

L^α SPACES FOR OPERATOR VALUED MEASURES

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RÉSUMÉ. Dans cet article on introduit les espaces L^α correspondant à une intégrale par rapport à une mesure à valeurs opérateurs dans un espace vectoriel topologique localement convexe. En particulier, la notion d'espace α -Radon-Nikodym est définie et caractérisée en terme de convergence de martingales.

ABSTRACT. The L^α spaces are introduced and studied for an integral with respect operator valued measures in the context of locally convex spaces. Also the α -Radon-Nikodym property with respect to an integral system is defined and a characterization is given for the space having it, by means of the convergence of martingales.

In [8] is developed a theory of integration of functions valued in locally convex spaces with respect to operator (also between locally convex spaces) valued measures, following the ideas of [9] and which extends the integration theory defined in [5] (I). This integration is not in general comparable with the integral studied in [7] as it is stated in [4]. The purpose of this paper is to define and to study the L^α spaces for the integral introduced in [8]; for this objective, the M^α spaces of the measurable functions of bounded (α, m) -semivariation are defined. Several standard questions are treated, as the density of the simple functions, the dominated convergence theorem and others. Also it is proved that the space L^α is Fréchet (or Polish) under certain conditions. Let us remark that these spaces L^α (in the context of locally convex spaces) are not in general complete.

The last part of the work is devoted to the study of the existence of the Radon-Nikodym derivative of a vector measure with respect to an operator valued measure, in the general context of the paper. The α -Radon-Nikodym property is defined using the concept of (α, m) -controlled measures and it is proved a characterization of the spaces having the α -Radon-Nikodym property by means of the convergence of martingales (the martingale and its convergence in this context are defined in a natural way).

Let us denote by E and F two locally convex Hausdorff spaces, being F complete and $\mathcal{P}$ and $\mathcal{R}$ two generating families of seminorms on E and F respectively. Let G be the space $L(E, F)$ of the linear and continuous mappings from E into F , endowed with the topology of the pointwise convergence. m will be a (countably additive) G -valued measure defined on a σ -algebra Σ of subsets of a set Ω which verifies the $*$ -condition, i.e. there exists a family of non negative and finite measures defined on Σ , $\{\nu_{p,r} \mid p \in \mathcal{P}, r \in \mathcal{R}\}$, such that $\|m\|_{p,r} \ll \nu_{p,r}$ for every pair $(p, r) \in \mathcal{P} \times \mathcal{R}$ such that $\|m\|_{p,r}(\Omega) < \infty$ being

$$\|m\|_{p,r}(A) = \sup r \left(\sum_{i=1}^n x_i m(A_i) \right)$$

for every $A \in \Sigma$, where the supremum is taken over all the measurable and finite partitions $\{A_i\}_{i=1}^n$ of A and all finite families $\{x_i\}_{i=1}^n \subset E$ such that $p(x_i) \leq 1$ for $i = 1, \dots, n$. We say that the measure m verifies the $\ast'$ -condition if for every continuous seminorm $r \in \mathcal{R}$ there exists a non negative and finite measure $\nu_r: \Sigma \rightarrow \mathbb{R}^+$ such that $\|m_{p,r}\| \ll \nu_r$ for every continuous seminorm $p \in \mathcal{P}$. As usual m is said to verify the $\ast\ast$ -condition if there is a non negative and finite measure ν on Σ such that $\|m_{p,r}\| \ll \nu$ for every pair $(p, r) \in \mathcal{P} \times \mathcal{R}$ such that $\|m\|_{p,r}(\Omega) < \infty$. In a standard way, a set $A \in \Sigma$ will be said to be m -null if $\|m\|_{p,r}(A) = 0$ for every pair of continuous seminorms $p \in \mathcal{P}$ and $r \in \mathcal{R}$.

Now on we will assume that the measure m is of bounded variation (i.e. for every seminorm $r \in \mathcal{R}$ there exists $p \in \mathcal{P}$ such that $\|m\|_{p,r}(\Omega) < \infty$).

DEFINITION 1. Let be a function $f: \Omega \rightarrow E$. We say that f is *u-simple* if it is a uniform limit of a net of simple functions (the simple functions and their integral, and the integral of the u-simple functions are defined in a standard way). S_0 and S will denote respectively the families of the simple functions and the u-simple functions.

The function f is said to be *measurable* (or *m-measurable*) if for every $\varepsilon > 0$ and $(p, r) \in \mathcal{P} \times \mathcal{R}$ with $\|m\|_{p,r}(\Omega) < \infty$, there exists $A \in \Sigma$ such that $\|m\|_{p,r}(\Omega \setminus A) \leq \varepsilon$ and the function $f \cdot \chi_A$ is u-simple.

We say that a function f is *m-integrable* (or *integrable*) if it is measurable and for every pair of seminorms $(p, r) \in \mathcal{P} \times \mathcal{R}$ with $\|m\|_{p,r}(\Omega) < \infty$, the (p, r) -semivariation of the function f is continuous, i.e. if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|m\|_{p,r}(f, A) \leq \varepsilon$ for every $A \in \Sigma$ which verifies that $\|m\|_{p,r}(A) \leq \delta$, being

$$\|m\|_{p,r}(f, A) = \sup r \left(\int_A g \, dm \right),$$

where the supremum is taken over all u-simple functions $g \in S$ such that $p[g(t)] \leq p[f(t)]$ for every $t \in A$.

If f is an integrable function then

$$m_f(A) = \int_A f \, dm = \lim_{K \in \Sigma(f)} \int_A f \cdot \chi_K \, dm$$

for every $A \in \Sigma$, being $\Sigma(f) = \{k \in \Sigma \mid f \cdot \chi_K \in S\}$.

DEFINITION 2. If $1 \leq \alpha < \infty$ and f is a measurable function, for every pair of continuous seminorms $p \in \mathcal{P}$ and $r \in \mathcal{R}$ with $\|m\|_{p,r}(\Omega) < \infty$, the (α, m, p, r) -semivariation of the function f will be the set function $m_{p,r}^\alpha(f, \cdot): \Sigma \rightarrow \mathbb{R}^+ \cup \{\infty\}$ such that

$$m_{p,r}^\alpha(f, A) = \sup \left[\int_A p^\alpha(f) \, d|y'm|_p \right]^{1/\alpha},$$

where the supremum is taken over all $y' \in F'$ (being F' the dual space of F) such that $|y'(z)| \leq r(z)$ for every $z \in F$ being $|y'm|_p$ the p -variation of the measure $y'm: \Sigma \rightarrow L(ER)$ (remark that

$$|y'm|_p(A) = \sup \sum_{B \in \pi} \left(\sup_{\substack{x \in E \\ p(x) \leq 1}} |y'm(B)(x)| \right) \quad (A \in \Sigma)$$

where the supremum is taken over all the measurable and finite partitions π of A).

It can be proved that the equality

$$m_{p,r}^\alpha(f, A) = \sup \left[r \left(\int_A g \, dm \right) \right]^{1/\alpha}$$

holds where the supremum is taken over all the u -simple functions g such that $p[g(t)] \leq p^\alpha[f(t)]$ for every $t \in A$.

The function f is said to be of *bounded (α, m) -semivariation* if $m_{p,r}^\alpha(f) = m_{p,r}^\alpha(f, \Omega) < \infty$ for every pair of seminorms $p \in \mathcal{P}$ and $r \in \mathcal{R}$ such that $\|m\|_{p,r}(\Omega) < \infty$.

DEFINITION 3. Let us denote by $\mathcal{M}^\alpha(E)$ ($1 \leq \alpha < \infty$) the vector space of all measurable functions of bounded (α, m) -semivariation endowed with the topology defined by the family of seminorms $\{m_{p,r}^\alpha(\cdot, \Omega) \mid (p, r) \in \mathcal{P} \times \mathcal{R} \text{ with } \|m\|_{p,r}(\Omega) < \infty\}$, and as usual $M^\alpha(E)$ will be the corresponding Hausdorff space of the equivalence classes modulus $\{f \in \mathcal{M}^\alpha(E) \mid m_{p,r}^\alpha(f) = 0 \text{ for every pair } (p, r) \in \mathcal{P} \times \mathcal{R} \text{ such that } \|m\|_{p,r}(\Omega) < \infty\}$.

$\mathcal{L}^\alpha(E)$ denotes the space of the functions $f \in \mathcal{M}^\alpha(E)$ such that their (α, m) -semivariation are $\|m\|_{p,r}$ -continuous ($m_{p,r}^\alpha(f, \cdot) \ll \|m\|_{p,r}$ for every pair of seminorms $(p, r) \in \mathcal{P} \times \mathcal{R}$ such that $\|m\|_{p,r}(\Omega) < \infty$, endowed with the topology induced by $\mathcal{M}^\alpha(E)$. The corresponding Hausdorff space of the equivalence classes will be denoted by $L^\alpha(E)$.

$\mathcal{L}^\infty(E)$ represents the space of the m -integrable functions f which are essentially bounded endowed with the topology defined by the family of seminorms $\{p_\infty \mid p \in \mathcal{P}\}$ being

$$p_\infty(f) = \text{essen sup}\{p[f(t)] \mid t \in \Omega\}.$$

$L^\infty(E)$ will denote the corresponding Hausdorff space of the equivalence classes.

It can be proved that $S_0 \subset S \subset L^\beta(E) \subset M^\beta(E) \subset L^\alpha(E) \subset M^\alpha(E)$, for $1 \leq \alpha < \beta \leq \infty$. Moreover S_0 (and therefore S) is a dense subspace of $L^\alpha(E)$ and $L^\alpha(E)$ is a closed subspace of $M^\alpha(E)$, for every $1 \leq \alpha < \infty$. From where it follows immediately that $L^\alpha(E)$ is separable if $1 \leq \alpha < \infty$, m verifies the $**$ -condition and the measure space (Ω, Σ, ν) is countably generated.

THEOREM 4 (DOMINATED CONVERGENCE THEOREM). Let be $1 \leq \alpha < \infty$ and $(f_n)_{n \in \mathbb{N}}$ a sequence of measurable functions which converge to a measure function. If for every continuous seminorm $p \in \mathcal{P}$ there exists a function $g_p \in L^\alpha(E)$ such that $p(f_n) \leq p(g_p)$ a.e. for every $n \in \mathbb{N}$, then $f_n, f \in L^\alpha(E)$ ($n \in \mathbb{N}$) and the sequence $(f_n)_{n \in \mathbb{N}}$ converges to f in $L^\alpha(E)$.

PROOF: In fact $\{f, f_n \mid n \in \mathbb{N}\} \subset L^\alpha(E)$ since $m_{p,r}^\alpha(f_n) \leq m_{p,r}^\alpha(g_p) < \infty$ and $m_{p,r}(f_n, A) \leq m_{p,r}(g_p)$ for every $A \in \Sigma$, $n \in \mathbb{N}$ and $(p, r) \in \mathcal{P} \times \mathcal{R}$ such that $\|m\|_{p,r}(\Omega) < \infty$ and

$$p[f(t)] = \lim_n p(f_n(t)) \leq p[g_p(t)] \text{ a.e..}$$

Moreover, for every $\varepsilon > 0$, $p \in \mathcal{P}$ and $r \in \mathcal{R}$ with $\|m\|_{p,r}(\Omega) < \infty$, it results from Egorov's theorem (since $\|m\|_{p,r} \ll \nu_{p,r}$) the existence of a set $A \in \Sigma$ such that $m_{p,r}^\alpha(f, \Omega \setminus A) \leq \varepsilon$ and $m_{p,r}^\alpha(f_n, \Omega \setminus A) \leq \varepsilon$ for every $n \in \mathbb{N}$, and therefore, there exists $n_0 \in \mathbb{N}$ such that $p[f(t) - f_n(t)] \leq \varepsilon$ and

$$\begin{aligned} m_{p,r}^\alpha(f - f_n) &\leq m_{p,r}^\alpha(f - f_n, A) + m_{p,r}^\alpha(f - f_n, \Omega \setminus A) \\ &\leq \varepsilon [\|m\|_{p,r}(\Omega)]^{1/\alpha} + m_{p,r}^\alpha(f, \Omega \setminus A) + m_{p,r}^\alpha(f_n, \Omega \setminus A) \\ &\leq \varepsilon \left[(\|m\|_{p,r}(\Omega))^{1/\alpha} + 2 \right] \end{aligned}$$

for every $t \in A$ and $n \geq n_0$, from where it follows immediately that the sequence (f_n) converges to f in $L^\alpha(E)$. $\square$

LEMMA 5. *If the measure m verifies the $*$ '-condition, the space E is metrizable and (f_n) is a sequence of measurable functions which converges a.e. to a function f , then the function f is also measurable.*

PROOF: Let us consider $p \in \mathcal{P} (= \{p_k \mid k \in \mathbb{N}\})$, $r \in \mathcal{R}$ and a m -null set $B \in \Sigma$ such that $\|m\|_{p,r}(\Omega) < \infty$ and $(f_n(t))$ converges to $f(t)$ for every $t \in \Omega \setminus B$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|m\|_{p,r}(A) \leq \varepsilon$ for every $A \in \Sigma$ with $\nu_r(A) \leq \delta$. It results from Egorov's theorem that for every $k \in \mathbb{N}$ there exists $C_k \in \Sigma$ such that $\nu_r(C_k) \leq \delta/2^k$ and $(p_k(f_n - f))_{n \in \mathbb{N}}$ is uniformly convergent to zero on $\Omega \setminus (C_k \cup B)$. If $A = B \cup (\bigcup_{k=1}^{\infty} C_k)$ then $A \in \Sigma$,

$$\|m\|_{p,r}(A) \leq \|m\|_{p,r}(B) + \|m\|_{p,r}\left(\bigcup_{k=1}^{\infty} C_k\right) \leq \varepsilon$$

and (f_n) is uniformly convergent to f on $\Omega \setminus A$. Therefore $f \cdot \chi_{\Omega \setminus A}$ is a u-simple function and f is measurable. $\square$

THEOREM 6. *If E is a Banach space and F is Fréchet, then the space $L^\alpha(E)$ is Fréchet for $1 \leq \alpha < \infty$.*

PROOF: Let us denote by p the norm on E and consider a non decreasing generating family of seminorms $\mathcal{R} = \{r_n \mid n \in \mathbb{N}\}$ on F . If $(f_n) \subset L^\alpha(E)$ is a Cauchy sequence, we can find a subsequence that we will denote by (g_n) , such that

$$m_{p,r_n}^\alpha(g_n - g_{n+1}) \leq \frac{1}{2^{n(1+1/\alpha)}}$$

for every $n \in \mathbb{N}$. Let

$$A_n = \left\{ t \in \Omega \mid p(g_n - g_{n+1}) > \frac{1}{2^n} \right\}$$

for every $n \in \mathbb{N}$ and $A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$.

Evidently, $A_n, A \in \Sigma$ ($n \in \mathbb{N}$) and A is a m -null set since for every $k \in \mathbb{N}$ we have that

$$\begin{aligned} \|m\|_{p,r_k}(A) &\leq \|m\|_{p,r_k}\left(\bigcup_{n=k}^{\infty} A_n\right) \\ &\leq \sum_{n \geq k} \|m\|_{p,r_n}(A_n) \leq \frac{1}{2^{k-1}} \end{aligned}$$

and so for every $\varepsilon > 0$ and $r \in \mathcal{R}$, there exists $n \in \mathbb{N}$ such that $1/2^{n-1} < \varepsilon$ and $r \leq r_n$ and consequently,

$$\|m\|_{p,r}(A) \leq \|m\|_{p,r_n}(A) \leq \frac{1}{2^{n-1}} < \varepsilon.$$

Moreover, if $t \notin A$ there exists $k \in \mathbb{N}$ such that $t \notin \bigcup_{n \geq k} A_n$ and for every $\varepsilon > 0$ there exists $n_k \geq k$ such that $1/2^{n_k-1} < \varepsilon$ and

$$p[g_n(t) - g_m(t)] \leq \sum_{i \geq n_k} \frac{1}{2^i} < \varepsilon$$

for $n, m \geq n_k$. Therefore for every $t \in \Omega \setminus A$, $(g_n(t))_{n \in \mathbb{N}}$ is a Cauchy sequence in E and there exists $g(t) = \lim_n g_n(t)$. It results from Lemma 5 that the function g is measurable (we consider $g|_A = 0$).

If $t \notin A^K = \bigcup_{n \geq k} A_n$, then

$$p[g(t) - g_k(t)] = \lim_{n \geq k} p[g_n(t) - g_k(t)] \leq \lim_{n \geq k} \sum_{i=k}^{n-1} p[g_i(t) - g_{i+1}(t)] \leq \sum_{i \geq k} \frac{1}{2^i} = \frac{1}{2^{k-1}}.$$

Moreover, for every $\varepsilon > 0$, $r_k \in \mathcal{R}$, $n \geq k$ and $s \in S_0$ verifying $p(s) \leq p^\alpha(g - g_n)$, there exists $q \in \mathbb{N}$ such that $q \geq n$ and

$$\frac{1}{2^{q-1}} < \min \left(\frac{1}{2^n} C, \frac{1}{2^n} \|m\|_{p, r_k}(\Omega) \right),$$

being $C = \max\{p[s(t)] \mid t \in \Omega\}$ (we can assume that $C > 0$ and $\|m\|_{p, r_k}(\Omega) > 0$, since in other case the question is trivial). Then,

$$\begin{aligned} r_k \left(\int_{\Omega} s \, dm \right) &\leq r_k \left(\int_{A^q} s \, dm \right) + r_k \left(\int_{\Omega \setminus A^q} s \, dm \right) \leq C \|m\|_{p, r_k}(A^q) + r_k \left(\int_{\Omega \setminus A^q} s \, dm \right) \\ &\leq \frac{1}{2^n} + \sup_{\substack{y' \in F' \\ |y'| \leq r_k}} \int_{\Omega \setminus A^q} p(s) \, d|y' m|_p \leq \frac{1}{2^n} + \sup_{\substack{y' \in F' \\ |y'| \leq r_k}} \int_{\Omega \setminus A^q} p^\alpha(g - g_n) \, d|y' m|_p \\ &\leq \frac{1}{2^n} + \sup_{\substack{y' \in F' \\ |y'| \leq r_k}} \left[\int_{\Omega \setminus A^q} p^\alpha(g - g_q) \, d|y' m|_p \right]^{1/\alpha} + \sup_{\substack{y' \in F' \\ |y'| \leq r_k}} \left[\int_{\Omega \setminus A^q} p^\alpha(g_p - g_n) \, d|y' m|_p \right]^{1/\alpha} \\ &\leq \frac{1}{2^n} + \frac{1}{2^{q-1}} \|m\|_{p, r_k}(\Omega) + m_{p, r}^\alpha(g_p - g_n) \leq \frac{1}{2^{n-1}} + \sum_{i=n}^{q-1} m_{p, r_k}(g_i - g_{i+1}) \\ &\leq \frac{1}{2^{n-1}} + \sum_{i=1}^{q-1} \frac{1}{2^{i(1+1/\alpha)}} \leq \frac{1}{2^{n-1}} + \sum_{i \geq n} \frac{1}{2^i} = \frac{1}{2^{n-2}}, \end{aligned}$$

from where it follows immediately that $g \in M^\alpha(E)$ and that (g_n) converges to g in $M^\alpha(E)$, and therefore, $g \in L^\alpha(E)$ and (f_n) converges to g in $L^\alpha(E)$. $\square$

COROLLARY 7. *If the measure m verifies the **-condition, the control measure space (Ω, Σ, ν) is countably generated, E is a Separable Banach space and F is a Fréchet space, then $L^\alpha(E)$ is a Polish space for every $1 \leq \alpha < \infty$.*

PROOF: It follows immediately from Theorem 6 (see also the last part of the Definition 3). $\square$

Radon-Nykodym property.

DEFINITION 8. Let $\theta: \Sigma \rightarrow F$ be a countably additive measure. It is said that θ is (α, m) -controlled ($1 \leq \alpha \leq \infty$) if there exists a subalgebra $\Sigma' \subset \Sigma$ and a bounded subset $(f_A)_{A \in \Sigma'}$ of $L^\alpha(E)$, such that the following assertions holds:

$$8.1 \quad \sigma(\Sigma') = \Sigma$$

$$8.2 \quad \theta(A) = \int_A f_A \, dm, \text{ for every } A \in \Sigma'.$$

It can be easily proved that if the measure θ is (α, m) -controlled, then it is (β, m) -controlled for every $1 \leq \beta \leq \alpha$. Moreover, if $1 < \alpha \leq \infty$ and θ is (α, m) -controlled, then

for every pair of continuous seminorms $p \in \mathcal{P}$ and $r \in \mathcal{R}$ for which $\|m\|_{p,r}(\Omega) < \infty$ holds, there exists $M_{p,r} > 0$ such that

$$r[\theta(A)] \leq [\|m\|_{p,r}(A)]^{1-1/\alpha} \cdot M_{p,r}$$

is verified for every $A \in \Sigma'$.

We say that a complete Hausdorff locally convex space F has the α -Radon-Nykodym property (we will write α -R.N.P.) ($1 \leq \alpha \leq \infty$) with respect an integral system $(\Omega, \Sigma, m, E, F)$, if every (α, m) -controlled measure $\theta: \Sigma \rightarrow F$ has a Radon-Nykodym derivative with respect to m belonging to $L^\alpha(E)$ (i.e. there exists $f \in L^\alpha(E)$ such that $\theta = m_f$). If the space F has the α -R.N.P. with respect to every integral system (of last type) whose measure m verifies the $**$ -condition, we will say that the space F has the R.N.P..

If F has the α -R.N.P. then it can be easily proved that every closed subspace of F has also the α -R.N.P..

DEFINITION 9. Let $(\Omega, \Sigma, m, E, F)$ be an integral system such that m verifies the $**$ -condition, $(\Sigma_i)_{i \in I}$ a non decreasing net of sub- σ -algebras of Σ and a net $(f_i)_{i \in I}$ in $L^\alpha(E)$ ($1 \leq \alpha \leq \infty$). $(f_i, \Sigma_i)_{i \in I}$ will be a martingale in $L^\alpha(E)$ if the following conditions are satisfied:

$$9.1 \quad \sigma\left(\bigcup_{i \in I} \Sigma_i\right) = \Sigma.$$

$$9.2 \quad \int_A f_i dm = \int_A f_j dm \text{ is verified for every } A \in \Sigma_i \text{ and } i, j \in I \text{ with } i \leq j$$

We say that the martingale $(f_i, \Sigma_i)_{i \in I}$ is a bounded martingale in $L^\alpha(E)$ if $(f_i)_{i \in I}$ is a bounded subset of $L^\alpha(E)$.

A martingale $(f_i, \Sigma_i)_{i \in I}$ in $L^\alpha(E)$ is said to be convergent (in $L^\alpha(E)$) if there exists a function $f \in L^\alpha(E)$ such that

$$\int_A f dm = \lim_i \int_A f_i dm$$

for every $A \in \bigcup_{i \in I} \Sigma_i$.

THEOREM 10. Let be $1 < \alpha \leq \infty$ and F a complete Hausdorff locally convex space, then the following assertions are equivalent:

10.1 F has the α -R.N.P..

10.2 Every Bounded martingale in $L^\alpha(E)$ is convergent (in $L^\alpha(E)$).

PROOF: Suppose that 10.1 is cerified and let $(\Omega, \Sigma, m, E, F)$ be a integral system such that m verifies the $**$ -condition and consider a bounded martingale in $L^\alpha(E)$. Then

$$\theta_0(A) = \lim_i \int_A f_i dm$$

exists for every $A \in \bigcup_{i \in I} \Sigma_i$ and for every pair of continuous seminorms $p \in \mathcal{P}$ and $r \in \mathcal{R}$ for wich $\|m\|_{p,r}(\Omega) < \infty$ holds, we have that

$$r[\theta_0(A)] = \lim_i \left(\int_A f_i dm \right) \leq \left[\sup_{i \in I} m_{p,r}^\alpha(f_i, A) \right] [\|m\|_{p,r}(A)]^{1-1/\alpha}$$

for every $A \in \bigcup_{i \in I} \Sigma_i$. Since m verifies the $**$ -condition, there exists a non negative and finite measure $\nu: \Sigma \rightarrow \mathbb{R}^+$ such that $\|m\|_{p,r} \ll \nu$ for every pair $(p, r) \in \mathcal{P} \times \mathcal{R}$ such that

$\|m\|_{p,r}(\Omega) < \infty$, and so $\theta_0 \ll \nu$ and proceeding like in the proof of the Carathéodory-Hahn-Kluvanek theorem, it is deduced the existence of a (α, m) -controlled measure $\theta: \Sigma \rightarrow F$ which is an extension of θ_0 , and since F has the α -R.N.P., there exists $f \in L^\alpha(E)$ such that $\theta = m_f$ and therefore

$$\int_A f \, dm = \theta(a) = \theta_0(a) = \lim_i \int_a f_i \, dm$$

for every $A \in \bigcup_{i \in I} \Sigma_i$, and the martingale $(f_i, \Sigma_i)_{i \in I}$ is convergent to f .

Conversely, suppose that 10.2 is verified and let us consider an integral system $(\Omega, \Sigma, m, E, F)$, such that m verifies the $**$ -condition, and a (α, m) -controlled measure $\theta: \Sigma \rightarrow F$. Then there exists a subalgebra $\Sigma' \subset \Sigma$ and a bounded subset $(f_A)_{A \in \Sigma'}$ of $L^\alpha(E)$ which satisfy 8.1 and 8.2. Let us consider the net $(f_\pi, \sigma(\pi))_{\pi \in \Pi}$ being Π the family of all finite and measurable partitions of Ω ordered by refinement and

$$f = \sum_{A \in \pi} f_A \chi_A$$

for every partition $\pi \in \Pi$. It easily proved that $(f_\pi, \sigma(\pi))_{\pi \in \Pi}$ is a bounded martingale in $L^\alpha(E)$ and therefore it is convergent to some function $f \in L^\alpha(E)$ which is a density with respect to m and consequently 10.1 holds. $\square$

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