

MAPPING THEOREMS AND GENERAL NEWTON-KANTOROVICH PROCESSES

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RÉSUMÉ. On donne un théorème de résolubilité pour les applications fermés dans le contexte des espaces métriques/Banach ordonnés. On établit ensuite que ce résultat englobe les cas abstraits dûs à Altman ainsi que les cas numériques obtenus par Amann.

ABSTRACT. A solvability theorem for closed mappings is given, in the context of ordered metric/Banach spaces. This result is shown to contain the abstract ones due to Altman as well as the numerical ones obtained by Amann.

0. Introduction. Let (X, d) be a complete metric space, $(Y, \|\cdot\|)$ a Banach space and $T: D \subset X \rightarrow Y$ a *closed mapping* (for each sequence (x_n) in D , $x_n \rightarrow x$ and $Tx_n \rightarrow y$ imply $x \in D$ and $Tx = y$). Concerning the solvability of the operator equation $Tx = 0$, the following local mapping theorem established by Altman [1, Theorem 1.1] (cf. also Cramer and Ray [10, Theorem 2.1]) is of a great importance for many applications.

THEOREM 1. Suppose the continuous increasing function $b: [0, \infty) \rightarrow [0, \infty)$ with $\int_0^t b(s)/s \, ds < \infty$, $t > 0$, the point $x_0 \in D$ and the constants $q \in (0, 1)$, $\alpha > 0$, $M > 0$ fulfilling $(1 - q)^{-1} \int_0^a (b(s)/s) \, ds \leq \alpha$ (where $a = M\|Tx_0\|e^{1-q}$) are such that, for each $x \in D$ with $d(x_0, x) \leq \alpha$, there exist $x' \in D$ and $\varepsilon \in (0, 1]$ with

$$\|Tx' - (1 - \varepsilon)Tx\| \leq \varepsilon q\|Tx\|, \quad d(x, x') \leq \varepsilon b(M\|Tx\|). \quad (1)$$

Then $Tz = 0$ for at least one $z \in D$ with $d(x_0, z) \leq \alpha$.

As already indicated by the above quoted authors, Theorem 1 is a common extension of some related contributions in this area due to Pokhozhaev [20], Browder [7], Kachurovskij [14], Zabreiko and Krasnoselskij [27], Kirk and Caristi [16], Downing and Kirk [13]. It is our first aim in the following to indicate a more general framework in which the conclusion of this result should remain valid and, at the same time, to clarify its relationships to another extension of Theorem 1 due to Ray and Walker [21], based on the Brézis-Browder ordering principle [6]. Furthermore, a constructive (Newton-Kantorovich) version of the main result (containing, in particular a related Altman's contribution [1, Theorem 2.1]) is presented. And, finally, as an argument for its usefulness, an existence and approximation result for a class of equations with accretive operators from a (reflexive) Banach space into itself is given obtaining, among others, a more general version of Amann's related statement [5, Theorem 6] with immediate applications to Hammerstein equations. Some further extensions to metrizable structures in the sense of Turinici [23] will be given elsewhere.

1. Main results. Let (X, d) be a metric space and let $\leq$ be a *quasi-ordering* (that is, a reflexive and transitive relation) on X . Given a sequence (x_n) in X , we shall term it *ascending* (*descending*) when $x_i \leq x_j$ ($x_i \geq x_j$) for $i \leq j$, and *bounded above* (*below*) provided that $x_n \leq x$ ($x_n \geq x$), $n \in \mathbb{N}$, in which case x will be termed an *upper* (*lower*) *bound* of the considered sequence. Under a convention precised in Turinici [24], call the ambient space X , $\leq$ -*complete* when each ascending Cauchy sequence converges, and the initial quasi-ordering $\leq$, *self-closed* if the limit of an ascending sequence is an upper bound of it. Finally, given the normed space $(Y, \|\cdot\|)$ and introducing a quasi-ordering $\leq$ on it by the convention $u \leq v$ if and only if $\|u\| \geq \|v\|$ (observe at this moment $\leq$ is *semi-closed* in the sense of Nachbin [18, Appendix] hence self-closed) let us call the mapping T from $D \subset \text{dom}(T)$ to Y , $(\leq, \leq)$ -*closed* (or, equivalently, after a terminology suggested by Browder [8], $(\leq, \leq)$ -*normally solvable*) when (x_n) ascending in D , (Tx_n) ascending in Y and $x_n \rightarrow x$, $Tx_n \rightarrow y$ imply $x \in D$ and $Tx = y$. Suppose further X is $\leq$ -complete, $\leq$ is self-closed and Y is $\leq$ -complete. Given the subset D of $\text{dom}(T)$, the $(\leq, \leq)$ -closed mapping $T: D \rightarrow Y$ and the point $x_0 \in D$ (which, actually, may be taken so as to satisfy $Tx_0 \neq 0$) we are interested to find local (modulo x_0) infinitesimal conditions (of the form (1)) under which the (operator) equation $Tx = 0$ should have a solution in a D -sphere around x_0 . In this direction, the main result of the present paper is

THEOREM 2. Suppose a couple of functions $a: (0, \infty) \rightarrow (0, \infty)$, $f: (0, \infty) \rightarrow (0, \infty)$ with

(i) $a(t) < t$ and $a^n(t) \rightarrow 0$ as $n \rightarrow \infty$, for all $t > 0$ (where a^n denotes the n^{th} iterate of the function a)

(ii) $\frac{t}{t-a(t)} \leq \frac{f(t)-f(s)}{t-s}$, $t, s > 0$, $a(t) \leq s < t$

as well as a couple of functions $b: (0, \infty) \rightarrow [0, \infty)$, $g: (0, \infty) \rightarrow [0, \infty)$ with

(iii) $\frac{b(t)}{t-a(t)} \leq \frac{g(t)-g(s)}{t-s}$, $t, s > 0$, $a(t) \leq s < t$

may be found so as, for each x in

$$D(x_0, g, f) = \left\{ x \in D \left| \begin{array}{l} x_0 \leq x, Tx_0 \leq Tx, d(x_0, x) \leq g(\|Tx_0\|), \\ \|Tx_0 - Tx\| \leq 2f(\|Tx_0\|) \end{array} \right. \right\}$$

with $Tx \neq 0$ there correspond x' in D and ε in $(0, 1]$ with $x \leq x'$ and

$$\|Tx' - (1 - \varepsilon)Tx\| \leq \varepsilon a(\|Tx\|), \quad d(x, x') \leq \varepsilon b(\|Tx\|). \quad (2)$$

Then, $Tx = 0$ has at least a solution in $D(x_0, g, f)$.

PROOF: By (2) (the first half) we have the a priori evaluations

$$\|Tx'\| \leq (1 - \varepsilon)\|Tx\| + \varepsilon a(\|Tx\|)$$

and

$$\|Tx - Tx'\| \leq \varepsilon(\|Tx\| + a(\|Tx\|)) < 2\varepsilon\|Tx\|.$$

Denoting for simplicity $t = \|Tx\|$, $s = (1 - \varepsilon)t + \varepsilon a(t)$ we clearly have $a(t) \leq s < t$, $\varepsilon = (t - s)/(t - a(t))$ and $\|Tx'\| \leq s$ (hence $Tx \leq Tx'$) in which case, by (ii)

$$\|Tx - Tx'\| \leq \frac{2t}{t - a(t)}(t - s) \leq 2(f(t) - f(s));$$

moreover, by the second part of (2) we have, combining with (iii)

$$d(x, x') \leq \frac{b(t)}{t - a(t)}(t - s) \leq g(t) - g(s).$$

Suppose by contradiction $Tx \neq 0$ for each $x \in D_0 = D(x_0, g, f)$ and let E stand for the subset of all $x \in D_0$ with $d(x_0, x) \leq g(\|Tx_0\|) - g(\|Tx\|)$, $\|Tx_0 - Tx\| \leq 2(f(\|Tx_0\|) - f(\|Tx\|))$. By the evaluations we just obtained it follows that $x \in E$ implies $x_0 \leq x'$, $Tx_0 \leq Tx'$ and

$$\begin{aligned} d(x_0, x') &\leq d(x_0, x) + d(x, x') \leq g(\|Tx_0\|) - g(s) \leq g(\|Tx_0\|) \\ \|Tx_0 - Tx'\| &\leq \|Tx_0 - Tx\| + \|Tx - Tx'\| \leq 2(f(\|Tx_0\|) - f(s)) \leq 2(f(\|Tx_0\|)) \end{aligned}$$

proving $x' \in D_0$; consequently, $0 < \|Tx'\| \leq s$ in which case, adding the fact that, by (i)–(iii), g and f are increasing on $(0, \infty)$, the above relations give

$$\begin{aligned} d(x, x') &\leq g(\|Tx\|) - g(\|Tx'\|) \\ \|Tx - Tx'\| &\leq 2(f(\|Tx\|) - f(\|Tx'\|)) \end{aligned}$$

or, in other words, $x' \in E$. Let e denote the product metric in D

$$e(x, y) = \max(d(x, y), \|Tx - Ty\|), \quad x, y \in D,$$

and $\leq$ the ordering on D_0 defined as

$$x \leq y \text{ if and only if } \begin{cases} x \leq y, Tx \leq Ty \\ d(x, x') \leq g(\|Tx\|) - g(\|Tx'\|) \\ \|Tx - Tx'\| \leq 2(f(\|Tx\|) - f(\|Tx'\|)) \end{cases}.$$

Suppose (x_n) is an ascending (modulo $\leq$) sequence in E , namely $x_n \leq x_m$, $Tx_n \leq Tx_m$ and

$$\begin{aligned} d(x_n, x_m) &\leq g(\|Tx_n\|) - g(\|Tx_m\|) \\ \|Tx_n - Tx_m\| &\leq 2(f(\|Tx_n\|) - f(\|Tx_m\|)), \end{aligned}$$

whenever $n \leq m$. As $(g(\|Tx_n\|); n \in \mathbb{N})$ and $(f(\|Tx_n\|); n \in \mathbb{N})$ are decreasing (hence Cauchy) sequences in $[0, \infty)$, (x_n) is an ascending (modulo $\leq$) Cauchy sequence in $E \subset D_0 \subset D$ and (as f is increasing (strictly) on $(0, \infty)$), (Tx_n) an ascending Cauchy sequence in Y so, by our completeness/closedness hypotheses $x_n \xrightarrow{e} x$ for some $x \in D$. Moreover, D_0 being closed with respect to ascending (modulo $\leq$) e -convergent sequences, $x \in D_0$ (hence $Tx \neq 0$) and $x_n \leq x$, $Tx_n \leq Tx$ for all $n \in \mathbb{N}$. So (again using the monotonicity properties of g and f)

$$\begin{aligned} d(x_n, x_m) &\leq g(\|Tx_n\|) - g(\|Tx\|) \\ \|Tx_n - Tx_m\| &\leq 2(f(\|Tx_n\|) - f(\|Tx\|)), \quad n \leq m, \end{aligned}$$

in which case, passing to limit as $m \rightarrow \infty$ one immediately gets $x_n \leq x \in E$, $n \in \mathbb{N}$. This shows all conditions involved in author's maximality principle [24, Theorem 1] are fulfilled in $(E, e, \leq)$; and, by the conclusion of that result, a maximal (modulo $\leq$ element

$z \in E \subset D_0$ may be chosen so that $x_0 \leq z$. However, as $Tz \neq 0$, the first part of the proof tells us an element $z' \in E$ may be found with $z \leq z'$ and $z \neq z'$. This contradiction proves the assumption $Tx \neq 0$ for each $x \in D_0$ is false and the result follows. $\square$

A useful remark about this result may be presented under the lines below. Let $c: [0, \infty) \rightarrow (0, \infty)$ be a decreasing function satisfying

$$h(\infty) > g(\|Tx_0\|) \quad (\text{where } h(t) = \int_0^t c(s) ds, \quad t \geq 0)$$

and assume that, in the statement of Theorem 2, $D(x_0, g, f)$ is to be replaced by

$$D(x_0, g, h, f) = \left\{ x \in D \left| \begin{array}{l} x_0 \leq x, \quad Tx_0 \leq Tx, \quad h(d(x_0, x)) \leq g(\|Tx_0\|), \\ \|Tx_0 - Tx\| \leq 2f(\|Tx_0\|) \end{array} \right. \right\}$$

and the second part of (2) by

$$c(d(x_0, x)) d(x, x') \leq \varepsilon b(\|Tx\|). \quad (2')$$

Then (the notations being the same) in view of

$$c(d(x_0, x)) d(x, x') \leq g(t) - g(s)$$

it is clear that, $E^{(h)}$ denoting the subset E above with D_0 substituted $D_0^{(h)} = D(x_0, g, h, f)$ and $d(x_0, x)$ by $h(d(x_0, x))$, we have at each $x \in E^{(h)}$

$$\begin{aligned} h(d(x_0, x')) &\leq h(d(x_0, x) + d(x, x')) - h(d(x_0, x)) + h(d(x_0, x)) \\ &\leq c(d(x_0, x)) d(x, x') + h(d(x_0, x)) \\ &\leq g(t) - g(s) + g(\|Tx_0\|) - g(t) \\ &= g(\|Tx_0\|) - g(s) \leq g(\|Tx_0\|). \end{aligned}$$

This proves $x' \in D_0^{(h)}$, as well as (in view of (2'))

$$d(x, x') \leq \varepsilon \frac{b(\|Tx\|)}{c(\lambda)} \quad (\text{where } \lambda = h^{-1}(g(\|Tx_0\|)))$$

if we take into account that, as an immediate consequence of definition

$$c(t) \leq \frac{h(t) - h(s)}{t - s} \leq c(s), \quad 0 \leq s < t.$$

From this point on, the argument of Theorem 2 is working on the structure $(E^{(h)}, e, \leq)$ where, in the definition of $\leq$, g is to be substituted by $g/c(\lambda)$; in other words, we proved

THEOREM 3. *Under the same hypotheses about X, D, Y and T , let $x_0 \in D$ with $Tx_0 \neq 0$ be such that, for each x in $D_0^{(h)} = D(x_0, g, h, f)$ with $Tx \neq 0$ there correspond x' in D and ε in $(0, 1]$ for which (2) (with second part substituted by (2')) is holding. Then the operator equation $Tx = 0$ has at least a solution in $D_0^{(h)}$.*

Concerning the second part of (i) (which, as indicated by Matkowski [17], surely contains the first part of (i) when a is increasing on $(0, \infty)$) remark that a practical circumstance assuring its validity is

$$(i') \quad \limsup_{s \rightarrow t+} a(s) < t, \quad \text{for all } t > 0,$$

the signifiacnce of the “lim sup” operator being the standard one (see, e.g., Turinici [22]). At the same time supposing that, in addition to (i), we accept

$$(iv) \quad t \mapsto t/(t - a(t)), \quad t \mapsto b(t)/(t - a(t)) \text{ are decreasing and } A(t) = \int_0^t s/(s - a(s)) ds, \\ B(t) = \int_0^t b(s)/(s - a(s)) ds \text{ are both finite for all } t > 0$$

conditions (ii) + (iii) will be clearly fulfilled with $f = A$, $g = B$. Alternatively, still admitting (i), plus

$$(v) \quad t \mapsto t - a(t) \text{ is increasing and a function } d: (0, \infty) \rightarrow (0, \infty) \text{ exists with the property:} \\ t, s > 0, a(t) \leq s < t \text{ imply } t \leq d(s)$$

assume (iv) is replaced by

$$(iv') \quad t \mapsto d(t), \quad t \mapsto b(t) \text{ are increasing and } A^*(t) = \int_0^t d(s)/(s - a(s)) ds, \quad B^*(t) = \\ \int_0^t b(d(s))/(s - a(s)) ds \text{ are both finite, for all } t > 0$$

then (ii) + (iii) hold with $f = A^*$, $g = B^*$. In particular, when $a(t) = qt$, $t > 0$, for some $q \in (0, 1)$, conditions (i) plus (v) will be satisfied; in this case, under an assumption of the form (iv'), our main result reduces to Theorem 1 if one adds the remark that, in the above reasonings, the information $\|Tx_0 - Tx\| \leq 2f(\|Tx_0\|)$ (used in the construction of $D(x_0, g, f)$ and $D(x_0, g, h, f)$) may be neglected without affecting them. Of course, there are many other choices of this function which cannot be covered by the above one, such as, e.g., that described by

$$a(t) = \frac{t}{1 + \mu t^r}, \quad t > 0, \quad \text{for some } \mu > 0, \quad r \in (0, 1),$$

the motivation of this assertion is a consequence of the property $a(t)/t \rightarrow 1$ as $t \rightarrow 0+$. Finally, passing to the basic assumptions expressed by the inequalities in (2), let us assume the metric structure on X (supposed to be a linear space) is introduced by a norm and let X_+ be a closed *quasi-cone* of X (that is a subset of X with the property $\lambda x + \mu y \in X_+$ whenever $\lambda, \mu \geq 0$ and $x, y \in X_+$); then, modulo the quasi-ordering $\leq$ on X induced by X_+ , it is clear that (if we denote by $T'(x)(\cdot)$ the usual Gâteaux derivative of T at x) the following one suffices for its validity

$$\|T'(x)(v) + Tx\| \leq a(\|T(x)\|) \text{ and } \|v\| \leq b(\|Tx\|) \text{ for some } v \in X_+;$$

indeed, it is enough to remark that if we put $x' = x + \varepsilon v$, with $\varepsilon > 0$ sufficiently small, the assertion follows by the definition of $T'(x)$; cf. also Cramer and Ray [10].

As already pointed out in a previous place Theorem 3 is technically equivalent with Theorem 2; in particular, this tells us the basic solvability results in Ray and Walker [21, Theorems 3.1 and 3.2] are in fact reducible to (hence equivalent with) Altman's result quoted as Theorem 1 (cf. also [3, Theorem 3.2]). In the light of these considerations, the remark in Park and Bae [19] that the fixed point result in Ray and Walker (see the above reference), claimed as an extension of the Caristi-Kirk theorem [9] is nothing but an equivalent formulation of it, seems to be not very surprising. To get an alternative proof of this fact, let (X, d) be a complete metric space, $\phi: X \rightarrow [0, \infty)$ a lower semicontinuous function, and $c: [0, \infty) \rightarrow (0, \infty)$ a continuous decreasing function. Let the self-mapping T of X be such that, for some $x_0 \in X$, $h(\infty) > \phi(x_0)$ (where $h: [0, \infty) \rightarrow (0, \infty)$) is taken as in Theorem 3) and

$$c(d(x, x_0)) d(x, Tx) \leq \phi(x) - \phi(Tx), \quad x \in X. \quad (4)$$

Introducing the subset Y of X by

$$Y = \{x \in X \mid h(d(x, x_0)) \leq \phi(x_0) - \phi(x)\}$$

it is manifest that it is nonempty ($x_0 \in Y$) and closed (as h is continuous and ϕ is lsc). Let $x \in Y$, that is, $h(d(x, x_0)) \leq \phi(x_0) - \phi(x)$. By (3) plus (4),

$$\begin{aligned} h(d(Tx, x_0)) &\leq h(d(x, Tx) + d(x, x_0)) \\ &= h(d(x, Tx) + d(x, x_0)) - h(d(x, x_0)) + h(d(x, x_0)) \\ &\leq c(d(x, x_0))d(x, Tx) + h(d(x, x_0)) \\ &\leq \phi(x) - \phi(Tx) + \phi(x_0) - \phi(x) \\ &= \phi(x_0) - \phi(Tx) \end{aligned}$$

or, in other words, $Tx \in Y$. Moreover, $h(d(x, x_0)) \leq \phi(x_0)$ gives at once (in view of $h(\infty) > \phi(x_0)$)

$$d(x, x_0) \leq t_0, \quad \text{for some } t_0 > 0$$

and the obtained relation implies $c(d(x, x_0)) \geq c(t_0) > 0$ so, again by (4),

$$d(x, Tx) \leq \frac{\phi(x) - \phi(Tx)}{c(t_0)}.$$

Summing up, Y is closed, invariant under T and the above inequality holds for all $x \in Y$; consequently the Caristi-Kirk fixed point statement (see above reference) applies and the assertion is proved. A slight generalization of the main result could be reached if one replaces the normed structure of the image space Y by a metrizable (Fréchet) structure $S = \{|\cdot|_i \mid i \in \mathbb{N}\}$; for a partial indication about the corresponding versions we refer to [23].

2. Newton-Kantorovich processes. A close analysis of the proof in the main result shows the process $x \mapsto x'$ (generated by (2)) under which a solution in $D_0 = D(x_0, g, f)$ of the operator equation Tx_0 is to be reached, has a (generally) non-constructive character because, as results from [25], the maximal element of the structure $(E, \leq)$ cannot be viewed as a last point of a denumerable chain. It is natural then to ask under which additional conditions may a solution in D_0 of the equation in question be attained through a denumerable process of this form. To be more precise, given $x_0 \in D_0$ with $Tx_0 \neq 0$, let $x_1 \in D_0$ and ε_0 in $(0, 1]$ be determined by (2) with $x = x_0$, supposing $Tx_1 \neq 0$, there exists a pair $x_2 \in D_0$, $\varepsilon_1 \in (0, 1]$ for which (2) holds with $x = x_1$ and so on. By an ordinary induction procedure one obtains (if we eliminate at each step the possibility of deriving, via (2), a solution of the considered equation) an increasing (modulo $\leq$) sequence (x_n) in D_0 and a sequence (ε_n) in $(0, 1]$ for which $t_n = \|Tx_n\| > 0$ for all $n \in \mathbb{N}$ and

$$\|Tx_{n+1} - (1 - \varepsilon_n)Tx_n\| \leq \varepsilon_n a(\|Tx_n\|), \quad d(x_n, x_{n+1}) \leq \varepsilon_n b(\|Tx_n\|), \quad n \in \mathbb{N}. \quad (5)$$

Our aim is now to find under what circumstances the sequence (x_n) which, from a technical reason explained by Altman [1] may be referred to as a *general Newton-Kantorovich process*, converges to a solution in D_0 of the operator equation $Tx = 0$. In this direction, the following answer may be formulated.

THEOREM 4. Suppose in the main result, condition (ii) were deleted and, in addition

- (vi) either $\varepsilon_n = 1$ for infinitely many ranks n and, in (i), $t \mapsto a(t)$ is increasing on $(0, \infty)$ or $\sum_n \varepsilon_n = \infty$ and the second part of (i) were substituted by (i').

Then, the general Newton-Kantorovich process (x_n) described by (5) converges to a solution $z \in D_0$ of the equation $Tx = 0$ under an error evaluation formula expressed as

$$d(x_n, z) \leq g(\|Tx_n\|) - g(0+), \quad n \in \mathbb{N}. \quad (6)$$

PROOF: As a consequence of (5),

$$t_{n+1} \leq (1 - \varepsilon_n)t_n + \varepsilon_n a(t_n), \quad n \in \mathbb{N}$$

so that, necessarily, (t_n) is decreasing and converges to a number $t \geq 0$. Now, in the first alternative of (vi), let $(n(k) \mid k \in \mathbb{N})$ be the (strictly increasing) sequence of ranks with $\varepsilon_{n(k)} = 1$; in view of

$$t_{n(k)} \leq a^k(t_{n(0)}), \quad k \in \mathbb{N},$$

it is clear that the case $t > 0$ cannot be accepted. In the second one, we note that, by (i'), $a(s) < \lambda s$ (hence $s - a(s) > (1 - \lambda)s > (1 - \lambda)t$), $t < s < t + \eta$, for some $\eta > 0$ sufficiently small and some λ in $(0, 1)$ sufficiently close to 1, if one assumes $t > 0$; this fact combined with

$$\varepsilon_n(t_n - a(t_n)) \leq t_n - t_{n+1}, \quad n \in \mathbb{N}$$

gives at once

$$(1 - \lambda)t\varepsilon_n \leq t_n - t_{n+1}, \quad n \geq n(t)$$

where $n(t) \in \mathbb{N}$ is large enough and therefore, as $\sum_n (t_n - t_{n+1})$ converges, the assumption about (ε_n) will be contradicted. Therefore, in any case we have $t_n \rightarrow 0$. At the same time, by the reasonings we developed in the proof of Theorem 2

$$d(x_n, x_m) \leq g(t_n) - g(t_m), \quad n \leq m$$

and this shows (x_n) is an ascending Cauchy (hence convergent) sequence in D_0 . Let z denotes its limit; we therefore have $x_n \rightarrow z \in D_0$ and $Tx_n \rightarrow 0$ so, by our closedness hypothesis, $Tz = 0$. Adding the fact that, as a consequence of the above relation

$$d(x_n, x_m) \leq g(t_n) - g(0+), \quad n \leq m,$$

relation (6) follows at once performing the limit as $m \rightarrow \infty$ and the proof is complete. $\square$

Concerning the key condition (vi), we note that, in case the required properties for (ε_n) fail to be satisfied, conclusion of Theorem 4 is still retainable when, alternatively, we stipulate that

$$(vi') \quad \sum_n \varepsilon_n > \sum_n (t_n - t_{n+1}) / (t_n - a(t_n))$$

provided the second series converges; in fact, it suffices to take into account the inequality encountered in the proof above. Of course, it might very well happen that (vi') be without object when the above mentioned series diverges but, in the convergence case, this procedure yields a Newton-Kantorovich process (x_n) whose associated sequence (ε_n) is, in general non-standard (in the sense of (vi)). In particular, when $a(t) = qt$, $t > 0$, for some $q \in (0, 1)$, the result we just exposed reduces to that of Altman [1, Theorem 2.1] proved by a different method.

3. An application. Let X be a Banach space X^* its (topological) dual and J the natural duality map of the couple (X, X^*) , that is

$$J(x) = \{x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X, \quad x \neq 0,$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between X and X^* . A self-mapping T of X will be called *accretive* when a constant $\alpha > 0$ exists with the property: for each couple $u, v \in X$, $u \neq v$,

$$\langle Tu - Tv, j \rangle \geq \alpha \|u - v\|^2, \quad \text{for some } j \in J(u - v).$$

Of course, under such an assumption, the operator equation $Tx = 0$ has at most a solution in X . Concerning the existence of this solution and its approximation by (general) Newton-Kantorovich processes, the following answer may be formulated.

THEOREM 5. Assume X is reflexive and the continuous accretive self-map T of X has a Fréchet derivative $T'(x) \in L(X)$ at each $x \in X$, with $x \mapsto T'(x)$ (from X to $L(X)$) lipschitzian on bounded subset of X . Then, given $x_0 \in X$ arbitrary fixed there exists a sequence (ε_n) in $(0, 1]$ with $\varepsilon_n = 1$ for $n \geq n(x_0)$ such that the (general) Newton-Kantorovich process

$$x_{n+1} = x_n - \varepsilon_n (T'(x_n))^{-1} T x_n, \quad n \in \mathbf{N}$$

converges to the unique solution $z \in X$ of the operator equation $Tx = 0$ under an error evaluation formula expressed as

$$\|x_n - z\| \leq \frac{1}{\alpha} \|T x_n\|, \quad n \in \mathbf{N}.$$

PROOF: The last assertion being evident by virtue of (7), the point is to prove the remaining ones. Firstly, as an easy consequence of (7) plus the the reflexivity of X we have at each pair $x, y \in X$, $y \neq 0$ the property

$$\langle T'(x)y, j \rangle \geq \alpha \|y\|^2, \quad \text{for some } j \in J(y);$$

so, by a result of Ray and Walker [21, Theorem 3.4] (see also Deimling [11]) $T'(x)$ is, for each $x \in X$, a homeomorphism of X onto X . Consequently $T'(x)^{-1}$ exists (as element of $L(X)$) and, moreover,

$$\|T'(x)^{-1}\| \leq \frac{1}{\alpha}, \quad \text{for all } x \in X.$$

Secondly, $x_0 \in X$ being arbitrary fixed put (for a given $q \in (0, 1)$)

$$D = \left\{ x \in X \mid \|x_0 - x\| \leq \frac{\|T x_0\| - \|T x\|}{\alpha(1 - q)} \right\}$$

and, indicating by λ the Lipschitz constant of $x \mapsto T'(x)$ on

$$D_0 = \left\{ x \in X \mid \|x_0 - x\| \leq \frac{\|T x_0\|}{\alpha(1 - q)} \right\}$$

denote

$$\varepsilon(x) = \min \left(1, \frac{q\alpha^2}{\lambda \|T x\|} \right), \quad x \in D, \quad T x \neq 0$$

$$h(x) = -T'(x)^{-1} T x, \quad x \in D.$$

Under these conventions, assume $x \in D$ is such that $Tx \neq 0$ and put $x' = x + \varepsilon(x)h(x)$. We have

$$\begin{aligned} \|x_0 - x'\| &\leq \|x_0 - x\| + \|x - x'\| \\ &\leq \frac{\|Tx_0\| - \|Tx\|}{\alpha(1-q)} + \varepsilon(x)\|h(x)\| \\ &\leq \frac{\|Tx_0\| - \|Tx\|}{\alpha(1-q)} + \frac{\|Tx\|}{\alpha(1-q)} \\ &= \frac{\|Tx_0\|}{\alpha(1-q)} \end{aligned}$$

that is, $x' \in D_0$. Moreover,

$$\lambda \varepsilon(x) \|h(x)\|^2 \leq \lambda \frac{q\alpha^2}{\lambda \|Tx\|} \frac{\|Tx\|^2}{\alpha^2} = q \|Tx\|;$$

so, taking into account the Lipschitz property

$$\|T'(x + \varepsilon h(x))h(x) - T'(x)h(x)\| \leq q \|Tx\|, \quad 0 \leq \varepsilon \leq \varepsilon(x)$$

in which case, by the well-known Dieudonné mean value theorem [12, Ch. 8, Section 5]

$$\|T(x + \varepsilon h(x)) - (1 - \varepsilon)Tx\| \leq \varepsilon q \|Tx\|, \quad 0 \leq \varepsilon \leq \varepsilon(x).$$

In particular, for $\varepsilon = \varepsilon(x)$ this relation becomes

$$\|Tx' - (1 - \varepsilon(x))Tx\| \leq \varepsilon(x)q \|Tx\|$$

(hence $\|Tx'\| - \|Tx\| \leq \varepsilon(x)q \|Tx\|$) and combining this with

$$\|x - x'\| \leq \frac{\varepsilon(x)\|Tx\|}{\alpha} \leq \frac{\|Tx\| - \|Tx'\|}{\alpha(q-1)}$$

it finally follows $x' \in D$, proving at this moment conditions of Theorem 2 will be fulfilled with

$$a(t) = qt, \quad b(t) = \frac{t}{\alpha}, \quad t > 0$$

and $D(x_0, g, f)$ taken as $D \wedge D_0 = D$ (with no information about condition $\|Tx_0 - Tx\| \leq 2f(\|Tx_0\|)$). so, by the conclusion of that result, combined with a previous remark, the operator equation $Tx = 0$ has a solution $z \in D$ which, in fact, is its unique solution in X . Let (x_n) be the (general) Newton-Kantorovich process described in (5), where $(\varepsilon_n = \varepsilon(x_n) \mid n \in \mathbb{N})$ is introduced by the above convention (if we eliminate at each step of this construction the alternative $x_n = z$). As $(\|Tx_n\| \mid n \in \mathbb{N})$ decreases, (ε_n) increases and therefore (vi) (the second alternative) holds, proving conditions of Theorem 4 are fulfilled too. By the conclusion of that statement, $x_n \rightarrow z$; moreover, as $Tx_n \rightarrow 0$, it is clear that $\varepsilon_n = 1$ for $n \geq n(x_0)$ = the smallest value of n with the property $\|Tx_n\| \leq q\alpha^2/\lambda$. The proof is complete. $\square$

In particular, assume X is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then, obviously, (7) has the form

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \quad u, v \in X \quad (7')$$

and the above result reduces to Theorem 6 in Amann [5] proved by a different method (see also [4]). As already indicated in the above quoted papers, the statements of this type found an interesting application to the iterative solvability of the linear problems

$$x_i + \sum_{j=1}^m k(s_i, s_j) f(s_j, x_j) \omega_j = 0, \quad i = 1, \dots, m, \quad (8)$$

arising from the discretization of the Hammerstein equations

$$x(t) + \int_a^b k(t, s) f(s, x(s)) ds = 0, \quad t \in [a, b] = I \quad (8')$$

where $k: I^2 \rightarrow \mathbf{R}$ and $f: I \times \mathbf{R} \rightarrow \mathbf{R}$ are continuous functions determined in such a way that (8') has at least a solution in $C(I)$ (e.g., that hypothesis of the main result be fulfilled with $X = Y = C(I)$ and $T: X \rightarrow Y$ defined by the left member of (8')). Concerning this last aspect, it is worth nothing that, by a technique involving *monotone operators* theory, Altman [2, Theorem 5.1] was able to derive, among others, an existence uniqueness and continuous dependence result for the nonlinear Hammerstein equations in $L^2(I)$ (comprising in general (8'))

$$x - A^*FA(x) = b \quad (= a \text{ fixed element in } L^2(I))$$

where $A: L^2(I) \rightarrow L^p(I)$ is a bounded linear operator, $A^*: L^q(I) \rightarrow L^2(I)$ (with $p^{-1} + q^{-1} = 1$) its adjoint, and $F: L^p(I) \rightarrow L^q(I)$ a nonlinear operator; see also Vainberg [26, Theorem 17.5]. A more detailed analysis of these problems in connection with a general class of Newton-Kantorovich processes in the sense of Kantorovich and Akilov [14, Ch. 18] will be discussed in a forthcoming paper.

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