

UNIQUENESS OF THE CAUCHY PROBLEM FOR THE EQUATION $Bu'(t) = Au(t)$ IN HILBERT SPACES

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RÉSUMÉ. Dans ce travail on obtient une extension du théorème 1.1 p. 52 dans la monographie [7]. Il s'agit d'un passage de l'équation $u'(t) = Au(t)$ à l'équation plus générale $Bu'(t) = Au(t)$, les deux opérateurs A et B étant linéaires et en général non-bornés dans l'espace de Hilbert H . Avec une condition de borne uniforme pour l'opérateur $(\lambda B - A)^{-1}$ on démontre que les solutions $u(t)$ dans un intervalle $[0, T]$ qui s'annulent dans l'origine, sont partout nulles dans $[0, T]$.

ABSTRACT. In this paper we obtain an extension of Theorem 1.1, p. 52 in the monograph [7]. It consists in the passage from the equation in normal form $u'(t) = Au(t)$ to the more general (implicit) equation $Bu'(t) = Au(t)$ where both A and B are linear unbounded operators in a Hilbert space H . Under an uniform boundedness condition for the operator $(\lambda B - A)^{-1}$ we prove that solutions $u(t)$ on the interval $[0, T]$ such that $u(0) = 0$, vanish identically on the whole interval.

Introduction. Uniqueness of Cauchy problem for abstract differential equations and also for abstract differential inequalities when existence and continuous dependence are not taken into consideration is an object of profound investigation: we mention here the Theorem 23.7.1 in the treatise Hille-Phillips [3] and the Theorem 1.2 in the memoir [1] of Agmon and Nirenberg. The vector-valued Laplace or Fourier transforms are the main tools therein. In our monograph [7] we presented, as Theorem 1.1 in Chapter 3, a simplified variant of the above mentioned result of Agmon-Nirenberg, and in the present paper we indicate a new extension of that theorem to a situation where a second (unbounded) linear operator appears, acting on the first derivative of the unknown function $u(t)$. Somehow, this kind of (implicit or singular) abstract differential problem of the first order was proposed in the classical book of Lions [5] and there is a discussion in Showalter [6] and also in Fattorini [2], but our results here appear unrelated to those exposed there.

1. Let H be a Hilbert space with scalar product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$. Let us consider two linear operators, both closed, A and B , which are defined respectively on $D(A) \subseteq H$ and on $D(B) \subseteq H$, with range also in H . We shall make the following

HYPOTHESIS 1. There exists a sequence of vertical lines in the complex plane $\{\operatorname{Re} \lambda = \sigma_n\}_{n=1}^{\infty}$ where $\sigma_n \rightarrow \infty$ as $n \rightarrow \infty$ and with the following property: the inverse operator $(\lambda B - A)^{-1} = R(\lambda; A, B)$ exists and belongs to $L(H)$ for $\operatorname{Re} \lambda = \sigma_n$, $n = 1, 2, \dots$, and the uniform bound: $\|R(\lambda; A, B)\| \leq M$ for $\operatorname{Re} \lambda = \sigma_n$, $n = 1, 2, \dots$, is verified.

We shall also introduce the

HYPOTHESIS 2. The operator B is invertible (that is, $Bh = 0 \iff h = 0$), but B^{-1} could be unbounded and it is possible that $D(B^{-1})$ is not the whole space H .

THEOREM. Let A, B be two linear operators which are closed in the Hilbert space H , satisfying both Hypotheses 1 and 2. Let also $u(t)$ be a function $0 \leq t \leq T \rightarrow D(A) \cap D(B)$,

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such that $u(t) \in C^1([0, T]; H)$, $u'(t) \in D(B)$, $Au(t) \in C([0, T]; H)$, $Bu'(t) = Au(t)$ on $[0, T]$ and $u(0) = 0$. Then $u(t) = 0$ for $0 \leq t \leq T$.

PROOF: We shall follow the method in [7] after doing the necessary changes. Consider the real-valued function $\xi_\delta(t) \in C^\infty(0, T)$, such that $\xi_\delta(t) = 1$ for $0 \leq t \leq \alpha < T$ and $\xi_\delta(t) = 0$ for $\alpha + \delta \leq t \leq T$, where $0 < \alpha < \alpha + \delta < T$. Next, for any real number σ , define the H -valued function $v_\sigma(t)$ which is $e^{-\sigma t} \xi_\delta(t) u(t)$ for $0 \leq t \leq T$ and equals zero for $t \notin (0, T)$. We see that $v_\sigma(0) = 0$ and that for $0 \leq t \leq \alpha$ the differential quotient $\frac{1}{t}[v_\sigma(t) - v_\sigma(0)]$ equals $e^{-\sigma t} \frac{1}{t}[u(t) - u(0)]$. Hence, we obtain that $\lim_{t \downarrow 0} \frac{1}{t}[v_\sigma(t) - v_\sigma(0)] = u'_+(0)$. On the other hand we have that $Bu'_+(0) = Au(0) = 0$ and using Hypothesis 2 we get that $u'_+(0) = 0$ too. Hence $v'_\sigma(0)$ exists and is null. Therefore the strong derivative $v'_\sigma(t)$ exists as a continuous function for any real number t and $v'_\sigma(t) = 0$ for $t \notin (0, T)$. After an elementary calculation we also get the relation:

$$v'_\sigma(t) = -\sigma v_\sigma(t) + \xi'_\delta(t) e^{-\sigma t} u(t) + \xi_\delta(t) e^{-\sigma t} u'(t).$$

Applying the operator B (which is obviously possible), we get the relation

$$\begin{aligned} Bv'_\sigma(t) &= -\sigma Bv_\sigma(t) + \xi'_\delta(t) e^{-\sigma t} Bu(t) + \xi_\delta(t) e^{-\sigma t} Bu'(t) \\ &= -\sigma Bv_\sigma(t) + \xi'_\delta(t) e^{-\sigma t} Bu(t) + \xi'_\delta(t) e^{-\sigma t} Au(t) \\ &= (A - \sigma B)v_\sigma(t) + \xi_\delta(t) e^{-\sigma t} Bu(t). \end{aligned}$$

Hence we can write

$$Bv'_\sigma(t) = (A - \sigma B)v_\sigma(t) + \xi'_\delta(t) e^{-\sigma t} Bu(t), \quad 0 \leq t \leq T.$$

Introducing the new function $f_\sigma(t)$ which is $\xi'_\delta(t) e^{-\sigma t} Bu(t)$ on $[0, T]$ and zero outside this interval we have on the whole real line, the equality

$$Bv'_\sigma(t) + (\sigma B - A)v_\sigma(t) = f_\sigma(t). \quad (1.1)$$

Consider now the Fourier transforms

$$\begin{aligned} \widehat{v}_\sigma(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} v_\sigma(t) dt, \\ \widehat{f}_\sigma(\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} f_\sigma(t) dt; \end{aligned}$$

also we note that

$$\widehat{v'_\sigma}(\tau) = i\tau \widehat{v}_\sigma(\tau).$$

In the equation (1.1) we multiply both sides by $\frac{1}{\sqrt{2\pi}} e^{-i\tau t}$, $\tau \in \mathbf{R}$ and then integrate on the real line. We obtain

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} Bv'_\sigma(t) dt + \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} (\sigma B - A)v_\sigma(t) dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} f_\sigma(t) dt. \quad (1.2)$$

As both integrals $\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} Bv'_\sigma(t) dt$ and $\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} (\sigma B - A)v_\sigma(t) dt$ exist, we deduce (using Theorem 1.3.5 in [4]), that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} v'_\sigma(t) dt \in D(B)$$

and

$$B \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} v'_\sigma(t) dt \right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} B v'_\sigma(t) dt.$$

In the same way considering separately σB and A , we find that $\frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} v_\sigma(t) dt \in D(A) \cap D(\sigma B)$ and $(\sigma B - A)\widehat{v}_\sigma(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-i\tau t} (\sigma B - A)v_\sigma(t) dt$. We have therefore $B(i\tau\widehat{v}_\sigma(\tau)) + (\sigma B - A)\widehat{v}_\sigma(\tau) = \widehat{f}_\sigma(\tau)$ or also

$$[(\sigma + i\tau)B - A]\widehat{v}_\sigma(\tau) = \widehat{f}_\sigma(\tau), \quad \forall \tau \in \mathbf{R}, \quad (1.3)$$

where $\widehat{v}_\sigma(\tau) \in D(A) \cap D(B)$ and $\widehat{v}_\sigma(\tau), \widehat{f}_\sigma(\tau)$ belong to $L^2(\mathbf{R}, H)$. Taking now $\sigma = \sigma_n$ as in Hypothesis 1 and using (1.3) we infer the relation

$$\widehat{v}_\sigma(\tau) = R(\lambda; A, B)\widehat{f}_\sigma(\tau), \quad \tau \in \mathbf{R}, \quad \lambda = \sigma_n + i\tau, \quad n = 1, 2, \dots \quad (1.4)$$

and the uniform estimate

$$\|\widehat{v}_{\sigma_n}(\tau)\| \leq M \|\widehat{f}_\sigma(\tau)\|, \quad \tau \in \mathbf{R}, \quad n = 1, 2, \dots \quad (1.5)$$

Using now the vector-valued Parseval equality we see that

$$\int_{\mathbf{R}} \|v_{\sigma_n}(t)\|^2 dt \leq M^2 \int_{\mathbf{R}} \|f_{\sigma_n}(t)\|^2 dt. \quad (1.6)$$

Now we remember the definitions of both $v_\sigma(t)$ and $f_\sigma(t)$ and derive easily from (1.6) the new estimate:

$$\int_0^\alpha e^{-2\sigma_n t} \|u(t)\|^2 dt \leq M^2 \int_\alpha^{\alpha+\delta} |\xi'_\delta(t)|^2 e^{-2\sigma_n t} \|Bu(t)\|^2 dt, \quad n = 1, 2, \dots \quad (1.7)$$

Take β such that $0 < \beta < \alpha$ and n_0 such that for $n \geq n_0$, $\sigma_n > 0$ (note that $\sigma_n \rightarrow +\infty$ as $n \rightarrow \infty$). Thus, for $n \geq n_0$ we have

$$e^{-2\sigma_n \beta} \int_0^\beta \|u(t)\|^2 dt \leq M_{1,\delta} \left(\int_\alpha^{\alpha+\delta} \|Bu(t)\|^2 dt \right) e^{-2\sigma_n \alpha} \quad (1.8)$$

($M_{1,\delta} = M^2 \sup_{\alpha \leq t \leq \alpha+\delta} |\xi'_\delta(t)|^2$). It follows that

$$\int_0^\beta \|u(t)\|^2 dt \leq M_{1,\delta} e^{2\sigma_n(\beta-\alpha)} \int_\alpha^{\alpha+\delta} \|Bu(t)\|^2 dt, \quad n \geq n_0. \quad (1.9)$$

As $n \rightarrow \infty$ we get, for $\beta < \alpha$, $\int_0^\beta \|u(t)\|^2 dt = 0$. Hence $u(t) = 0$ on $[0, \beta]$ for all $\beta < \alpha$ and $u(t) = 0$ on $[0, \alpha]$. This is also true for any $\alpha < T$, hence $u(t) = 0$ on $[0, T]$. $\square$

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