

## FIRST AND SECOND ORDER OPTIMALITY CONDITIONS IN VECTOR OPTIMIZATION

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**RÉSUMÉ.** Cet article présente des conditions nécessaires d'optimalité de type Kuhn et Tucker pour un problème d'optimisation vectorielle avec contraintes d'égalité et d'inégalité. Les hypothèses de différentiabilité utilisées sont suffisamment larges pour être vérifiées en particulier par les fonctions Fréchet différentiables ainsi que les fonctions convexes continues. Le principal outil est une généralisation du classique lemme de Farkas Minkowski faisant intervenir des applications et des multiapplications convexes. Comme conséquence, moyennant un renforcement des hypothèses, on montre qu'un optimum local faible est aussi un minimum local pour un Lagrangien augmenté. Enfin en ajoutant aux conditions nécessaires des hypothèses de différentiabilité du second ordre, on obtient des conditions suffisantes d'optimalité.

**ABSTRACT.** A multiplier rule is presented for a vector optimization problem with equality and inequality constraints. The results are established for a class of mappings which contains Fréchet differentiable mappings and continuous convex mappings, using a version for convex mappings of the Farkas/Minkowski lemma. In the case of a reflexive space, adding to the necessary conditions a second order differentiability assumption, we get sufficient optimality conditions.

**1. Introduction.** In this paper we are interested in constrained vector (multiobjective, Pareto) optimization problems for which we want to obtain necessary and sufficient optimality conditions. Several results of this type can be found in the literature, see for instance [Oettli], [Craven], [Minami], [Jahn], [Jahn/Sachs] among others. Recently the results of Jahn for Fréchet differentiable functions were extended to the class of regularly locally convex functions by Khanh/Nuong. When an equality constraint is present, necessary conditions are generally carried out by using the Ljusternik's Theorem. In particular the equality constraint must be of class  $C^1$  with a surjective derivative. Here we consider an objective mapping belonging to a class which contains in particular the regularly locally convex mappings. We show that under a regularity assumption, it is sufficient, in order to get a multiplier rule, to assume that the equality constraint admits a Fréchet derivative, the range of which is a direct factor. It is well known that necessary optimality conditions are not sufficient unless convexity assumptions are added. In this paper we give an other type of sufficiency dealing with second order conditions.

In the first part we introduce extended notions of differentiability and in the second section we discuss the regularity assumption. Under the regularity assumption, the solution set of our minimization problem is included into an other subset which can be characterized in terms of directional derivatives. In the third section we present a result of Farkas/Minkowski type for convex mappings, from which we derive the Fritz-John optimality conditions for our problem and then we get a multiplier rule. Furthermore strengthening the conditions we show that each solution appears to be a local minimum

of a perturbed Lagrangian function. In the last part we show that adding a second order differentiability assumption to the necessary conditions previously obtained, we get sufficient optimality conditions, without convexity nor regularity.

Unless otherwise specified,  $X, W$  are Banach spaces and  $Y, Z$  are normed vector spaces ordered by closed convex cones  $C$  and  $D$  respectively. Consider  $f: X \rightarrow Y, g: X \rightarrow Z, p: X \rightarrow W$  and a convex subset  $\Omega$  of  $X$ . The problem we are interested in, is stated under the following general form:

$$\min\{f(x) \mid x \in S\} \quad (\mathcal{P})$$

where  $S = \{x \in \Omega \mid g(x) \leq 0, p(x) = 0\}$  is the feasible set. An element  $x_0 \in S$  is called a *local minimum* for  $(\mathcal{P})$  if there exists a neighborhood  $N \in \mathcal{N}(x_0)$  of  $x_0$  such that for every  $x \in N \cap S$ ,  $f(x_0) - f(x) \notin C \setminus (-C)$ . If we have for every  $x \in N \cap S$ ,  $f(x_0) - f(x) \notin \text{ri } C$ , where  $\text{ri } C$  denotes the relative interior of  $C$ , we say that  $x_0$  is a *local weak minimum*.

As in [Iofe/Tihomirov],  $f$  is said to be *locally convex* at  $x \in X$ , if its directional derivative  $f'(x) \cdot h$  exists for all  $h \in X$  and is convex in the sense that

$$tf'(x) \cdot h + (1-t)f'(x) \cdot k - f'(x) \cdot [th + (1-t)k] \in C,$$

for all  $h, k \in X$  and  $t \in [0, 1]$ .

Note that  $f'(x)$  is a positively homogeneous mapping as soon as it is defined. The mapping  $f$  is said *uniformly differentiable* in the direction  $h \in X$  at  $x \in X$  if  $f'(x) \cdot h$  exists and for every  $U \in \mathcal{N}(0)$  in  $Y$  there exists  $N \in \mathcal{N}(h)$  in  $X$  and  $t_0 > 0$  such that

$$t^{-1}[f(x + tk) - f(x)] - f'(x) \cdot h \in U,$$

whenever  $k \in N$  and  $t \in (0, t_0)$ .

If  $f$  is locally convex at  $x$  and uniformly differentiable in all directions at  $x$ , then we say that  $f$  is *regularly locally convex* at  $x$ . In this case  $f'(x)$  is a continuous function. If the closed convex cone  $C$  of  $Y$  is Daniell (i.e. each decreasing net which has a lower bound converges to its infimum), then every convex mapping from  $X$  to  $Y$  is locally convex. Moreover every Fréchet differentiable mapping is regularly locally convex. We have also:

**1.1 PROPOSITION.** *Let  $C$  be a closed convex cone of  $Y$  which is Daniell and normal (i.e. there is a base of neighborhoods  $U$  of 0 with  $U = (U - C) \cap (C - U)$ ) and let  $f: X \rightarrow (Y, C)$  be a convex map. Then  $f$  is regularly locally convex at  $x$  if and only if it is continuous at  $x$ .*

**PROOF:** The necessity is obvious from the definition of the regular local convexity. Moreover,  $f$  being convex and  $C$  being Daniell,  $f$  is automatically locally convex. Thus, we have only to check uniform differentiability at  $x$  in all directions. Since  $f$  is continuous at  $x$  it is also continuous on a neighborhood  $V$  of this point [Borwein, Cor. 2.4]. Let  $h \in X$ ,  $U = (U - C) \cap (C - U)$  a neighborhood of 0 in  $Y$  and  $U_0 \subset U$  such that  $U_0 + U_0 \subset U$ . We choose  $t_0 > 0$  such that  $x + t_0 h \in U_0$  and

$$t_0^{-1}[f(x + t_0 h) - f(x)] - f'(x) \cdot h \in U_0. \quad (1.1)$$

By the continuity of  $f$  at  $x + t_0 h$ , there exists  $N \in \mathcal{N}_Y(h)$ ,  $N$  symmetric with respect to  $h$  such that for all  $k \in N$

$$t_0^{-1}[f(x + t_0 k) - f(x + t_0 h)] \in U_0. \quad (1.2)$$

On the other hand, from the convexity of  $f$  we have for all  $k \in N$

$$\forall t \in (0, t_0) \quad t_0^{-1}[(t_0 - t)f(x) + tf(x + t_0k)] - f(x + tk) \in C$$

which gives

$$\forall t \in (0, t_0) \quad t_0^{-1}[f(x + t_0k) - f(x)] - t^{-1}[f(x + tk) - f(x)] \in C. \quad (1.3)$$

From (1.1), (1.2) and (1.3) we get for all  $k \in N$  and  $t \in (0, t_0)$

$$t^{-1}[f(x + tk) - f(x)] - f'(x) \cdot h \in U - C. \quad (1.4)$$

Setting  $k' = 2h - k$  we have  $k' \in N$  since  $N$  is symmetric with respect to  $h$  and by the convexity of  $f$

$$\forall t \in (0, t_0) \quad f(x + tk) - 2(f(x + th) + f(x + tk')) \in C \quad (1.5)$$

and

$$\forall t \in (0, t_0) \quad t^{-1}[f(x + th) - f(x)] - f'(x) \cdot h \in C. \quad (1.6)$$

It follows from (1.5) and (1.6) that for all  $k \in N$  and  $t \in (0, t_0)$

$$t^{-1}[f(x + tk) - f(x)] - f'(x) \cdot h \in t^{-1}[f(x + th) - f(x) + f(x) - f(x + tk')] + C$$

and from (1.6)

$$t^{-1}[f(x + tk) - f(x)] - f'(x) \cdot h \in f'(x) \cdot h - t^{-1}[f(x + tk') - f(x)] + C.$$

As  $k' \in N$  we get from (1.4)

$$t^{-1}[f(x + tk) - f(x)] - f'(x) \cdot h \in -U + C \quad (1.7)$$

and the normality of  $C$ , with (1.4) and (1.7), entails that

$$\forall k \in N \quad \forall t \in (0, t_0) \quad t^{-1}[f(x + tk) - f(x)] - f'(x) \cdot h \in U. \quad \square$$

Let us introduce now an extended notion of differentiability. For each  $x \in X$  and  $h \in X$  we define  $f'(x, h)$  as the subset of all  $y \in Y$  such that for every  $U \in \mathcal{N}(0)$  in  $Y$ , there exist  $N \in \mathcal{N}(h)$  in  $X$  and  $t_0 > 0$  satisfying

$$t^{-1}[f(x + tk) - f(x)] - y \in U - C,$$

whenever  $k \in N$  and  $t \in (0, t_0)$ . If  $f'(x, h) \neq \emptyset$  we say that  $f$  is *lower uniformly differentiable* at  $x$  in the direction  $h$ . Furthermore  $f$  is said to be *regularly lower uniformly differentiable* at  $x$  if the multiapplication  $f'(x, \cdot): X \rightarrow 2^Y$  has non empty values and is convex in the following sense:

$$tf'(x, h) + (1 - t)f'(x, k) \subset f'(x, th + (1 - t)k) + C$$

for all  $h, k \in X$  and  $t \in [0, 1]$ . Obviously every uniformly differentiable mapping is lower uniformly differentiable,  $f'(x, h)$  being the singleton  $\{f'(x) \cdot h\}$ . The following example shows that this inclusion may be strict.

1.2 EXAMPLE. Consider  $f: \mathbf{R} \rightarrow \mathbf{R}$  defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0: \\ x \sin(1/x) & \text{if } x > 0. \end{cases}$$

The quantity  $t^{-1}f(t)$  oscillates between 1 and  $-1$  when  $t$  tends to  $0^+$  so that  $f'(0) \cdot 1$  does not exist. However if  $h \geq 0$ ,  $f'(0, h) = [1, \infty[$  and if  $h < 0$ ,  $f'(0, h) = [0, \infty[$ .

1.3 LEMMA. If  $f: X \rightarrow (Y, C)$  is lower uniformly differentiable at  $x \in X$  in the direction  $h \in X$  then for every  $\lambda > 0$ ,  $f'(x, \lambda h) = \lambda f'(x, h)$

PROOF: Let be  $y \in f'(x, h)$ ,  $\lambda > 0$  and  $U \in \mathcal{N}(0)$ . Since  $\lambda^{-1}U \in \mathcal{N}(0)$ , there exist  $N \in \mathcal{N}(h)$  and  $t_0 > 0$  such that for every  $t \in (0, t_0)$  and every  $k \in N$

$$t^{-1}[f(x + tk) - f(x)] - y \in \lambda^{-1}U - C.$$

Setting  $r = \lambda^{-1}t$  we have

$$r^{-1}[f(x + r\lambda k) - f(x)] - \lambda y \in U - C$$

which means  $\lambda y \in f'(x, \lambda h)$ . Thus we have proved  $\lambda f'(x, h) \subset f'(x, \lambda h)$ . In the same way we have  $\lambda^{-1}f'(x, \lambda h) \subset f'(x, \lambda^{-1}\lambda h)$  implying the equality.  $\square$

1.4 LEMMA. If  $f: x \rightarrow (Y, C)$  is lower uniformly differentiable at  $x \in X$  in all directions and  $\text{int } C \neq \emptyset$  then the multimapping  $f'(x, \cdot): X \rightarrow 2^Y$  is lower semi continuous.

PROOF: Let be  $y \in f'(x, h)$  and  $U \in \mathcal{N}(0)$  in  $Y$ . Let be  $c \in \text{int } C \cap U$  and consider the neighborhood of 0,  $Q = U \cap (c - C)$ . Since  $f$  is lower uniformly differentiable at  $x$ , there exists  $N \in \mathcal{N}(0)$  in  $X$  such that  $N/2 + N/2 \subset N$  and  $t_0 > 0$  such that for every  $t \in (0, t_0)$  and every  $k \in h + N$

$$t^{-1}[f(x + tk) - f(x)] - y \in Q - C.$$

Then, for every  $k \in h + N/2$ , every  $b \in k + N/2$  and every  $t \in (0, t_0)$

$$t^{-1}[f(x + tb) - f(x)] - y \in Q - C. \quad (1.8)$$

Consider  $y_1 = y + c$ . Since  $Q \subset c - C$  we have for  $V \in \mathcal{N}(0)$  in  $Y$ ,  $y + Q - C \subset y_1 + V - C$ . Thus for every  $k \in h + N/2$ , it follows from (1.8) that  $y_1 \in f'(x, k)$  which entails  $f'(x, k) \cap (y + Q) \neq \emptyset$ .  $\square$

If  $S$  is a non empty subset of a real separated topological vector space  $X$  and  $x \in S$ , we say that  $h$  belongs to the *contigent cone*  $T_x S$  of  $S$  at  $x$ , whenever there are nets  $(x_i)_{i \in I}$ ,  $x_i \in S$  and  $(\lambda_i)_{i \in I}$ ,  $\lambda_i > 0$  such that:

$$h = \lim_i \lambda_i^{-1}(x_i - x), \quad \lim_i \lambda_i = 0.$$

We denote by  $Y^*$  (resp.  $Z^*$ ,  $W^*$ ) the spaces of linear continuous functionals on  $Y$ , (resp.  $Z$ ,  $W$ ) and we define the polar cones of  $C \subset Y$  and  $D \subset Z$  by:

$$C^* = \{\lambda^* \in Y^* \mid \lambda^* y \geq 0 \text{ for all } y \in C\}$$

and

$$D^* = \{\mu^* \in Z^* \mid \mu^* z \geq 0 \text{ for all } z \in D\}.$$

For  $x \in X$  we set

$$D_x^* = \{\mu^* \in D^* \mid \mu^* g(x) = 0\}$$

and

$$(D_x^*)^* = \{z \in Z \mid \mu^* z \geq 0 \text{ for all } \mu^* \in D_x^*\}.$$

Obviously we have  $D \subset (D_x^*)^*$ . Recall (see *e.g.* [Jahn, Lemma 3.21 a]) that if  $Z$  is locally convex and  $D$  a closed convex cone of  $Z$  then  $D = \{z \in Z \mid \mu^* z \geq 0 \text{ for all } \mu^* \in D^*\}$ . When the directional derivatives  $g'(x) \cdot h$  and  $p'(x) \cdot h$  exist, we introduce the subset of  $X$ :

$$H_x = \{h \in (\text{int } \Omega) - x \mid g'(x) \cdot h \in -\text{int}((D_x^*)^*), p'(x) \cdot h = 0\}.$$

If  $g$  is locally convex and  $p$  Fréchet differentiable then  $H_x$  is a (possibly empty) convex set.

**2. Regularity condition.** We begin with a result showing that  $T_x S$  is included into the convex cone defined with the aid of the directional derivatives of effective constraints at  $x \in S$ .

**2.1 PROPOSITION.** *If  $g$  is uniformly differentiable at  $x \in S$  and  $p$  is Fréchet differentiable at  $x$  then*

$$T_x S \subset \{h \in T_x \Omega \mid g'(x) \cdot h \in -(D_x^*)^*, p'(x) \cdot h = 0\}.$$

**PROOF:** Let be  $h = \lim_{n \rightarrow \infty} t_n^{-1}(x_n - x) \in T_x S$  with  $x_n \in S$ ,  $t_n > 0$  and  $\lim_{n \rightarrow \infty} t_n = 0$ . The mapping  $g$  being regularly locally convex at  $x$ , for every  $U \in \mathcal{N}_Y(0)$ , there exist  $t_0 > 0$  and  $V \in \mathcal{N}_X(h)$  such that

$$\forall t \in (0, t_0) \quad \forall k \in V \quad t^{-1}[g(x + tk) - g(x)] - g'(x) \cdot h \in U$$

taking  $t = t_n$  and  $k = t_n^{-1}(x_n - x)$  there exists  $n_0$  such that

$$\forall n > n_0 \quad t_n^{-1}[g(x_n) - g(x)] - g'(x) \cdot h \in U$$

and for each  $\mu \in D_x^*$ ,  $\|\mu\| = 1$  we get

$$\forall n > n_0 \quad t_n^{-1} \mu^* g(x_n) - \mu^* g'(x) \cdot h \in \mu^*(U).$$

Since  $g(x_n) \leq 0$  and  $\mu^* \in D_x^* \subset D^*$  we deduce that for every  $U \in \mathcal{N}_Y(0)$ ,  $\mu^* g'(x) \cdot h \in -\mu^*(U) - \mathbf{R}^+$  which implies,  $U$  being arbitrarily small,  $\mu^* g'(x) \cdot h \leq 0$ . Thus  $g'(x) \cdot h \in -(D_x^*)^*$ . Now  $p$  being Fréchet differentiable at  $x$ ,

$$\forall \varepsilon > 0 \exists n_0 > 0 \forall n > n_0 \quad \|t_n^{-1}[p(x_n) - p(x) - p'(x)(x_n - x)]\| + \|p'(x)(t_n^{-1}(x_n - x) - h)\| < \varepsilon.$$

Since  $p(x_n) = p(x) = 0$  and  $\|p'(x) \cdot h\| \leq \|p'(x)(t_n^{-1}(x_n - x))\| + \|p'(x)(t_n^{-1}(x_n - x) - h)\|$  it yields  $p'(x) \cdot h = 0$ .  $\square$

**2.2 REMARK.** The inclusion in Proposition 2.1 can be strict. Consider for example:

$$S = \{(x_1, x_2) \in \mathbf{R}^2 \mid x_1^2 + x_2^2 - 1 \leq 0, x_1 + 1 = 0\}$$

and the point  $(-1, 0) \in S$ . In fact  $S = \{(-1, 0)\}$  and  $T_{(-1, 0)} S = \{(0, 0)\}$ . However  $g'(-1, 0) \cdot (h_1, h_2) = -2h_1$ ,  $p'(-1, 0) \cdot (h_1, h_2) = h_1$  and

$$\{h = (h_1, h_2) \in \mathbf{R}^2 \mid g'(-1, 0) \cdot h \leq 0, p'(-1, 0) \cdot h = 0\} = \{h \in \mathbf{R}^2 \mid h_1 = 0\}.$$

Let be  $x \in S$ . We shall use the following regularity condition:

$$H_x \subset T_x S. \quad (\text{R1})$$

**2.3 LEMMA.** *If  $g$  is uniformly differentiable at  $x \in S$  and  $g'(x) \cdot h \in -\text{int}((D_x^*)^*)$  with  $h = \lim_{n \rightarrow \infty} t_n^{-1}(x_n - x)$ ,  $x_n \in X$ ,  $t_n > 0$  and  $\lim_{n \rightarrow \infty} t_n = 0$ , then there exists  $n_0 > 0$  such that, for each  $n > n_0$ ,  $g(x_n) \leq 0$ .*

**PROOF:** Since  $D$  is a closed convex cone,  $g(z) \leq 0$  is equivalent to  $\mu^* g(z) \leq 0$  for every  $\mu^* \in D^*$ . Suppose that there exist sequences  $n_k \in \mathbb{N}$ ,  $\lim_{k \rightarrow \infty} n_k = \infty$  and  $\mu_k^* \in D^*$ ,  $\|\mu_k^*\| = 1$  such that for each  $k \in \mathbb{N}$

$$\mu_k^* g(x_{n_k}) > 0. \quad (2.1)$$

We may suppose that  $\mu_k^*$  weakly converges to  $\mu^* \in D^*$  with  $\|\mu^*\| = 1$ . The mapping  $g$  being uniformly differentiable at  $x$  we set for each  $n_k$ :

$$\alpha_k = t_{n_k}^{-1}(g(x_{n_k}) - g(x)) - g'(x) \cdot h$$

and we have  $\lim_{k \rightarrow \infty} \alpha_k = 0$ . By linearity of  $\mu_k^*$

$$\mu_k^* g(x_{n_k}) = \mu_k^* g(x) + t_{n_k} \mu_k^* (g'(x) \cdot h + \alpha_k).$$

If we suppose  $\mu^* g(x) \leq -\eta < 0$  then there exists  $k_0 \in \mathbb{N}$  such that  $k > k_0$  implies  $\mu_k^* g(x) < -\eta/2$  and  $t_{n_k} \|g'(x) \cdot h + \alpha_k\| < \eta/2$ . Therefore,  $k > k_0$  implies  $\mu_k^* g(x_{n_k}) < -\eta/2 + \eta/2 = 0$ , a contradiction with (2.1). Thus,  $\mu^* g(x) = 0$  and  $\mu^* \in D_x^*$ . As  $g'(x) \cdot h \in -\text{int}((D_x^*)^*)$  we have  $\mu^* g'(x) \cdot h < -\varepsilon < 0$  and there exists  $k_0$  such that  $k > k_0$  entails  $\mu^* (g'(x) \cdot h + \alpha_k) < -\varepsilon/2$ . Then for  $k > k_0$

$$\mu_k^* (g'(x) \cdot h + \alpha_k) < (\mu_k^* - \mu^*) g'(x) \cdot h + \|\mu_k^* - \mu^*\| \|\alpha_k\| - \frac{\varepsilon}{2}.$$

Since  $\lim_{k \rightarrow \infty} \alpha_k = 0$ ,  $\|\mu_k^*\| = \|\mu^*\| = 1$  and  $\mu_k^*$  being weakly convergent to  $\mu^*$  there exists  $k_1 > 0$  such that for all  $k > k_1$

$$t_{n_k} < 1 \text{ and } \mu_k^* (g'(x) \cdot h + \alpha_k) < -\frac{\varepsilon}{4}$$

which implies  $\mu_k^* g(x_{n_k}) < -\varepsilon/4$  and a contradiction with (2.1).  $\square$

**2.4 LEMMA.** *Suppose that  $\lim_{n \rightarrow \infty} t_n^{-1}(x_n - x) \in (\text{int } \Omega) - x$ , with  $x_n \in X$ ,  $t_n > 0$ ,  $\lim_{n \rightarrow \infty} t_n = 0$ . Then there exists  $n_0 > 0$  such that for all  $n > n_0$ ,  $x_n \in \Omega$ .*

**PROOF:** We set  $h = \lim_{n \rightarrow \infty} t_n^{-1}(x_n - x)$  and for each  $n \in \mathbb{N}$ ,  $\varepsilon_n = h - t_n^{-1}(x_n - x)$  with  $h = \omega - x$ ,  $\omega \in \text{int } \Omega$  and  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . Therefore  $x_n = t_n(\omega - \varepsilon_n) + (1 - t_n)x$ , the convexity of  $\Omega$  implying  $x_n \in \Omega$  as soon as  $\omega - \varepsilon_n \in \Omega$  (i.e. for  $n$  large enough).  $\square$

**2.5 REMARK.** From these two lemmas, if we consider a problem  $(\mathcal{P})$  without equality constraints and  $g$  a uniformly differentiable mapping at  $x \in S$  then  $x$  satisfies the regularity condition (R1). Indeed, for  $h \in H_x$  we have  $h = \lim_{n \rightarrow \infty} n(x + (1/n)h - x)$ ,  $h \in (\text{int } \Omega) - x$  and  $g'(x) \cdot h \in -\text{int}((D_x^*)^*)$  which gives  $(x + (1/n)h) \in S$  for  $n$  large enough. Actually, the assumption “ $g$  admits a directional derivative in all directions at  $x$ ” would be sufficient in this particular case, because we apply Lemma 2.3 with the added information that  $x_n$  is of the form  $x + (1/n)h$ .

**2.6 PROPOSITION.** *If  $g$  is uniformly differentiable at  $x \in S$  and  $p$  continuously Fréchet differentiable at  $x$  with  $p'(x)$  surjective, then  $x$  satisfies the regularity condition (R1).*

**PROOF:** Let be  $h \in H_x$ . From the Ljusternik theorem [Jahn, Th. 3.49] or [Ioffe/Tihomirov, § 0.2], we have  $h = \lim_{n \rightarrow \infty} t_n^{-1}(x_n - x)$  with  $p(x_n) = 0$ ,  $t_n > 0$  and  $\lim_{n \rightarrow \infty} t_n = 0$ . By Lemmas 2.3 and 2.4 it exists  $n_0 > 0$  such that  $n > n_0$  implies  $x_n \in \Omega$  and  $g(x_n) \leq 0$ . Therefore  $x_n \in S$  for  $n$  large enough and then  $h \in T_x S$ .  $\square$

**3. Necessary optimality conditions.** We suppose from now that  $C$  and  $D$  are proper closed convex cones with non-empty interiors and that the convex set  $\Omega \subset X$  has also a non-empty interior.

**3.1 PROPOSITION.** *If  $f$  is a lower uniformly differentiable mapping then every local weak solution  $x$  of  $(\mathcal{P})$  satisfies*

$$\forall h \in T_x S \quad f'(x, h) \cap -\text{int } C = \emptyset.$$

PROOF: Suppose that  $y \in f'(x, h) \cap -\text{int } C$  with  $h = \lim_{n \rightarrow \infty} t_n^{-1}(x_n - x)$ ,  $x_n \in S$ ,  $t_n > 0$ ,  $\lim_{n \rightarrow \infty} t_n = 0$ . Let be  $U \in \mathcal{N}_Y(0)$  a neighborhood of 0 such that  $y + U \subset -\text{int } C$ . As  $f$  is lower uniformly differentiable at  $x$ , there is  $N \in \mathcal{N}(h)$  and  $t_0 > 0$  such that for each  $k \in N$  and  $t \in (0, t_0)$

$$t^{-1} [f(x + tk) - f(x)] - y \in U - C$$

from which

$$f(x + tk) - f(x) \in t[y + U] - C \subset -\text{int } C.$$

Choosing  $t = t_n$  and  $k = t_n^{-1}(x_n - x)$  we get for  $n$  large enough

$$f(x_n) - f(x) \in -\text{int } C,$$

a contradiction with  $x$  local weak solution of  $(\mathcal{P})$ .  $\square$

Proposition 3.1 implies that for a local weak solution  $x$ , satisfying the regularity condition (R1) we have

$$\forall h \in H_x \quad f'(x, h) \cap -\text{int } C = \emptyset. \quad (3.1)$$

We denotes by  $\Theta$  the subset of  $X$  consisting of all points of  $S$  satisfying (3.1).

$$\Theta = \{x \in S \mid f'(x, h) \cap -\text{int } C = \emptyset \text{ for all } h \in H_x\}.$$

**3.2 PROPOSITION.** *We suppose that  $f$  is regularly lower uniformly differentiable at  $x \in S$ ,  $g$  is locally convex at  $x$  and  $p$  is Fréchet differentiable at  $x$ . Then  $x \in \Theta$  if and only if there exists  $\lambda^* \in C^* \setminus \{0\}$  such that  $h \in H_x$  implies  $\lambda^* f'(x, h) \subset \mathbf{R}^+$ .*

PROOF: Since  $y \in f'(x, h) \cap -\text{int } C$  would imply  $\lambda^* y < 0$ , the condition is obviously sufficient. Now, let be  $x \in \Theta$ . If  $H_x = \emptyset$  the result is obvious. In the other case the convex subset  $\text{conv}(f'(x, H_x) + \text{int } C)$  has a non-empty interior and does not contain 0. Indeed, suppose that  $\sum_{i=1}^n \lambda_i(y_i + c_i) = 0$  with  $\sum_{i=1}^n \lambda_i = 1$ ,  $\lambda_i > 0$ ,  $y_i \in f'(x, h_i)$ ,  $h_i \in H_x$  and  $c_i \in -\text{int } C$ . The multimapping  $f'(x)$  being convex, there exists  $y \in f'(x, \sum_{i=1}^n \lambda_i h_i)$  such that  $\sum_{i=1}^n \lambda_i y_i \in y + C$  which implies  $y \in -\text{int } C$ , contradicting  $x \in \Theta$  since  $H_x$  is convex. By the topological version of Eidelheit's separation theorem (see e.g. [Jahn, Th. 3.16]) there exists a continuous linear functional  $\lambda^* \in Y^* \setminus \{0\}$  such that for all  $y \in f'(x, H_x)$  and  $c \in \text{int } C$

$$\lambda^*(y + c) > 0.$$

Therefore  $\lambda^* \in C^* \setminus \{0\}$  and  $\lambda^* y \geq 0$  for each  $y \in f'(x, H_x)$ .  $\square$

In order to derive a generalized multiplier rule from the Proposition 3.2 we need a version of the Farkas/Minkowski's lemma involving convex mapping instead of linear mappings.

**3.3 THEOREM.** Consider a convex non empty valued lower semi continuous multimapping  $F: X \rightarrow 2^Y$ , a convex continuous mapping  $G: x \rightarrow Z$  and  $P: X \rightarrow W$  linear continuous with  $P(X)$  closed. If  $\Delta \subset X$  is a convex subset such that  $\text{int } \Delta \neq \emptyset$  and  $N^*$  a closed convex cone in  $Z^*$  such that  $N^* \subset D^*$ , then the implication

$$h \in \text{int } \Delta \quad G(h) \in -\text{int}((N^*)^*) \quad P(h) = 0 \implies F(h) \cap -\text{int } C = \emptyset \quad (3.2)$$

entails

$$\exists(\lambda^*, \mu^*, \nu^*) \in C^* \times N^* \times W^* \setminus (0, 0, 0) \quad \forall h \in \Delta \quad (\lambda^* F + \mu^* G + \nu^* P)h \subset \mathbf{R}^+. \quad (3.3)$$

Moreover if  $P(X)$  is a direct factor in  $W$  and  $\{h \in \text{int } \Delta \mid G(h) \in -\text{int}((N^*)^*), P(h) = 0\} \neq \emptyset$  then we can choose  $\lambda^* \neq 0$ .

PROOF: If  $P(X)$  is a proper subspace of  $W$ , there is  $\nu^* \in W^* \setminus \{0\}$  such that  $\nu^* P = 0$ . Then we get (3.3) with  $\lambda^* = \mu^* = 0$ . In the other case  $P(X) = W$ . Consider the subset  $Q$  of  $Y \times Z \times W$  defined by

$$Q = \{(F, G, P) \text{int } \Delta\} + \{\text{int } C \times \text{int}((N^*)^*) \times \{0\}\}.$$

Note that  $\text{int}((N^*)^*) \supset \text{int}((D^*)^*) \supset \text{int } D \neq \emptyset$ . We want to show that  $Q$  is open in  $Y \times Z \times W$ . Let  $(y + c, G(h) + d, P(h)) \in Q$  with  $h \in \text{int } \Delta$ ,  $y \in F(h)$ ,  $c \in \text{int } C$ ,  $d \in \text{int}((D^*)^*)$ . By the lower semi continuity of  $F$  and the continuity of  $G$  there exists  $U \in \mathcal{N}(h)$ ,  $U \subset \text{int } \Delta$  such that for each  $k \in U$

$$F(k) \cap \left(y + \frac{c}{2} - \text{int } C\right) \neq \emptyset, \quad G(k) \in G(h) + \frac{d}{2} - \text{int}((N^*)^*).$$

Then, for each  $k \in U$  there exists  $y_1 \in F(k)$  such that  $y_1 \in y + c/2 - \text{int } C$  and

$$\left[y + \frac{c}{2} + \text{int } C, G(h) + \frac{d}{2} + \text{int}((N^*)^*), P(k)\right] \subset [y_1 + \text{int } C, G(k) + \text{int}((N^*)^*), P(k)] \subset Q.$$

As  $P$  is an open linear map, the subset  $[y + c/2 + \text{int } C, G(h) + d/2 + \text{int}((N^*)^*), P(U)]$  is an open neighborhood of  $[y + c, G(h) + d, P(h)]$  in  $Y \times Z \times W$ . Thus we have proved that  $Q$  is open and that  $\text{conv } Q$  is a convex set with a non empty interior. In the case where  $0 \notin \text{conv } Q$ , then by the topological version of the Eidelheit's separation theorem there exists  $(\lambda^*, \mu^*, \nu^*) \in Y^* \times Z^* \times W^* \setminus (0, 0, 0)$  such that

$$\begin{aligned} \forall(h, c, d) \in \text{int } \Delta \times \text{int } C \times \text{int}((N^*)^*) \quad \forall y \in F(h) \\ \lambda^*(y + c) + \mu^*(G(h) + d) + \nu^* P(h) > 0. \end{aligned} \quad (3.4)$$

This implies that  $\lambda^* \in C^*$ ,  $\mu^* \in N^*$  and

$$\forall h \in \text{int } \Delta \quad \forall y \in F(h) \quad \lambda^* y + \mu^* G(h) + \nu^* P(h) \geq 0.$$

Let be  $\varepsilon > 0$ . If  $k \in \Delta$  and  $z \in F(k)$ , from the lower semi continuity of  $F$  and the continuity of  $G$  and  $P$ , there exists  $h \in \text{int } \Delta$  such that

$$F(h) \cap (z + O) \neq \emptyset, \quad \mu^* G(k) + \nu^* P(k) \geq \mu^* G(h) + \nu^* P(h) - \frac{\varepsilon}{2}$$

where  $O = \{y \in Y \mid |\lambda^* y| < \varepsilon/2\}$ . Therefore we may choose  $y \in F(h)$  such that  $y \in z + O$  and then  $\lambda^* z \geq \lambda^* y - \varepsilon/2$ . Since  $\varepsilon$  is arbitrarily small we have

$$\forall k \in \Delta \quad \forall z \in F(k) \quad \lambda^* z + \mu^* G(k) + \nu^* P(k) \geq 0.$$

Now we suppose that  $0 \in \text{conv } Q$ , then there exist  $h_i \in \text{int } \Delta$ ,  $\gamma_i \geq 0$   $i = 1, \dots, k$  with  $\sum_{i=1}^k \gamma_i = 1$  such that:

$$\sum_{i=1}^k \gamma_i F(h_i) \cap -\text{int } C \neq \emptyset \quad \sum_{i=1}^k \gamma_i G(h_i) \in -\text{int}((N^*)^*) \quad \sum_{i=1}^k \gamma_i P(h_i) = 0.$$

From the convexity of both the multimapping  $F$  and the mapping  $G$ , the linearity of  $P$  and the inclusion  $D \subset (N^*)^*$  we have:

$$F\left(\sum_{i=1}^k \gamma_i h_i\right) \cap -\text{int } C \neq \emptyset \quad G\left(\sum_{i=1}^k \gamma_i h_i\right) \in -\text{int}((N^*)^*) \quad P\left(\sum_{i=1}^k \gamma_i h_i\right) = 0.$$

contradicting (3.2) since  $\sum_{i=1}^k \gamma_i h_i \in \text{int } \Delta$ . In the case where  $P(X)$  is a direct factor in  $W$ , we denote  $P(X)$  by  $W_1$  and in a same way as previously we can prove the existence of  $(\lambda^*, \mu^*, \nu_1^*) \in Y^* \times Z^* \times W_1^* \setminus (0, 0, 0)$  such that

$$\forall h \in \Delta \quad (\lambda^* F + \mu^* G + \nu_1^* P)h \in \mathbf{R}^+.$$

Setting  $\nu^* = \nu_1^* \text{proj}_{W_1}$  we have  $\nu^* \in W^*$ , since  $\text{Proj}_{W_1}$  is continuous, thus we get (3.3). Moreover if it exists  $h_0 \in \text{int } \Delta$  such that  $G(h_0) \in -\text{int}((N^*)^*)$  and  $P(h_0) = 0$  then, from (3.4) (with  $\nu_1^*$  instead of  $\nu^*$ ),

$$\forall c \in \text{int } C \quad \lambda^*(F(h_0) + c) > 0$$

from which  $\lambda^* \neq 0$ .  $\square$

In the proof above the continuity assumptions on  $F$ ,  $G$  and  $\text{int } \Delta \neq \emptyset$  are needed in order to ensure the openness of  $Q$ . In the case of a problem without equality constraint these assumptions can be removed and we get the following result (see [Holmes, § 14 F]).

**3.4 THEOREM.** Consider  $F: X \rightarrow 2^Y$  a convex non empty valued multimapping,  $G: X \rightarrow Z$  a convex mapping, a closed convex cone  $N^*$ , a convex subset  $\Delta$  with  $N^* \subset D^*$  and  $\Delta \subset X$ . Then the implication

$$\{h \in \Delta \text{ and } G(h) \in \text{int}((N^*)^*)\} \implies F(h) \cap -\text{int } C = \emptyset$$

entails

$$\exists (\lambda^*, \mu^*) \in C^* \times N^* \setminus (0, 0) \quad \forall h \in \Delta \quad (\lambda^* F + \mu^* G)h \in \mathbf{R}^+.$$

Moreover if it exists  $h_0 \in \Delta$  such that  $G(h_0) \in -\text{int}((N^*)^*)$  then we can choose  $\lambda^* \neq 0$ .

Now we are ready to present the announced multiplier rule.

**3.5 THEOREM.** Suppose that  $f: X \rightarrow (Y, C)$  is regularly lower uniformly differentiable at  $x$ ,  $g: X \rightarrow (Z, D)$  is regularly locally convex at  $x$  and  $p: X \rightarrow W$  Fréchet differentiable at  $x$  with  $p'(x)X$  closed. Suppose that  $x$  satisfies the regularity condition (R1). Then

- (i) If  $x$  is a weak solution then  $x \in \Theta$ .
- (ii) If  $x \in \Theta$  then there exists  $(\lambda^*, \mu^*, \nu^*) \in (C^* \times D_x^* \times W^*) \setminus (0, 0, 0)$  such that for all  $h \in T_x \Omega$ ,  $\lambda^* f'(x, h) + \mu^* g'(x) \cdot h + \nu^* p'(x) \cdot h \in \mathbb{R}^+$ .
- (iii) Furthermore if  $p'(x)X$  is a direct factor in  $W$  and  $H_x \neq \emptyset$  then we can add  $\lambda^* \neq 0$  in the conclusion of (ii).

PROOF: (i) is a direct consequence of the condition (R1) and the Proposition 3.1. (ii) comes from the Theorem 3.3. Indeed taking  $F = f'(x, \cdot)$ ,  $G = g'(x)$ ,  $P = p'(x)$ ,  $N^* = D_x^*$  and  $\Delta = \Omega - x$ , there exists  $\lambda^* \in C^*$ ,  $\mu^* \in D_x^*$ ,  $\nu^* \in W^*$  not all zer such that

$$\forall h \in \Omega - x \quad \lambda^* f'(x, h) + \mu^* g'(x) \cdot h + \nu^* p'(x) \cdot h \in \mathbb{R}^+ \quad (3.5)$$

moreover under the assumptions in (iii) we can choose  $\lambda^* \neq 0$ . Now we want to show that (3.5) is valid for every  $h \in T_x \Omega$ . In this case, we have  $h = \lim t_n^{-1} h_n$ ,  $h_n \in \Omega - x$ ,  $t_n > 0$ ,  $\lim t_n = 0$ . Then for every  $y \in f'(x, h)$  and  $\varepsilon > 0$  consider  $U = \{y \in Y \mid |\lambda^* y| < \varepsilon/2\}$ . From the lower semi continuity of  $f'(x, \cdot)$  there exists a neighborhood  $N \in \mathcal{N}(h)$  such that for every  $k \in N$

$$f'(x, h) \cap (U + y) \neq \emptyset.$$

Therefore, it exists  $n_0$  such that for each  $n > n_0$  we can find  $y_n \in f'(x, t_n^{-1} h_n) \cap (U + y)$ . Since  $g'(x)$  and  $p'(x)$  are positively homogeneous and  $f'(x, t_n^{-1} h_n) = t_n^{-1} f'(x, h_n)$  (Lemma 1.3), it follows from (3.5) that

$$\lambda^* y_n + \mu^* g'(x) \cdot t_n^{-1} h_n + \nu^* p'(x) \cdot t_n^{-1} h_n \geq 0. \quad (3.6)$$

Since  $y_n \in U + y$  we have  $\lambda^* y_n \leq \lambda^* y + \varepsilon/2$  and from the continuity of  $g'(x)$  and  $p'(x)$  we get for  $n$  large enough

$$[\mu^* g'(x) + \nu^* p'(x)](t_n^{-1} h_n) \leq [\mu^* g'(x) + \nu^* p'(x)] h + \frac{\varepsilon}{2}.$$

Therefore it follws from (3.6) that

$$\lambda^* y + \mu^* g'(x) \cdot h + \nu^* p'(x) \cdot h \geq -\varepsilon$$

for every  $y \in f'(x, h)$  and every  $\varepsilon \geq 0$ . Thus we have proved (3.5) for every  $h \in T_x \Omega$ .  $\square$

If we suppose  $p$  continuously Fréchet differentiable at  $x$  we may remove the regularity assumption (R1) in the Theorem 3.5 to obtain:

**3.6 THEOREM.** Let be  $f: X \rightarrow (Y, C)$  regularly lower uniformly differentiable at  $x$ ,  $g: X \rightarrow (Z, D)$  regularly locally convex at  $x$  and  $p: X \rightarrow W$  continuously Fréchet differentiable at  $x$  with  $p'(x)X$  closed. Then

- (i) If  $x$  is a weak minimum then there exists  $(\lambda^*, \mu^*, \nu^*) \in (C^* \times D_x^* \times W^*) \setminus (0, 0, 0)$  such that for all  $h \in T_x \Omega$ ,  $\lambda^* f'(x, h) + \mu^* g'(x) \cdot h + \nu^* p'(x) \cdot h \in \mathbb{R}^+$ .
- (ii) Futhermore if  $p'(x)X$  is a direct factor in  $W$  and  $H_x \neq \emptyset$  then we can add  $\lambda^* \neq 0$  in (i).

PROOF: If  $p'(x)X$  is a proper subspace of  $Y$ , we take  $\nu^* \neq 0$ , such that  $\nu^*p'(x) = 0$  and  $\mu^* = \lambda^* = 0$ . In the other case  $p'(x)$  is surjective and from the Proposition 2.6, (R1) is satisfied. So we have (i) from the Theorem 3.5. For (ii), we first apply Theorem 3.5 (iii), with  $p'(x)X$  instead of  $W$ , since in this case (R1) is satisfied. We obtain, setting  $W_1 = p'(x)X$ , the existence of  $(\lambda^*, \mu^*, \nu_1^*) \in (C^* \times D_x^* \times W^*) \setminus (0, 0, 0)$  such that for all  $h \in T_x\Omega$ ,

$$\lambda^* f'(x, h) + \mu^* g'(x) \cdot h + \nu^*(p'(x) \cdot h) \in \mathbf{R}^+.$$

The result follows taking  $\nu^* = \nu_1^* \text{proj}_{W_1} \in W^*$ .  $\square$

In the particular case of a problem  $(\mathcal{P})$  without equality constraint we get from Remark 2.5 and Theorem 3.4:

**3.7 THEOREM.** Consider  $f: X \rightarrow (Y, C)$  regularly lower uniformly differentiable at  $x$  and  $g: X \rightarrow (Z, D)$  regularly locally convex at  $x$ .

- (i) If  $x$  is a weak optimal solution then  $x \in \Theta$ .
- (ii) If  $x \in \Theta$  then there exists  $(\lambda^*, \mu^*) \in (C^* \times D_x^*) \setminus (0, 0)$  such that for all  $h \in T_x\Omega$ ,  $\lambda^* f'(x, h) + \mu^* g'(x) \cdot h \in \mathbf{R}^+$ .
- (iii) Furthermore if  $\{h \in (\text{int } \Omega) - x \mid g'(x) \cdot h \in -\text{int}((D_x^*)^*)\} \neq \emptyset$  then we may add  $\lambda^* \neq 0$  in the conclusion of (ii).

We know from Theorem 3.5 that under suitable assumptions a local weak optimum  $x$  is a critical point for the Lagrangian function  $\lambda^* f + \mu^* g + \nu^* p$ , but in the non-convex case  $x$  is not generally a minimum point for that functional. In Theorem 3.9 we shall give assumptions, under which a kind of augmented Lagrangian function achieves its minimum at  $x$ . In order to prove this result we need a generalization of the Ljusternik's Theorem [Dolecki], for which we give a proof for sake of completeness.

**3.8 THEOREM.** Let  $X, Y$  be Banach spaces and  $F: X \rightarrow Y$  a continuously differentiable mapping at  $x \in X$  and  $C$  a closed cone in  $Y$ . If

$$Y = F'(x)X + C \tag{3.7}$$

then there exist  $k > 0$  and  $\delta > 0$  such that for every  $t \in B(x, \delta)$  (i.e. the closed ball with center  $x$  and radius  $\delta$ ) and  $y \in B(F(x), \delta)$

$$\text{dist}(t, F^-(y - C)) \leq k \text{dist}(y, F(t) + C).$$

PROOF: Consider the following multimapping  $\Delta: X \rightarrow 2^X$

$$\Delta z = z + F'(x)^-(y + F(t + z) - C)$$

From the continuity of  $F'(x)$  and (3.7),  $\Delta z$  is a non empty, closed subset of  $X$ . We are going to prove that  $\Delta$  is a contraction in a neighborhood of 0. Consider  $z_1, z_2 \in X$ , let  $H$  be the Hausdorff distance and denote  $F_1 = y - F(t + z_1)$  and  $F_2 = y - F(t + z_2)$ . From the definition of the Hausdorff distance  $H(\Delta z_1, \Delta z_2) = \max(\text{ecc}(\Delta z_1, \Delta z_2), \text{ecc}(\Delta z_2, \Delta z_1))$ , where

$$\text{ecc}(\Delta z_1, \Delta z_2) = \sup_{u \in F'(x)^-[F_1 - C]} \inf_{v \in F'(x)^-[F_2 - C]} \|z_1 + u - z_2 - v\|. \tag{3.8}$$

Note that for each  $m \in F'(x)^-[F_2 - F_1 - C]$  and each  $u \in F'(x)^-[F_1 - C]$  we have  $m + u \in F'(x)^-[F_2 - C]$ .

Therefore using (3.8)

$$\text{ecc}(\Delta z_1, \Delta z_2) \leq \text{dist}(z_1 - z_2, F'(x)^-[F_2 - F_1 - C]).$$

The multimapping  $(F'(x) + C)$  being convex, graph closed and satisfying (3.7), it follows from the generalized open mapping theorem [Borwein/Zhuang, Th. 34, Th. 2.5] that there exists  $\xi > 0$  and  $\beta > 0$  such that

$$\text{dist}(z_1 - z_2, F'(x)^-[F_2 - F_1 - C]) \leq \xi \text{dist}(F'(x)[z_1 - z_2], F_2 - F_1 - C) \quad (3.9)$$

as soon as the right hand side is less than  $\beta$ .

From the mean value theorem there exist  $\theta \in ]0, 1[$  such that

$$F_2 - F_1 = F(t + z_1) - F(t + z_2) = F'(t + z_1 + \theta(z_2 - z_1))(z_1 - z_2).$$

Thus using the continuity of  $F'$  at  $x$ , there exists  $r$ ,  $0 < r < \beta/2\xi$  such that for  $(x - t)$ ,  $z_1, z_2 \in B(0, r)$  the right hand side of (3.9) is less than  $\xi\|z_2 - z_1\| \leq \beta$ . Consequently

$$\forall z_1, z_2 \in B(0, r) \quad \text{ecc}(\Delta z_1, \Delta z_2) \leq \xi\|z_2 - z_1\|.$$

Swapping  $z_1$  and  $z_2$  we can replace  $\text{ecc}$  by the Hausdorff distance  $H$ .

Note that we have also from the generalized open mapping theorem

$$\text{dist}(0, \Delta 0) = \text{dist}(0, F'(x)[y - F(t) - C]) \leq \xi \text{dist}(y - F(t), C)$$

as soon as  $\xi \text{dist}(y - F(t), C) \leq \beta$ . Then there exists  $\delta > 0$  such that for  $y \in B(F(x), \delta)$  and  $t \in B(x, \delta)$  we get  $\text{dist}(0, \Delta 0) < r(1 - \xi)$ . This allows to apply the contraction multimapping principle [Ioffe/Tihomirov, p. 31] to  $\Delta$ . Thus for every  $s$  such that  $\text{dist}(0, \Delta 0) < s < r(1 - \xi)$  there exists  $z \in B(0, s)$  such that  $z \in \Delta z$  and  $\|z\| \leq (2/(1 - \xi)) \text{dist}(0, \Delta 0)$ . From the expression of  $\Delta$  it follows that  $F(t + z) \in y - C$  and

$$\text{dist}(t, F^-[y - C]) \leq \|z\| \leq \frac{2\xi}{1 - \xi} \text{dist}(y, F(t) + C). \quad \square$$

**3.9 THEOREM.** Let  $X, Y, Z, W$  Banach spaces and  $f: X \rightarrow Y, g: X \rightarrow Z, p: X \rightarrow W$  continuously differentiable mappings at  $x_0 \in S$ . We suppose that  $x_0$  is a weak optimal solution belonging to  $\text{int } \Omega$  and that  $Y \times Z \times W = (f, g, p)'(x_0) \cdot X + -C \times \mathbf{R}g(x_0) \times \{0\}$ .

Then there exists  $(\lambda^*, \mu^*, \nu^*) \in [C^* \setminus \{0\}] \times D_{x_0}^* \times W^*$  such that for each  $\delta > 0$  there exists  $V_\delta \in \mathcal{N}_X(x_0)$  satisfying:

$$\begin{aligned} \min_{x \in V_\delta} \lambda^* f(x) + \mu^* g(x) + \nu^* p(x) + \delta [\|f(x) - f(x_0)\|^2 + \|g(x) - g(x_0)\|^2 + \|p(x)\|^2]^{1/2} \\ = \lambda^* f(x_0). \end{aligned}$$

**PROOF:** We set  $F = (f - f(x_0), g - g(x_0), p)$ . One can see easily that the condition  $Y \times Z \times W = (f, g, p)'(x_0) \cdot X + -C \times \mathbf{R}g(x_0) \times \{0\}$  implies that  $p'(x_0)$  is surjective and that the condition  $H_x \neq \emptyset$  is satisfied. Then from the Proposition 2.4 (R1) is also satisfied and by the Theorem 3.5 we get  $(\lambda^*, \mu^*, \nu^*) \in (C^* \setminus \{0\}) \times D_{x_0}^* \times W^*$  such that

$$\forall x \in \Omega \quad (\lambda^* f + \mu^* g + \nu^* p)'(x_0) \cdot (x - x_0) \geq 0. \quad (3.10)$$

Let us use the notation  $r = \lambda^* f + \mu^* g + \nu^* p$ . As  $x_0 \in \text{int } \Omega$ , (3.10) with the linearity of  $r'(x_0)$  means that  $r'(x_0) = 0$ . Consider the closed hyperplan  $H = \{z \in Z \mid \mu^* z = 0\}$ . We have  $\text{Rg}(x_0) \subset H$  and denoting by  $k$  the closed convex cone  $-C \times H \times \{0\} \subset Y \times Z \times W$  we have obviously  $Y \times Z \times W = F'(x_0) \cdot X + K$ . It follows from Theorem 3.8 that there exist  $\varepsilon > 0$ ,  $k > 0$  such that

$$\forall x \in B(x_0, \varepsilon) \quad \text{dist}[x, F^-(-K)] \leq k \text{dist}[0, F(x) + K].$$

Thus for each  $x \in B(x_0, \varepsilon)$ , there exists  $a(x)$  with

$$f(a(x)) - f(x_0) \in C \quad g(a(x)) - g(x_0) \in H \quad p(a(x)) = 0 \quad (3.11)$$

and

$$\|x - a(x)\| \leq k \text{dist}[(f(x) - f(x_0), g(x) - g(x_0), p(x)), C \times H \times \{0\}]. \quad (3.12)$$

On the other hand we have by the mean value theorem:

$$\begin{aligned} \forall x \in B(x_0, \varepsilon) \quad r(x) &= r(a(x)) + r'(\xi)(a(x) - x) \\ &\geq r(a(x)) - k\|r'(\xi)\| [\|f(x) - f(x_0)\|^2 + \|g(x) - g(x_0)\|^2 + \|p(x)\|^2]^{1/2} \end{aligned} \quad (3.13)$$

where  $\xi \in ]x, a(x)[$ . Since  $r'(x_0) = 0$  and  $r$  is continuously differentiable at  $x_0$ , for every  $\delta > 0$  there exists  $\eta > 0$  such that

$$\forall x \in B(x_0, \eta) \quad \|r'(x)\| < \frac{\delta}{k}.$$

Now using (3.12) and the continuity of  $f$ ,  $g$  and  $p$  let be  $\varepsilon_1 < \inf(\varepsilon, \eta/2)$  such that

$$\forall x \in B(x_0, \varepsilon_1) \quad \|x - a(x)\| < \frac{\eta}{2}.$$

It yields

$$\forall x \in B(x_0, \varepsilon_1) \quad r(x) \geq r(a(x)) - \delta [\|f(x) - f(x_0)\|^2 + \|g(x) - g(x_0)\|^2 + \|p(x)\|^2]^{1/2}.$$

From (3.11)

$$r(a(x)) = \lambda^* f(a(x)) \geq \lambda^* f(x_0) = r(x_0)$$

which gives

$$\forall x \in B(x_0, \varepsilon_1) \quad r(x) \geq r(x_0) - \delta [\|f(x) - f(x_0)\|^2 + \|g(x) - g(x_0)\|^2 + \|p(x)\|^2]^{1/2}. \quad \square$$

**4. Sufficient optimality conditions whitout convexity.** In this section we use a slightly less restrictive notion of optimality. A point  $x \in S$  is said to be a *local weak solution in the direction*  $d \in X$  if there exists a neighborhood  $N \in \mathcal{N}(x)$  such that if  $t > 0$  and  $x + td \in S \cap N$  then we have  $f(x + td) - f(x) \notin -\text{int } C$ .

At first we note that the necessary conditions obtained in Theorem 3.5 (ii) and 3.6 (iii) are sufficient to obtain  $x \in \Theta$ , without any regularity condition:

**4.1 THEOREM.** *Let  $f$  be regularly lower uniformly differentiable at  $x \in S$ ,  $g$  regularly locally convex at  $x$  and  $p$  Fréchet differentiable at  $x$ . If there exists  $(\lambda^*, \mu^*, \nu^*) \in (C^* \setminus \{0\}) \times D_x^* \times W^*$  such that*

$$\forall h \in \Omega - x \quad [\lambda^* f'(x) + \mu^* g'(x) + \nu^* p'(x)] h \geq 0$$

*then  $x \in \Theta$ .*

**PROOF:** Suppose that  $x \notin \Theta$ , then there exists  $h_0 \in H_x$  such that  $f'(x, h_0) \cap -\text{int } C \neq \emptyset$ ,  $g'(x) \cdot h_0 \in -\text{int}((D^*)^*)$  and  $p'(x) \cdot h_0 = 0$ . Thus taking  $y_0 \in f'(x, h_0) \cap -\text{int } C$  we have

$$\lambda^* y_0 + \mu^* g'(x) \cdot h_0 + \nu^* p'(x) \cdot h_0 \leq \lambda^* y_0 < 0. \quad \square$$

Theorem 4.1 gives sufficient conditions for  $x$  to belong to  $\Theta$  but not to be an optimal solution. In order to answer this question we introduce a second order assumption ensuring that a point  $x \in \Theta$  be an optimal solution of the problem  $(\mathcal{P})$ .

**4.2 PROPOSITION.** *Let  $f$  and  $g$  be regularly locally convex at  $x \in S$  and  $p$  Fréchet differentiable at  $x$ . Suppose that there exists  $(\lambda^*, \mu^*, \nu^*) \in (C^* \setminus \{0\}) \times D_x^* \times W^*$  such that*

$$\forall h \in \Omega - x \quad [\lambda^* f'(x) + \mu^* g'(x) + \nu^* p'(x)] h \geq 0 \quad (4.1)$$

*and that*

$$\lambda^* f'(x) \cdot h = 0 \implies \lim_{t \rightarrow 0^+} \frac{\lambda^* f(x + th) - \lambda^* f(x)}{t^2} \in \mathbf{R}^+ \setminus \{0\} \quad (4.2)$$

*then  $x$  is a local weak minimum in all directions.*

**PROOF:** Let  $d \in X$  be a direction. Suppose that there exists  $t_n > 0$  such that  $x + t_n d \in S$ ,  $\lim_{n \rightarrow \infty} t_n = 0$  with  $f(x + t_n d) - f(x) \in -\text{int } C$ . Setting  $x_n = x + t_n d \in S$ , we have for each  $n \in \mathbf{N}$ ,  $d = t_n^{-1}(x_n - x) \in T_x S$ . From Proposition 2.1,  $g'(x) \cdot d \in -(D_x^*)^*$ ,  $p'(x) \cdot d = 0$  and

$$\lambda^* f'(x) \cdot d \geq -\mu^* g'(x) \cdot d - \nu^* p'(x) \cdot d \geq 0.$$

On the other hand,  $f$  being locally convex,  $\lambda^*$  a continuous functional and  $C$  closed,

$$\lambda^* f'(x) \cdot d = \lim_{n \rightarrow \infty} \lambda^* \left[ \frac{f(x + t_n d) - f(x)}{t_n} \right] \leq 0.$$

Therefore  $\lambda^* f'(x) \cdot d = 0$  and from (4.2), for  $n$  large enough

$$\frac{\lambda^* f(x + t_n d) - \lambda^* f(x)}{t^2} > 0$$

a contradiction with  $f(x + t_n d) - f(x) \in -\text{int } C$  and  $\lambda^* \in C^* \setminus \{0\}$ .  $\square$

**4.3 PROPOSITION.** *Let  $X, Y, Z, W$  be Banach spaces,  $f, g, p$  Fréchet differentiable mappings at  $x \in S$  and  $X$  a reflexive space. If (4.1) and (4.2) hold then  $x$  is a local weak solution of  $(\mathcal{P})$ .*

**PROOF:** Suppose there exist  $x_n \in S$ ,  $\lim_{n \rightarrow \infty} x_n = x$  with  $f(x_n) - f(x) \in -\text{int } C$ . We denote by  $h$  a weak limit of  $\|x_n - x\|^{-1}(x_n - x)$ . From the Fréchet differentiability of  $f, g$  and  $p$  we have

$$0 \geq \mu^* g(x_n) = \mu^* g'(x) \cdot (x_n - x) + \mu^* \|x_n - x\| \varepsilon_1(x_n - x) \quad (4.3)$$

$$0 = \nu^* p'(x) \cdot (x_n - x) + \nu^* \|x_n - x\| \varepsilon_2(x_n - x) \quad (4.4)$$

$$\lambda^* f(x_n) = \lambda^* f(x) + \lambda^* f'(x) \cdot (x_n - x) + \lambda^* \|x_n - x\| \varepsilon_3(x_n - x) \quad (4.5)$$

where  $\lim_{\|t\| \rightarrow \infty} \varepsilon_i(t) = 0$ ,  $i = 1, 2, 3$ .

From (4.1)

$$\lambda^* f'(x) \cdot (x_n - x) \geq -\mu^* g'(x) \cdot (x_n - x) - \nu^* p'(x) \cdot (x_n - x).$$

Thus for each  $\varepsilon > 0$  there exists  $n_0 > 0$  such that for  $n > n_0$

$$\lambda^* f'(x) \cdot \left[ \frac{x_n - x}{\|x_n - x\|} \right] \geq \mu^* \varepsilon_1(x_n - x) + \nu^* \varepsilon_2(x_n - x) \geq -\varepsilon.$$

As  $\lambda^* f'(x)$  is a linear continuous function,

$$\lambda^* f'(x) \cdot h = \lim_{n \rightarrow \infty} \lambda^* f'(x) \cdot \left[ \frac{x_n - x}{\|x_n - x\|} \right] \geq 0.$$

On the other hand by (4.5)

$$\lambda^* f'(x) \cdot \left[ \frac{x_n - x}{\|x_n - x\|} \right] = \frac{\lambda^* \cdot (f(x_n) - f(x))}{\|x_n - x\|} - \lambda^* \varepsilon_3(x_n - x)$$

which implies  $\lambda^* f'(x) \cdot h \leq 0$ , since  $f(x_n) - f(x) \in -\text{int } C$ ,  $\lambda^* \in C^* \setminus \{0\}$ . Therefore  $\lambda^* f'(x) \cdot h = 0$  and from (4.2) we get, as in the previous proof a contradiction with  $\lambda^*[f(x_n) - f(x)] < 0$ .  $\square$

## REFERENCES

1. Borwein, J.M., *Continuity and differentiability properties of convex operators*, Proc. London Math. Soc. **3** (1982), 420-444.
2. Borwein, J.M., Zhuang, D.M., *Verifiable Necessary and Sufficient Conditions for Openness and Regularity of Set-Valued and Single-Valued Maps*, Jour. Math. Anal. Appl. **134** (1988), 441-459.
3. Craven, B.D., *Vector-Valued Optimization*, in "Generalized Concavity in Optimization and Economics," Schaible, S., Ziemba, W.T. (eds.), Academic Press, New York, 1981, pp. 661-687.
4. Dolecki, S., Personal communication.
5. Holmes, R.B., "Geometric Functional Analysis and its Application," Springer-Verlag, 1975.
6. Ioffe, A.D., Tihomirov, V.M., "Theory of Extremal Problems," North-Holland, Amsterdam, 1979.
7. Jahn, J., "Mathematical Vector Optimization in Partially Ordered Linear Spaces," Verlag Peter Lang, Frankfurt am Main, 1986.
8. Jahn, J., Sachs, E., "Generalized Convex Mappings and Vector Optimization," Research Report, North Carolina State University, Raleigh, 1983.
9. Khank, P.C., Nuong, T.H., *On Necessary Optimality Conditions in Vector Optimization Problems*, J. Optim. Theory Appl. **58** (1988).
10. Malivert, C., *Optimisation de plusieurs fonctions sur une variété à bord*, Bulletin de Mathématiques Économiques **13** (1975), 60-88.
11. Minami, M., *Weak Pareto Optimality Multiobjective Problems in a Banach Space*, Bull. of Mathematical Statistics **19** (1981), 19-23.
12. Minami, M., *Weak Pareto Optimality Multiobjective Problems in a Locally Convex Linear Topological Space*, J. Optim. Theory Appl. **34** (1981), 469-484.
13. Minami, M., *Weak Pareto Optimal Necessary Conditions in a Nondifferentiable Multiobjective Program on a Banach Space*, J. Optim. Theory Appl. **41** (1983), 451-461.
14. Oettli, W., *Optimality Conditions for Programming Problems Involving Multivalued Mappings*, in "Modern Applied Mathematics: Optimization and Operations Research," Korte, B. (ed.), North-Holland, Amsterdam, 1980.
15. Smale, S., "Optimizing Several Functions," International Symposium on Manifolds, Tokyo, 1973.

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