of functions $\Phi^*(t)$, $[0, T] \rightarrow D(A^*)$, X^* -continuously differentiable, $A^* \Phi^* \in C([0, T]; X^*)$, $\Phi^* = 0$ near $t = 0$, but not necessarily near $t = T$). We note that in this way the relation (4.5) subsumes the joint condition $u(T) = \theta$ and equality (4.2) in the above Theorem 4.

Thus, validity of (4.5) is the (ultra)weak form of saying that

$$“u(T) = \theta \text{ and } \frac{1}{i} \frac{du}{dt} - Au = \theta \text{ on } [0, T]”.$$

We shall prove below that *under the additional hypothesis that $D(A^*)$ is dense in X^* , the function $u(t)$ equals zero almost-everywhere on the interval $[\alpha, T]$, for $\alpha < T$.*

To demonstrate this statement, let us first consider the function $\tilde{u}(t)$ which is $u(t)$ on $[0, T]$ and is zero for $t > T$, as a function from $[0, \infty)$ to X . We see immediately that the equality

$$\int_0^\infty < \frac{1}{i} \frac{d\Phi^*}{dt} + (A^* \Phi^*)(t), \tilde{u}(t) > dt = 0, \forall \Phi^* \in K_{A^*}(0, \infty) \quad (4.6)$$

holds, where $K_{A^*}(0, \infty)$ is the class of test-functions appearing in Theorem 1.

Also, in an obvious manner, equality (4.6) is the same as

$$\int_0^\infty < \frac{d\Phi^*}{dt} + (iA)^*, \tilde{u}(t) > dt = 0, \forall \Phi^* \in K_{A^*}(0, \infty) \quad (4.7)$$

Take now a modifying sequence as previously, $\{\alpha_n(t)\}_1^\infty$ and then form the convolutions $(\tilde{u} * \alpha_n)(t)$ which are $C^1([1/n, \infty); X)$ -functions.

It results, as proved in [8], that, denoting with J the canonical mapping of X into its bidual space X^{**} , the X^{**} -valued functions $J(\tilde{u} * \alpha_n)(t)$ belong to the domain $D(A^{**})$, for $t \geq 1/n$ and also the equations

$$\frac{1}{i} \left(\frac{d}{dt} \right) J(\tilde{u} * \alpha_n) = A^{**} J(\tilde{u} * \alpha_n), \quad t \geq \frac{1}{n} \quad (4.8)$$

are satisfied. Note also the relations

$$J(\tilde{u} * \alpha_n) \left(T + \frac{1}{n} \right) = \theta, n = 1, 2, \dots \quad (4.9)$$

which are a consequence of the vanishing of $\tilde{u}(t)$ for $t \geq T$.

Make now in (4.8) the substitutions $t = \sigma + 1/n$ and $J(\tilde{u} * \alpha_n)(\sigma + 1/n) = v_n(\sigma)$, for $\sigma \geq 0$. We obtain accordingly the new equations

$$\frac{1}{i} \left(\frac{d}{d\sigma} \right) v_n(\sigma) = A^{**} v_n(\sigma), \quad 0 \leq \sigma < \infty \text{ and } v_n(T) = \theta \quad (4.10)$$

On the other hand, as seen in [7], the resolvent of the operator A^{**} , that is the operator $(\lambda I - A^{**})^{-1}$ is well-defined on the curve Γ and satisfies there the estimate

$$\|R(\lambda; A^{**})\|_{L(X^{**})} = O(\exp(\alpha \operatorname{Im} \lambda)).$$

Then it can be inferred from the paper [1] that, for all $n = 1, 2, \dots$, the equality $v_n(\sigma) = \theta$, for $\alpha \leq \sigma \leq T$ is satisfied. Hence, we obtained that $J(\tilde{u} * \alpha_n)(\sigma + 1/n) = \theta$ for $\alpha \leq \sigma \leq T$ and accordingly we see that $J(\tilde{u} * \alpha_n)(t) = \theta$ for $\alpha + 1/n \leq t \leq T + 1/n$ which implies, as well-known, that, in the original space X , the equality $(\tilde{u} * \alpha_n)(t) = \theta$ holds for $\alpha + 1/n \leq t \leq T + 1/n$, hence also $(\tilde{u} * \alpha_n)(t) = \theta$ for all $t \geq \alpha + 1/n$. Let us define now a function $w(t)$, $R \rightarrow X$, by the formula: $w(t) = \tilde{u}(t)$ for $t \geq 0$ and $w(t) = \theta$, for $t < 0$. It follows that w belongs to the Bochner space $L^p(R; X)$. It results, as well-known, that $w * \alpha_n \rightarrow w$ as $n \rightarrow \infty$ in the sense of the space $L^p(R; X)$. Take now an interval $[\alpha', T]$ where $\alpha < \alpha' < T$. We see that:

$$\lim_{n \rightarrow \infty} \int_{\alpha'}^T \|w * \alpha_n - w\|^p dt = 0.$$

On the other hand we may note that

$$(w * \alpha_n)(t) = \int_{t-1/n}^{t+1/n} w(\sigma) \alpha_n(t-\sigma) d\sigma = \int_{t-1/n}^{t+1/n} \tilde{u}(\sigma) \alpha_n(t-\sigma) d\sigma = (\tilde{u} * \alpha_n)(t), \quad \text{for } t \geq \alpha'$$

and large enough n . Also, we have that $w(t) = u(t)$ on the interval $[\alpha', T]$. Therefore we obtain

$$\lim_{n \rightarrow \infty} \int_{\alpha'}^T \|(\tilde{u} * \alpha_n)(t) - u(t)\|^p dt = 0.$$

But we know already that $(\tilde{u} * \alpha_n)(t) = \theta$ for $t \geq \alpha'$ and $n \geq n_0$. It follows that $\int_{\alpha'}^T \|u(t)\|^p dt = 0$ and accordingly: $u(t) = \theta$, almost-everywhere on $[\alpha', T]$. As the number $\alpha' > \alpha$ is arbitrarily chosen, we get $u(t) = \theta$ a.e. on $[\alpha, T]$.

REFERENCES

1. Agmon, S., Nirenberg, L., *Properties of Solutions of Ordinary Differential Equations in Banach Space*, Comm. Pure and Appl. Math. **XVI** (1963), 121-239.
2. Hille, E., Phillips, R. S., *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. **31** (1957).
3. Kato, T., Tanabe, H., *On the abstract evolution equation*, Osaka Math. Journal **14** (1962), 107-133.
4. Krein, S. G., *Linear Differential Equations in Banach Space*, Translations of Mathematical Monographs (AMS) **29** (1971).
5. Ladas, G.E., Lakshmikantham, V., "Differential Equations in Abstract Spaces," Academic Press, New York, 1972.
6. Lions, J. L., *Equations différentielles à coefficients opérateurs non-bornés*, Bull. Soc. Math. France **86** (1958), 321-330.
7. Zaidman, S., *A remark on the weak backward Cauchy problem*, in "Romanian-Finnish Seminar on Complex Analysis," Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1979.
8. Zaidman S., *The weak Cauchy problem for abstract differential equations*, Rend. Semin. Math. Univ. Padova **56** (1977), 1-21.
9. Zaidman S., *Remarks on the well-posed weak Cauchy problem*, Boll. UMI (5) **17-B** (1980), 1012-1022.
10. Zaidman S., *A note on the well-posed ultraweak Cauchy problem*, Ann. sc. math. Québec **11** (1987), 371-378.
11. Zaidman S., *Remarks on weak solution of Differential Equations in Banach Spaces*, Boll. UMI (4) **9** (1974), 638-643.
12. Zaidman S., *Improving regularity of weak solutions of abstract differential equations*, Ann. sc. math. Québec **10** (1986), 103-108.
13. Zaidman S., *Remarks on very weak solutions of operator-differential equations*, Appl. Analysis **22** (no. 2) (1986), 149-156.

S. Zaidman

Centre de recherches mathématiques

et

Département de mathématiques et de statistiques

Université de Montréal

C.P. 6128, Succ. A

Montréal, Québec H3C 3J7